On Darboux selections

by

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Abstract. The main result is the following: if F is a planar set hitting each vertical line in a non-void countable set with the property that F hits each horizontal open interval having points of F above and below it, then there exists a subset G of F having the same property and which hits each vertical line exactly once.

In Bruckner and Ceder [1] the following question was posed: Given real-valued functions $f_1, f_2, \ldots, f_k$ each having domain $(-\infty, \infty)$ such that the union of their graphs is a connected set, does there exist a connected function g on $(-\infty, \infty)$ such that for all x, $g(x) = f_i(x)$ for some i?

This question remains unsolved (except, of course, when $k = 1$). Example 1 below shows that there need not be any such function g when the number of functions is countably infinite. However, in the way of a partial solution, if the condition of connectedness is relaxed to the "property of Darboux" then a significant positive result is obtained (Theorem 1).

A real-valued function f on $(-\infty, \infty)$ is Darboux provided it maps intervals into intervals and it is connected if its graph is connected. It turns out that a Darboux function can be characterized by the fact that its graph over any subinterval cannot be separated by a horizontal line whereas a connected function can be characterized by the (obviously stronger) fact that its graph over any subinterval cannot be separated by a continuum. (See [1] for a fuller explanation.)

The concept of a union of graphs being Darboux is formulated in a more general way as follows: A set valued function φ from $E = (-\infty, \infty)$ onto the non-void subsets of E is called a *carrier*. The sets $\{x\} \times \varphi(x)$ will be called *cross sections* of φ. By the *graph* of φ we mean the set $\mathrm{gr}\,\varphi = \bigcup\{\{x\} \times \varphi(x)\colon x \in E\}$.

A carrier φ is called *Darboux* if for each open subinterval I of E the set $\{y\colon y \in \varphi(x)$ for some $x \in I\}$ is an interval. A carrier φ is called *connected* if $\bigcup\{\{x\} \times \varphi(x)\colon x \in E\}$ is a connected set. A carrier φ is called *strongly connected* if for each open subinterval I of E the set $\bigcup\{\{x\} \times \varphi(x)\colon x \in I\}$ is connected. A strongly connected carrier is easily seen to be Darboux. However, a connected carrier need not be Darboux as shown by Example 1. Finally, a function f on E is called a *selection for* φ provided $f(x) \in \varphi(x)$ for all $x \in E$.

The above unsolved question, the cited example and theorem can be phrased in carrier-selection terminology as follows:

QUESTION 1. Does there exist a connected selection for a connected carrier on E with finite cross sections?

EXAMPLE 1. There exists a connected carrier on E with denumerable cross sections having no connected selection.

It is unknown whether one can make the carrier strongly connected in the statement of Example 1.

THEOREM 1. *A Darboux carrier in E with countable cross sections admits a Darboux selection.*

In other words, if F is a planar set hitting each vertical line in a non-void countable set with the property that F hits each horizontal open interval having points of F above and below it, then there exists a subset G of F having the same property and which hits each vertical line exactly once.

The above theorem cannot be improved to apply to arbitrary cross sections as shown by the following example.

EXAMPLE 2. There exists a Darboux and strongly connected carrier with intervals as cross sections admitting neither a Darboux selection nor a connected selection.

Proof. Decompose E into disjoint sets A, B and C where A is the set of rationals and B and C are each c-dense in E. Then define

$$\varphi(x)=\begin{cases} E & \text{if} \quad x\in A,\\ [0,1] & \text{if} \quad x\in B,\\ [2,3] & \text{if} \quad x\in C.\end{cases}$$

Clearly φ is strongly connected and Darboux. However, if f is any selection for φ then the image under f of any interval contains $[0,1]$ and $[2,3]$ but only countably many points of $(1,2)$. Hence, f cannot be connected or Darboux.

The proofs of Example 1 and Theorem 1 appear later.

Theorem 1 gives a sufficient condition on a Darboux carrier to admit a Darboux selection in terms of the cardinality of the cross sections. It says that if the cross sections are small enough cardinality-wise, then there is a Darboux selection. If, however, the cross sections of an arbitrary carrier have cardinality c the carrier may not admit a Darboux selection (e.g., Example 2), but there will be a Darboux selection provided the cross sections fit together in a nice enough way. One well-known result in this direction is that a carrier φ on E defined by $\varphi(x)=\big(h(x),g(x)\big)$ where g is lower-semicontinuous and h is upper-semicontinuous, admits a continuous (hence, connected and Darboux) selection.

Another sufficient condition for a given carrier to admit a Darboux section is the following

THEOREM 2. *If φ is a carrier with the property that each horizontal open interval which has points of $\operatorname{gr}\varphi$ above and below it intersects c cross sections, then φ admits a Darboux selection.*

Proof. Well-ordered the collection β of all open horizontal intervals which have points of $\operatorname{gr}\varphi$ above and below by the ordinal c so that $\beta=\{B_\alpha\colon \alpha<c\}$. By transfinite induction choose $b_\alpha\in(\operatorname{gr}\varphi\cap B_\alpha)-\bigcup\{V(b_\beta)\colon \beta<\alpha\}$ where $V(b_\beta)$ is the vertical line passing through b_β. Let $F=\{b_\alpha\colon \alpha<c\}$. For $x\notin\operatorname{dom}F$ choose d_x arbitrarily in $\varphi(x)$. Put $f=F\cup\{(x,d_x)\colon x\notin\operatorname{dom}F\}$. Then f is obviously a Darboux selection for φ.

A related previously known result is the following

THEOREM 3 (Ceder and Weiss [4]). *If the cross sections of a carrier φ are open intervals of the form $\{x\}\times(g(x),h(x))$ and each horizontal open interval having points of $\operatorname{gr}\varphi$ above and below it intersects c cross sections, then there exists a Darboux selection f of φ such that $f^{-1}(\lambda)$ has cardinality 0 or c for each λ.*

The condition of either Theorem 3 (or 2) is clearly not necessary. Moreover, it is met, for example, when both g and h are Darboux functions in Baire class 1. Incidentally, if g and h are Darboux functions in Baire class 2 the condition of Theorem 3 need not hold and there does not necessarily exist a Darboux selection. See [4] and [2] for details.

For related results regarding selections which are open and Darboux or Darboux and in Baire class α see Ceder and Weiss [4], Ceder and Pearson [3] and Bruckner, Ceder and Pearson [2]. For results regarding continuous selections in a more general topological setting see Michael [5], for example.

Before presenting the construction of Example 1 and the proof of Theorem 1, we need to establish the following terminology. A function will be identified with its graph. The symbol $|A|$ will denote the cardinality of A. Cardinal numbers will be taken to be ordinal numbers which are not in one-to-one correspondence with smaller ordinals. If $z\in E^2$, then $H(z)$ and $V(z)$ will denote the horizontal and vertical line through z respectively.

EXAMPLE 1. There exists a connected carrier on E with denumerable cross sections admitting no Darboux (and hence connected) selection.

Proof. Let $\mathscr{G}$ denote the family of all non-void nowhere dense closed subsets of the plane which cannot be covered by a small (i.e., less than c) number of horizontal or vertical lines. Well-order $\mathscr{G}$, union the set of all horizontal lines and all rational vertical open intervals by the ordinal c and write it as $\{A_\alpha\}_{\alpha<c}$. By transfinite induction on c we will define a set $\{a_\alpha\colon \alpha<c\}$ with $a_\alpha\in A_\alpha$ as follows: choose a_0 to be any point in A_0. Now suppose a_α has been choosen for each $\alpha<\beta$. Pick a_β to be any point in the set

$$\begin{array}{lll} A_\beta-\bigcup\{H(a_\alpha)\cup V(a_\alpha)\colon \alpha<\beta\}, & \text{if} & A_\beta\in\mathscr{G},\\ \{a_\alpha\colon A_\beta\subseteq H(a_\alpha)\}, & \text{if} & A_\beta-\bigcup\{H(a_\alpha)\colon \alpha<\beta\}=\Lambda,\\ A_\beta-\bigcup\{H(a_\alpha)\colon \alpha<\beta\}, & \text{if} & A_\beta-\bigcup\{H(a_\alpha)\colon \alpha<\beta\}\neq\Lambda.\end{array}$$

It is easily verified that the selection of a_β is always possible.

Let us define φ by $\{x\}\times\varphi(x) = \{a_\alpha\colon a_\alpha \in V(x)\}$. Then clearly $|\varphi(x)| = \aleph_0$ so each cross section is denumerable and dense. Also it is clear that each horizontal line hits $\operatorname{gr}\varphi$ exactly once, i.e., $|\{x\colon y\in\varphi(x)\}| = 1$. Moreover, $\operatorname{gr}\varphi$ hits each member of $\mathscr{G}$ by construction.

To show $\operatorname{gr}\varphi$ is connected, let us suppose that $\operatorname{gr}\varphi$ is disconnected by two non-void open sets each hitting $\operatorname{gr}\varphi$ and having the property that the union of their closures is E^2. Let G denote the boundary of one of these. By construction $\operatorname{gr}\varphi$ hits G if $G\in\mathscr{G}$. Hence $G\notin\mathscr{G}$ and G is contained in the union of a small number of rational and horizontal lines. Moreover, since $\operatorname{gr}\varphi$ hits each vertical interval, $\operatorname{gr}\varphi$ must be contained in the union of a small number of horizontal lines. From this it follows that some horizontal line is contained entirely in G. But this is in contradiction to the fact that $\operatorname{gr}\varphi$ hits each horizontal line. Therefore, $\operatorname{gr}\varphi$ is connected.

Now suppose there exists a connected selection f for φ. Since $\operatorname{gr}\varphi$ hits each horizontal line exactly once f must be one-to-one. Now suppose $a<c<b$. If $f(c)$ is not between $f(a)$ and $f(b)$, then a vertical open half ray with end at $(c, f(c))$ union a horizontal open half ray with end at $(a, f(a))$ or $(b, f(b))$ would separate f. Therefore, f is strictly monotonic. A similar argument shows that f is also continuous.

Thus the image under f of the unit open interval $(0, 1)$ is an open interval J. Let K be an open interval in J centered at $f(\frac{1}{2})$. Then the projection of K upon $V(\frac{1}{2}, f(\frac{1}{2}))$ contains another point z of $\operatorname{gr}\varphi$. Hence, $H(z)$ hits $\operatorname{gr}\varphi$ at z and a point of f, which contradicts the fact that each horizontal line hits $\operatorname{gr}\varphi$ once. Hence no connected selection exists.

Obviously the φ of Example 1 is not strongly connected. It is unknown if such an example exists for a strongly connected carrier φ.

Before starting the proof of Theorem 1 we introduce the following terminology. Let φ denote any Darboux carrier with countable cross sections. We say that a point z of the graph of φ is a *singleton* point of $\operatorname{gr}\varphi$ provided $|V(z)\cap\operatorname{gr}\varphi| = 1$. We say that $z\in\operatorname{gr}\varphi$ is a *density* point of $\operatorname{gr}\varphi$ if there exists a dense-in-itself $A\subseteq H(z)\cap\cap\operatorname{gr}\varphi$ for which $z\in A$. We say that $z\in\operatorname{gr}\varphi$ is a *segregated* point of $\operatorname{gr}\varphi$ if z is not a singleton or density point.

First we need the following three lemmas, the first of which is well known.

LEMMA 1. *Each linear set L can be decomposed into two disjoint sets one of which, $D(L)$, is dense-in-itself and the other, $C(L)$, is a countable scattered set (i.e., it contains no dense-in-itself subset).*

Proof. Let L be a linear set and $\mathscr{A}$ be the set of all dense-in-itself subsets of L. By Zorn's lemma we may choose a maximal member $B(L)$ of $\mathscr{A}$. Put $C(L) = L-B(L)$. Then $C(L)$ has no dense-in-itself subset. Moreover, if $|C(L)|>\aleph_0$, then the set of condensation points of $C(L)$ forms a dense-in-itself set. Hence, $C(L)$ is countable and scattered.

In the next lemma we prove that Theorem 1 is valid for a special kind of Darboux carrier.

LEMMA 2. *A Darboux carrier ψ with denumerable cross sections having no segregated point admits a Darboux selection.*

Proof. Let $\mathscr{H} = \{H(z)\colon z\in\operatorname{gr}\psi\}$. Then for each $H\in\mathscr{H}$ the linear set $H\cap\operatorname{gr}\psi$ can be decomposed into a dense-in-itself set $D(H)$ and a countable scattered set $C(H)$. Moreover, each $D(H)$ can be decomposed into a family of disjoint sets each $\aleph_0$-dense-in-itself, $\{D_\alpha(H)\colon \alpha\in\Gamma(H)\}$ where $\Gamma(H)$ is some ordinal $\leqslant c$. Hence, $H\cap\operatorname{gr}\psi = \bigcup\{D_\alpha(H)\colon \alpha\in\Gamma(H)\}\cup C(H)$.

Now for any planar set A put

$$M(A) = \bigcup\{D_\alpha(H)\colon \text{there exist } H\in\mathscr{H},\ \alpha\in\Gamma(H) \text{ such that } D_\alpha(H)\cap\bigcup\{V(z)\colon z\in A\}\neq\Lambda\}.$$

By induction define $G_0(A) = M(A)$ and $G_{n+1}(A) = M(G_n(A))$ and put $G(A) = \bigcup_{n=0}^{\infty} G_n(A)$. The sets $G(D_\alpha(H))$ are called *networks* and we put $\mathscr{N} = \{G(D_\alpha(H))\colon H\in\mathscr{H}, \alpha\in\Gamma(H)\}$. Then the following facts are easy to prove: Each network $G(D_\alpha(H))$ is countable; $N_1, N_2\in\mathscr{N}$ imply $N_1 = N_2$ or $N_1\cap N_2 = \Lambda$; each non-empty $N\cap H$ is $\aleph_0$-dense in each $D(H)$ for $H\in\mathscr{H}$ and distinct x-projections of members of $\mathscr{N}$ are disjoint.

If $z\in C(H)$, then $V(z)\cap\operatorname{gr}\psi = \{z\}$. Hence, letting $\mathscr{N}$ be the system of networks for ψ, $\operatorname{gr}\psi = \bigcup\mathscr{N}\cup\bigcup\{C(H)\colon H\in\mathscr{H}\}$.

Next we define for each network $N\in\mathscr{N}$ a function $f_N\subseteq N$ such that f_N is dense in any non-empty $H\cap N$ for $H\in\mathscr{H}$. To this end let $\{O_n\}_{n=0}^{\infty}$ be the set of all rational open horizontal intervals which hit N. Pick $w_0\in O_0\cap N$ and $w_{n+1}\in O_{n+1}\cap N-\bigcup_{i=0}^{n} V(w_i)$. Put $f_N = \{w_i\colon i\in\omega\}$.

Finally put

$$f(x) = f_N(x) \quad\text{if}\quad x\in\operatorname{dom}f_N \text{ for some } N,$$
$$f(x) \in \psi(x) \quad\text{if}\quad x\notin\bigcup\{\operatorname{dom}f_N\colon N\in\mathscr{N}\}.$$

It is easily checked that f is a Darboux selection for ψ.

We now proceed to take care of the general case of a carrier with segregated points. The procedure will be to judiciously remove points from the cross sections in order to obtain a Darboux "subcarrier" without segregated points and then apply Lemma 3.

LEMMA 3. *Let S denote the set of segregated points of φ and let F be a countable dense subset of $\operatorname{gr}\varphi$.*

Then there exists a function $g\subseteq S$ such that

(1) *If B is an open interval in some $H(z)$ having endpoints in $\bigcup\{V(x)\colon w\in F\}$ for which $|B\cap S|\geqslant\aleph_0$, then $B\cap g\neq\Lambda$.*

(2) *$D_\alpha(H)-\bigcup\{V(w)\colon w\in g\}$ is dense-in-itself for each α and H.*

Proof. Define networks as in the proof of Lemma 2 except one combines $D_0(H)$ and $C(H)$ into a new $D_0(H)$. Call the system of networks $\mathscr{M}$. Then as before one can show that each $M \in \mathscr{M}$ is countable and the x-projections of distinct members of $\mathscr{M}$ are disjoint.

For each $M \in \mathscr{M}$, let $\{B_{2i}\}_{i=0}^{\infty}$ be an enumeration of the rational, horizontal open intervals which intersect some $D(H \cap M)$ for $H \in \mathscr{H}$. Let $\{B_{2i+1}\}_{i=0}^{\infty}$ be an enumeration of all horizontal open intervals having endpoints in $\bigcup \{V(w)\colon w \in F\}$ and containing infinitely many points of $S \cap M$. (Without loss of generality we may assume that there is such a sequence.)

Pick by induction

$$b_i \in S \cap B_i - \bigcup \{V(b_j)\colon j<i\} \quad \text{if} \quad i \text{ is odd},$$
$$b_i \in M \cap B_i - \bigcup \{V(b_j)\colon j<i\} \quad \text{if} \quad i \text{ is even}.$$

Clearly the choice of b_i is always possible.

Put $g_M = \{b_{2i+1}\colon i \in \omega\}$. Now define $g = \bigcup \{g_M\colon M \in \mathscr{M}\}$. Clearly g is a function contained in S. Then g satisfies conditions (1) and (2). To show (1), let B be such an open interval in $H(z)$. Let M be the network containing $D_0(H)$ (and hence $B \cap S$). Then by construction g_M hits B.

To show (2), consider any $D_\alpha(H)$. Let M be the network containing $D_\alpha(H)$. By construction the corresponding set $\{b_{2i}\colon i \in \omega\}$ forms a dense subset of $D_\alpha(H) - \bigcup \{V(w)\colon w \in g_M\}$ and hence of $D_\alpha(H) - \bigcup \{V(w)\colon w \in g\}$, finishing the proof of Lemma 3.

We now define a new subcarrier ψ of φ by the following procedure: from a cross section $\{x\} \times \varphi(x)$ we remove

$$\begin{array}{lll} \text{all points except } (x, g(x)) & \text{if} & x \in \operatorname{dom} g, \\ \text{all points in } S & \text{if} & x \in \operatorname{dom} S - \operatorname{dom} g \text{ and} \\ & & (\{x\} \times \varphi(x)) - S \neq \Lambda, \\ \text{all points but one} & \text{if} & x \in \operatorname{dom} S - \operatorname{dom} g \text{ and} \\ & & \{x\} \times \varphi(x) \subseteq S. \end{array}$$

Hence,

$$\psi(x) = \begin{cases} \operatorname{range}(g \cap V(x)) & \text{if} \quad x \in \operatorname{dom} g, \\ \varphi(x) & \text{if} \quad x \notin \operatorname{dom} S, \\ \operatorname{range}[(\{x\} \times \varphi(x)) - S] & \text{if} \quad x \in \operatorname{dom} S - \operatorname{dom} g \text{ and} \\ & \qquad (\{x\} \times \varphi(x)) - S \neq \Lambda, \\ \{y\}, \text{ where } (x, y) \text{ is any point in } S \cap \operatorname{gr}\varphi & \text{if} \quad x \in \operatorname{dom} S - \operatorname{dom} g \text{ and} \\ & \qquad \{x\} \times \varphi(x) \subseteq S. \end{cases}$$

Then we may prove the following:

(a) $g \subseteq \operatorname{gr}\psi \subseteq \operatorname{gr}\varphi$,

(b) ψ is Darboux,

(c) ψ has no segregated points.

Part (a) is obvious. To prove ψ is Darboux, let us suppose ψ is not Darboux. Then there are points (a, b) and (c, d) in $\operatorname{gr}\psi$ and an open segment B above the open interval (a, c) and between the two points which fails to intersect $\operatorname{gr}\psi$. Since $\operatorname{gr}\psi \subseteq \operatorname{gr}\varphi$ and F is a countable dense subset of $\operatorname{gr}\varphi$ and φ is Darboux, we may take the points (a, b) and (c, d) to be in $\operatorname{gr}\varphi$ and assume B has endpoints in $\bigcup \{V(w)\colon w \in F\}$. Then B must contain points of $\operatorname{gr}\psi$ which have been removed. First of all B contains no density points of φ because of part (2) of Lemma 3. Hence, B contains only segregated points of $\operatorname{gr}\varphi$.

By part (1) of Lemma 3, B cannot contain infinitely many segregated points of $\operatorname{gr}\varphi$. Hence, B only contains a finite number of segregated points of $\operatorname{gr}\varphi$ which we order in ascending horizontal order $z_1, z_2, \ldots, z_k$. Suppose without loss of generality that the point (c, d) is above B and (a, b) is below it. Then, because φ is Darboux there can be no point of $\{z_k\} \times \varphi(z_k)$ below B and there is a point of $\{z_k\} \times \varphi(z_k)$ above B since z_k is not a singular point. The same can be said of z_{n-1}. Continuing in this way it is also true for z_1. Hence there is no point of $\{z_1\} \times \varphi(z_1)$ below B and there is a point, say w, in $\{z_1\} \times \varphi(z_1)$ above B. Now applying the Darboux condition $\operatorname{gr}\varphi$ crosses B between (a, b) and w, a contradiction to the fact that z_1 was the first such point.

To show ψ has no segregated points let us suppose $z = (x, y)$ is a segregated point of ψ. Then z is not a singleton point of ψ so that $\psi(x) = \varphi(x)$ or $\psi(x) = \operatorname{range}[(\{x\} \times \varphi(x)) - S]$ in which case $x \notin \operatorname{dom} S$ or $x \in \operatorname{dom} S - \operatorname{dom} g$ where $(\{x\} \times \varphi(x)) - S \neq \Lambda$ respectively. In either of these cases, it follows that z is a density point of $\operatorname{gr}\varphi$. There exists an α such that $z \in D_\alpha(H(z))$.

However, the only points in $D_\alpha(H(z))$ which were removed were points in $\bigcup \{V(w)\colon w \in g\}$ by the definition of ψ. But by part (2) of Lemma 3 $D_\alpha(H) - \bigcup \{V(w)\colon w \in g\}$ is dense in itself and moreover contains z. Therefore z is a density point for ψ, a contradiction.

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Accepté par la Rédaction le 29. 4. 1974