On the Whitehead theorem in shape theory I

by

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Abstract. Recently M. Moszyńska [17] has proved for finite-dimensional metric compact spaces and shape maps $f\colon (X, x_0)\to(Y, y_0)$ an analogue of the classical Whitehead theorem. In this paper another analogue of the Whitehead theorem in shape theory is established. (X, x_0) and (Y, y_0) are allowed to be arbitrary topological spaces of finite covering dimension. However, $f\colon (X, x_0)\to(Y, y_0)$ is required to be a continuous map. If f induces isomorphisms of the homotopy pro-groups, then f is shown to be a shape equivalence (in the sense of [12]). At the same time a shorter proof of Moszyńska's original theorem is given.

1. Introduction

Of great importance in homotopy theory is this classical theorem of J. H. C. Whitehead: *Let $f\colon (X, x_0)\to(Y, y_0)$ be a map of connected (pointed) CW-complexes and let $f_i\colon \pi_i(X, x_0)\to\pi_i(Y, y_0)$ be an isomorphism for $i<n_0 = \max\{1+\dim X, \dim Y\}$ and an epimorphism for $i = n_0$. Then f is a homotopy equivalence.* An analogue of this theorem in shape theory has been recently proved by M. Moszyńska [20] (1). Her result reads as follows:

THEOREM (M. Moszyńska). *Let $f\colon (X, x_0)\to(Y, y_0)$ be a shape map of connected (pointed) metric finite-dimensional compacta and let $f_k\colon \pi_k(X, x)\to\pi_k(Y, y)$ be the induced maps of homotopy pro-groups. If f_k is a bimorphism for $0\leqslant k<n_0+1 = \max\{1+\dim X, \dim Y\}+1$ and an epimorphism for $k = n_0+1$, then f is a shape equivalence.*

In this paper we extend Moszyńska's theorem to the case of arbitrary topological spaces and *continuous maps* $f\colon (X, x_0)\to(Y, y_0)$ (see Theorem 6). At the same time we obtain a simpler proof of the original result, which consists entirely of steps analogous to corresponding steps in the proof of the classical Whitehead theorem.

The notion of shape used in this paper is that described in [12]. It has been shown by K. Morita [16] that also in the case of topological spaces shape can be treated essentially as in [14], i.e. that shape reduces to pro-homotopy category of CW-complexes. This made the present generalization to topological spaces possible.

(1) In [20] Moszyńska also exhibited a simpler theorem for the case when (X, x_0) and (Y, y_0) are assumed to be movable. However, the proof contains a gap. Moszyńska has recently modified this second theorem in a note of correction to appear in Fund. Math.

The question of whether Moszyńska's theorem holds for topological spaces and *shape maps* remains open. The difficulty consists in setting up a satisfactory mapping cylinder. It should also be said that in this paper we do not discuss the case of movable spaces.

2. Category pro ($\mathscr{K}$)

2.1. With every category $\mathscr{K}$ one can associate a new category pro($\mathscr{K}$) introduced in [7] (see also [1]). One first considers a category $\underline{\mathscr{K}}$ whose objects are inverse systems $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, where $(\Lambda, \leqslant)$ is a directed set (²) and X_λ and $p_{\lambda\lambda'}$, $\lambda \leqslant \lambda'$, are objects and morphisms from $\mathscr{K}$ respectively. A morphism $\underline{f}: X \to Y = (Y_\mu, q_{\mu\mu'}, M)$ in $\underline{\mathscr{K}}$, called a *mapping of systems*, consists of a map $f: M \to \Lambda$ and of a collection of morphisms $f_\mu: X_{f(\mu)} \to Y_\mu$ such that for $\mu \leqslant \mu'$ there exists a $\lambda \geqslant f(\mu), f(\mu')$ for which $f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}$. If $\underline{g}: Y \to Z = (Z_\nu, r_{\nu\nu'}, N)$, then the composition $\underline{h} = \underline{gf}$: $X \to Z$ is given by $h = fg: N \to \Lambda$ and $h_\nu = g_\nu f_{g(\nu)}: X_{h(\nu)} \to Z_\nu$. The identity $\underline{1}: X \to X$ is given by $1: \Lambda \to \Lambda$ and $1_\lambda = 1: X_\lambda \to X_\lambda$. Two mappings of systems $\underline{f}, \underline{g}: X \to Y$ are considered to be equivalent, $\underline{f} \simeq \underline{g}$, provided for every $\mu \in M$ there is a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), g(\mu)$, such that $f_\mu p_{f(\mu)\lambda} = g_\mu p_{g(\mu)\lambda}$. This is an equivalence relation on every set $\underline{\mathscr{K}}(X, Y)$. Moreover, $\underline{f} \simeq \underline{f}'$ and $\underline{g} \simeq \underline{g}'$ implies $\underline{gf} \simeq \underline{g'f'}$. We thus obtain a quotient category $\underline{\mathscr{K}}/\simeq$ called *pro*($\mathscr{K}$). It has the same objects X, Y, etc. as $\underline{\mathscr{K}}$ and its morphism are equivalence classes of morphisms $\underline{f}$ from $\underline{\mathscr{K}}$; the class containing $\underline{f}$ will be denoted by f (cf. [18], [20], [12]).

2.2. Especially simple are maps of systems $\underline{f}: X \to Y$ where both X and Y are indexed by the same set Λ, $f = 1: \Lambda \to \Lambda$ and $f_\lambda p_{\lambda\lambda'} = q_{\lambda\lambda'} f_{\lambda'}$ for $\lambda \leqslant \lambda'$. We refer to such maps as to *special maps of systems*.

For every map of systems $\underline{f}: X \to Y$ there exist systems X', Y' and maps of systems $\underline{i}: X \to X'$, $\underline{j}: Y \to Y'$, $\underline{f}': X' \to Y'$ such that $f'i = jf$, i and j are isomorphisms in pro($\mathscr{K}$) and $\underline{f}'$ is a special map of systems. In order to see this, one considers the set N of all morphisms $n_\mu^\lambda \in \mathscr{K}(X_\lambda, Y_\mu)$ which admit a $\lambda' \geqslant f(\mu)$, λ such that $n_\mu^\lambda p_{\lambda\lambda'} = f_\mu p_{f(\mu)\lambda'}$. Note that λ and μ run through cofinal subsets of Λ and M respectively as n_μ^λ runs through N. Putting $n_\mu^\lambda \leqslant n_{\mu'}^{\lambda'}$ provided $\lambda \leqslant \lambda'$, $\mu \leqslant \mu'$ and $n_\mu^\lambda p_{\lambda\lambda'} = q_{\mu\mu'} n_{\mu'}^{\lambda'}$, N becomes a directed set. Now one defines $X' = (X'_n, p'_{nn'}, N)$, $Y' = (Y'_n, q'_{nn'}, N)$ by putting $X'_{n_\mu^\lambda} = X_\lambda$, $Y'_{n_\mu^\lambda} = Y_\mu$, $p'_{n_\mu^\lambda n_{\mu'}^{\lambda'}} = p_{\lambda\lambda'}$, $q'_{n_\mu^\lambda n_{\mu'}^{\lambda'}} = q_{\mu\mu'}$ and $\underline{f}' \in \underline{\mathscr{K}}(X', Y')$ by $f'_{n_\mu^\lambda} = n_\mu^\lambda$. Furthermore, $\underline{i}$ consists of a map $i: N \to \Lambda$, where $i(n_\mu^\lambda) = \lambda$ and of identity morphisms $i_{n_\mu^\lambda} = 1: X_\lambda \to X'_{n_\mu^\lambda} = X_\lambda$, $\underline{j}$ consists of a map $j: N \to M$, where $j(n_\mu^\lambda) = \mu$ and of $j_{n_\mu^\lambda} = 1: Y_\mu \to Y'_{n_\mu^\lambda} = Y_\mu$ (cf. Corollary 3.2, p. 160 of [1]).

2.3. $(M, \leqslant)$ is said to be *closure-finite* provided every $\mu \in M$ has only finitely many predecessors. In this case every function $f: M \to \Lambda$ into an ordered set admits an increasing function $f': M \to \Lambda$ such that $f \leqslant f'$ (Lemma 5, [14]).

(²) We do not need the greater generality as in [1].

Every system $X = (X_\lambda, p_{\lambda\lambda}, \Lambda)$ admits a system $Y = (Y_\mu, q_{\mu\mu'}, M)$ isomorphic to X in pro($\mathscr{K}$) and such that M is closure-finite. Moreover, every Y_μ is some X_λ, and every $q_{\mu\mu'}: Y_{\mu'} \to Y_\mu$ is some $p_{\lambda\lambda'}: X_\lambda \to X_{\lambda'}$ (cf. Theorem 10 of [12]). Indeed, one can take for $(M, \leqslant)$ the set of all finite subsets $\mu = \{\lambda_1, \ldots, \lambda_n\}$ of Λ ordered by inclusion. One defines then an increasing function $f: M \to \Lambda$ such that $f(\{\lambda\}) = \lambda$ for every $\lambda \in \Lambda$. Let $Y_\mu = X_{f(\mu)}$ and $q_{\mu\mu'} = p_{f(\mu)f(\mu')}$, $\mu \leqslant \mu'$, $Y = (Y_\mu, q_{\mu\mu'}, M)$. One defines maps of systems $\underline{f}: X \to Y$ and $\underline{g}: Y \to X$ by $f_\mu = 1: X_{f(\mu)} \to Y_\mu = X_{f(\mu)}$, $g(\lambda) = \{\lambda\}$, $g_\lambda = 1: Y_{g(\lambda)} = X_\lambda \to X_\lambda$. Clearly, $fg(\lambda) = \lambda$, $g_\lambda f_{g(\lambda)} = 1: X_\lambda \to X_\lambda$, so that $\underline{gf} = \underline{1}$. On the other hand, $gf(\mu) = \{f(\mu)\}$ and $f_\mu g_{f(\mu)} = 1: Y_{gf(\mu)} = X_{f(\mu)} \to X_{f(\mu)} = Y_\mu$. If $\mu' \geqslant gf(\mu)$, μ, then

$$f_\mu g_{f(\mu)} q_{gf(\mu)\mu'} = p_{f(\mu)f(\mu')} = q_{\mu\mu'}: Y_{\mu'} = X_{f(\mu')} \to X_{f(\mu)} = Y_\mu,$$

so that $\underline{fg} = \underline{1}$.

3. Pro-homotopy category and shapes

3.1. Let $\mathscr{W}$ denote the category whose objects are topological spaces having the homotopy type of a CW-complex and whose morphisms are homotopy classes of maps. Note that a topological space has the homotopy type of a CW-complex if and only if it has the homotopy type of a simplicial CW-complex (with the weak or with the metric topology) or equivalently of an ANR for metric spaces (see e.g. [12]).

Following K. Morita [16], we say that an inverse system $X = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathscr{W}$ is associated with a topological space X provided there are homotopy classes $p_\lambda: X \to X_\lambda$ such that $p_{\lambda\lambda'} p_{\lambda'} = p_\lambda$ for $\lambda \leqslant \lambda'$, each homotopy class $m: X \to P$, where P is an object of $\mathscr{W}$, admits a factorization $m = m_\lambda p_\lambda$, where, $m_\lambda \in \mathscr{W}(X_\lambda, P)$ and whenever $m_\lambda p_\lambda = m_{\lambda'} p_\lambda$, $m_\lambda, m_{\lambda'} \in \mathscr{W}(X_\lambda, P)$, then there is a $\lambda' \geqslant \lambda$ such that $m_\lambda p_{\lambda\lambda'} = m_{\lambda'} p_{\lambda\lambda'}$ (compare with conditions (i), (ii) in Theorem 5.2 of [12] and with [9]).

3.2. If Y is isomorphic with X in pro($\mathscr{W}$) and X is associated with a space X then so is Y. Indeed, let $f: X \to Y$, $g: Y \to X$ be inverse isomorphisms. We define homotopy classes $q_\mu: X \to Y_\mu$ by $q_\mu = f_\mu p_{f(\mu)}$. Clearly, $\mu \leqslant \mu'$ implies $q_\mu = q_{\mu\mu'} q_{\mu'}$. If $m: X \to P$ is a homotopy class of maps, $P \in \operatorname{Ob} \mathscr{W}$, then a factorization exists $m = m_\lambda p_\lambda$, where $m_\lambda: X_\lambda \to P$. Since $\underline{gf} = \underline{1}$, there is a $\lambda' \geqslant \lambda$, $fg(\lambda)$ such that $g_\lambda f_{g(\lambda)} p_{fg(\lambda)\lambda'} = p_{\lambda\lambda'}$ and therefore, $g_\lambda f_{g(\lambda)} p_{fg(\lambda)} = p_\lambda$. Consequently, $(m_\lambda g_\lambda) q_{g(\lambda)} = m_\lambda p_\lambda = m$. Similarly, $n_\mu q_\mu = n'_\mu q_\mu$ implies $n_\mu f_\mu p_{f(\mu)} = n'_\mu f_\mu p_{f(\mu)}$ and therefore $n_\mu f_\mu p_{f(\mu)\lambda} = n'_\mu f_\mu p_{f(\mu)\lambda}$ for some $\lambda \geqslant f(\mu)$. Since $\underline{fg} = \underline{1}$, there is a $\mu' \geqslant gf(\mu)$, μ, $g(\lambda)$ such that

$$q_{\mu\mu'} = f_\mu g_{f(\mu)} q_{gf(\mu)\mu'} = f_\mu p_{f(\mu)\lambda} g_\lambda q_{g(\lambda)\mu'}.$$

Consequently, $n_\mu q_{\mu\mu'} = n'_\mu q_{\mu\mu'}$.

3.3. Morita has shown [16] that every topological space X is associated with the inverse system X in $\mathscr{W}$ formed by the nerves of all open locally-finite normal (³)

(³) By definition, an open covering λ of X is normal provided there is a sequence of open coverings λ_n, $n \in N$, such that $\lambda_0 = \lambda$ and λ_{n+1} is a star-refinement of λ_n. The existence of canonical mappings shows that open locally-finite normal coverings coincide with open locally finite numerable coverings as defined in [4].

coverings of X. For $p_\lambda\colon X\to X_\lambda$ one takes (unique) homotopy classes determined by canonical mappings, i.e. mappings $\varphi_\lambda\colon X\to X_\lambda$ such that $(\varphi_\lambda)^{-1}(\mathrm{St}(U, X_\lambda))\subset U$ for every element U of the covering λ.

3.4. Following Morita [15] we say for a topological space X that $\dim X\leqslant n<\infty$ provided every finite open normal covering admits a finite open normal refinement of order $\leqslant n+1$. Notice that $\dim X\leqslant n$ implies that every locally finite open normal covering $\mathscr{U}=(U_\alpha, \alpha\in A)$ admits a locally finite open normal refinement of order $\leqslant n+1$. Indeed, let $\varphi\colon X\to N(\mathscr{U})$ be a canonical map, where $N(\mathscr{U})$ is the nerve of $\mathscr{U}$ provided with the metric topology. Then there is a metric space M, $\dim M\leqslant n$, and a factorization $\varphi=\chi\psi$ through M ([15], Lemma 2.2). If $V_\alpha=\chi^{-1}(\mathrm{St}(U_\alpha, N(\mathscr{U})))$, then $\mathscr{V}=(V_\alpha, \alpha\in A)$ is an open covering of M such that $\Psi^{-1}(V_\alpha)\subset U_\alpha$. Since M is metric and $\dim M\leqslant n$, $\mathscr{V}$ admits an open locally finite refinement $\mathscr{W}$ of order $\leqslant n+1$ ([21], Theorem II. 6, p. 22). $\mathscr{W}$ is normal because every open covering of a metric space is normal. Now $\psi^{-1}(\mathscr{W})$ is an open covering of X with all the desired properties.

Since the nerve of an open covering of a connected space is connected, we see that every connected space X with $\dim X\leqslant n$ admits an associated system $\boldsymbol{X}$ in $\mathscr{W}$ all of whose members are connected simplicial complexes of dimension $\leqslant n$.

3.5. In this paper shape theory for topological spaces is understood in the sense of [12]. It thus coincides with Borsuk's theory [2] on metric compacta (see [12]) with Fox's theory [6] on metric spaces (see [13], [16]) and with the ANR-system approach [14] on compact Hausdorff spaces (see [12], [16]).

Generalizing results from [14] and [12] Morita has shown ([16], Theorem 2.4) that there is a functorial bijection between shape maps $\underline{\varphi}\colon X\to Y$ of topological spaces and morphisms $\boldsymbol{f}\colon \boldsymbol{X}\to\boldsymbol{Y}$ from $\mathrm{pro}(\mathscr{W})$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are systems associated with X and Y respectively. $\underline{\varphi}$ and $\boldsymbol{f}$ correspond to each other provided $\underline{\varphi}(q_\mu)=f_\mu p_{f(\mu)}$ for every $\mu\in M$. This fact enables us to use the same notation $\boldsymbol{f}$ for morphisms $\boldsymbol{X}\to\boldsymbol{Y}$ from $\mathrm{pro}(\mathscr{W})$ and for the corresponding shape maps $X\to Y$. A similar approach to shape was studied by T. Porter [22], [23].

3.6. In this paper we also consider the homotopy category $\mathscr{W}_0$ of pointed spaces (X, x_0) having the homotopy type of a pointed CW-complex (the base-point is a 0-cell of the CW-decomposition). We also deal with the category $\mathscr{W}_0^2$ of pairs of pointed spaces (X, A, x_0), $x_0\in A\subset X$, having the homotopy type of a pointed CW-complex and a subcomplex. The shape of pointed spaces and pairs of pointed spaces can be described in terms of $\mathrm{pro}(\mathscr{W}_0)$ and $\mathrm{pro}(\mathscr{W}_0^2)$ following the same pattern as in the absolute case [16].

4. Pro-category of groups

4.1. Let $\mathscr{G}$ be the category of groups and homomorphisms. Objects of the corresponding pro-category $\mathrm{pro}(\mathscr{G})$ are called *pro-groups*. In this section we recall some facts about $\mathrm{pro}(\mathscr{G})$ essentially established already by Moszyńska in [20].

$\mathrm{pro}(\mathscr{G})$ is a category with zero objects. Indeed, a system $\boldsymbol{0}$ consisting of a single trivial group is obviously a zero-object in $\mathrm{pro}(\mathscr{G})$. A pro-group $\boldsymbol{G}=(G_\lambda, p_{\lambda\lambda'}, \Lambda)$

is a zero-object of $\mathrm{pro}(\boldsymbol{G})$ if and only if it is isomorphic with $\boldsymbol{0}$. This is the case if and only if every λ admits a $\lambda'\geqslant\lambda$ such that $p_{\lambda\lambda'}=0$. Indeed, if $\underline{f}\colon \boldsymbol{G}\to\boldsymbol{0}$ and $\underline{g}\colon \boldsymbol{0}\to\boldsymbol{G}$ are maps of systems and $\underline{gf}\simeq\underline{1}$, then every λ admits a $\lambda'\geqslant\lambda$ such that $0=g_\lambda f_{g(\lambda)}p_{fg(\lambda)\lambda'}p_{\lambda\lambda'}=p_{\lambda\lambda'}$.

4.2. Let $\boldsymbol{G}=(G_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{H}=(H_\lambda, q_{\lambda\lambda'}, \Lambda)$ be pro-groups and let $\underline{f}\colon \boldsymbol{G}\to\boldsymbol{H}$ be a special map of pro-groups consisting of homomorphisms $f_\lambda\colon G_\lambda\to H_\lambda$. If $N_\lambda=(f_\lambda)^{-1}(0)$, then $p_{\lambda\lambda'}(N_{\lambda'})\subset N_\lambda$ for $\lambda\leqslant\lambda'$ so that $\boldsymbol{N}=(N_\lambda, p_{\lambda\lambda'}|N_{\lambda'}, \Lambda)$ is a pro-group. Let $\underline{i}\colon \boldsymbol{N}\to\boldsymbol{G}$ consist of inclusions $i_\lambda\colon N_\lambda\to G_\lambda$. Then $\boldsymbol{i}$ is the kernel of $\boldsymbol{f}$ in $\mathrm{pro}(\mathscr{G})$ ([20], § 1, 3.3). Indeed, $f_\lambda i_\lambda=0$ so that $\boldsymbol{fi}=0$. Moreover, if $\boldsymbol{M}=(M_\alpha, s_{\alpha\alpha'}, \Lambda)$ and $\boldsymbol{m}\colon \boldsymbol{M}\to\boldsymbol{G}$ is a morphism in $\mathrm{pro}(\mathscr{G})$, $\boldsymbol{fm}=0$, then there is no loss of generality in assuming that $\boldsymbol{m}$ is determined by $m\colon \Lambda\to\Lambda$ and by homomorphisms $m_\lambda\colon M_{m(\lambda)}\to G_\lambda$ such that $f_\lambda m_\lambda=0$ for every $\lambda\in\Lambda$. Consequently, m_λ factors uniquely through N_λ and we obtain a unique morphism of pro-groups $\boldsymbol{m}'\colon \boldsymbol{M}\to\boldsymbol{N}$ such that $\boldsymbol{im}'=\boldsymbol{m}$. Now it follows by 2.2, that every morphism in $\mathrm{pro}(\mathscr{G})$ has a kernel, i.e. that $\mathrm{pro}(\mathscr{G})$ is a category with zero-objects and kernels.

4.3. Notice that for a special map of pro-groups $\underline{f}\colon \boldsymbol{G}\to\boldsymbol{H}$, $\boldsymbol{f}$ is an epimorphism in $\mathrm{pro}(\mathscr{G})$ if all $f_\lambda\colon G_\lambda\to H_\lambda$ are epimorphisms ([20], § 1, 3.1). Indeed, let $\boldsymbol{g}, \boldsymbol{g}'\colon \boldsymbol{H}\to\boldsymbol{K}$ be morphisms in $\mathrm{pro}(\mathscr{G})$ and let $\boldsymbol{gf}=\boldsymbol{g}'\boldsymbol{f}$. There is no loss of generality in assuming that $g=g'\colon M\to\Lambda$, $\boldsymbol{K}=(K_\mu, r_{\mu\mu'}, M)$. Then there is a $\lambda\geqslant fg(\mu)$ such that $g_\mu f_{g(\mu)}p_{fg(\mu)\lambda}=g'_\mu f_{g(\mu)}p_{fg(\mu)\lambda}$. Consequently, $g_\mu q_{g(\mu)\lambda}f_\lambda=g'_\mu q_{g(\mu)\lambda}f_\lambda$, which implies $g_\mu q_{g(\mu)\lambda}=g'_\mu q_{g(\mu)\lambda}$, because f_λ is an epimorphism. This proves that $\boldsymbol{g}=\boldsymbol{g}'$, and $\boldsymbol{f}$ is indeed an epimorphism in $\mathrm{pro}(\mathscr{G})$.

4.4. In general in a category $\mathscr{K}$ with zero-objects and kernels one can define exactness of sequences ([24], p. 114). A sequence

$$G\xrightarrow{f}H\xrightarrow{g}K$$

in $\mathscr{K}$ is said to be exact at H provided: (i) $gf=0$, (ii) in the unique factorization $f=if'$, where $i\colon N\to H$ is the kernel of g, the morphism f' is an epimorphism.

Now let us establish the following fact ([20], § 1, Corollary 3.6): Let $\underline{f}\colon \boldsymbol{G}\to\boldsymbol{H}$, $\underline{g}\colon \boldsymbol{H}\to\boldsymbol{K}$ be special maps of pro-groups with the property that

$$G_\lambda\xrightarrow{f_\lambda}H_\lambda\xrightarrow{g_\lambda}K_\lambda$$

is exact at H_λ in $\mathscr{G}$. Then the sequence

$$\boldsymbol{G}\xrightarrow{\boldsymbol{f}}\boldsymbol{H}\xrightarrow{\boldsymbol{g}}\boldsymbol{K}$$

is exact at $\boldsymbol{H}$ in $\mathrm{pro}(\mathscr{G})$. Indeed, $g_\lambda f_\lambda=0$ implies $\boldsymbol{gf}=0$. Let $(N_\lambda, i_\lambda)=\mathrm{Ker}\, g_\lambda$ so that $\boldsymbol{N}=(N_\lambda, q_{\lambda\lambda'}|N_{\lambda'}, \Lambda)$ and $\boldsymbol{i}$ given by the inclusions $i_\lambda\colon N_\lambda\to H_\lambda$ define the kernel of $\boldsymbol{g}$. By the exactness assumption, f_λ admits a unique factorization $f_\lambda=i_\lambda f'_\lambda$, where $f'_\lambda\colon G_\lambda\to N_\lambda$ is an epimorphism. The homomorphisms f'_λ define a morphism

$f'\colon G\to N$ of pro-groups for which $if' = f$. However, f' is an epimorphism in $\mathrm{pro}(\mathscr{G})$ because all f'_λ are epimorphisms.

4.5. In this paper we shall also need the category $\mathscr{E}_0$ of pointed sets and base-point preserving maps. Its zero-objects are singletons. The corresponding pro-category $\mathrm{pro}(\mathscr{E}_0)$ also is a category with zero-objects and kernels. The analogues of the above results are established for $\mathscr{E}_0$ in the same way as for $\mathscr{G}$.

5. Homotopy pro-groups

5.1. Let $(X, x) = ((X, x)_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an object of $\mathrm{pro}(\mathscr{W}_0)$ (4). For every $k\geqslant 0$ we define systems $\pi_k(X, x) = (\pi_k(X, x)_\lambda, p_{\lambda\lambda' k}, \Lambda)$ called the *k-th homotopy pro-group of* (X, x); note that $\pi_0(X, x)$ is an object of $\mathrm{pro}(\mathscr{E}_0)$.

Every morphism $f\colon (X, x)\to(Y, y)$ in $\mathrm{pro}(\mathscr{W}_0)$ determines morphisms of pro-groups $f_k\colon \pi_k(X, x)\to\pi_k(Y, y)$, $k\geqslant 0$. If f is given by $f\colon M\to\Lambda$ and by morphisms $f_\mu\colon (X, x)_{f(\mu)}\to(Y, y)_\mu$ in $\mathscr{W}_0$, then f_k is given by $f\colon M\to\Lambda$ and by homomorphisms $f_{\mu k}\colon \pi_k(X, x)_{f(\mu)}\to\pi_k(Y, y)_\mu$. Clearly, $(gf)_k = g_k f_k$, $1_k = 1$, so that we have a covariant functor. Consequently, systems isomorphic in $\mathrm{pro}(\mathscr{W}_0)$ have isomorphic homotopy pro-groups.

Similarly, one defines relative homotopy pro-groups $\pi_k(X, A, x)$, $k\geqslant 1$, for systems (X, A, x) in $\mathscr{W}_0^2$. They belong to $\mathrm{pro}(\mathscr{G})$ for $k\geqslant 2$ and to $\mathrm{pro}(\mathscr{E}_0)$ for $k = 1$.

5.2. Along with a system $(X, A, x) = ((X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathscr{W}_0^2$ we also consider systems (A, x), (X, x) and morphisms $i\colon (A, x)\to(X, x)$ in $\mathrm{pro}(\mathscr{W}_0^2)$ given by the inclusions $i_\lambda\colon (A, x)_\lambda\to(X, x)_\lambda$, and by $j\colon (X, x, x)\to(X, A, x)$ given by the inclusions $j_\lambda\colon (X, x, x)_\lambda\to(X, A, x)_\lambda$. They define morphisms of pro-groups $i_k\colon \pi_k(A, x)\to\pi_k(X, x)$, $k\geqslant 0$ and $j_k\colon \pi_k(X, x)\to\pi_k(X, A, x)$, $k\geqslant 1$. Finally, we define morphisms $\partial_k\colon \pi_k(X, A, x)\to \pi_{k-1}(A, x)$, $k\geqslant 1$, by $\partial_\lambda\colon \pi_k(X, A, x_\lambda)\to\pi_{k-1}(A, x)_\lambda$. The naturality of the boundary operator ∂_λ proves that ∂_λ, $\lambda\in\Lambda$, form a morphism in $\mathrm{pro}(\mathscr{G})$. We thus obtain the *homotopy sequence of pro-groups of* (X, A, x):

$$\ldots\to\pi_k(A, x)\xrightarrow{i_k}\pi_k(X, x)\xrightarrow{j_k}\pi_k(X, A, x)\xrightarrow{\partial_k}\pi_{k-1}(A, x)\ldots$$
$$\ldots\xrightarrow{j_1}\pi_1(X, A, x)\xrightarrow{\partial_1}\pi_0(A, x)\xrightarrow{i_0}\pi_0(X, x)\,.$$

Since the corresponding sequence for each λ is exact in $\mathscr{G}$ (or $\mathscr{E}_0$), we conclude, by 3.4, that this sequence is exact in $\mathrm{pro}(\mathscr{G})$ (or in $\mathrm{pro}(\mathscr{E}_0)$) ([20], § 2, 1.1).

5.3. In the proof of the Whitehead theorem we need the following result ([20], § 2, 1.3):

Let (X, A, x) be a system in $\mathscr{W}_0^2$. If for a given $k\geqslant 1$, $i_k\colon \pi_k(A, x)\to\pi_k(X, x)$ is an epimorphism and $i_{k-1}\colon \pi_{k-1}(A, x)\to\pi_{k-1}(X, x)$ is a monomorphism in $\mathrm{pro}(\mathscr{G})$, then $\pi_k(X, A, x) = 0$.

(4) We simplify the notations (X_λ, x_λ), $(X_\lambda, A_\lambda, x_\lambda)$, etc. to $(X, x)_\lambda$, $(X, A, x)_\lambda$, etc.

This is an immediate consequence of the following assertion:
Let $\mathscr{K}$ be a category with zero-objects and kernels and let

$$A_1\xrightarrow{f_1}A_2\xrightarrow{f_2}A_3\xrightarrow{f_3}A_4\xrightarrow{f_4}A_5$$

be an exact sequence. If f_1 is an epimorphism and f_4 is a monomorphism, then A_3 is a zero-object.

Proof. Since $f_4 f_3 = 0$, f_3 factors through $\mathrm{Ker} f_4$. However, f_4 being a monomorphism, $\mathrm{Ker} f_4 = 0$ and thus $f_3 = 0$. Let $\mathrm{Ker} f_3 = (K, k)$. By exactness at A_3, there is an epimorphism $g\colon A_2\to K$ such that $kg = f_2$. Now $f_3 = 0$ implies $k = 1$, so that $f_2 = g$ is an epimorphism. By assumption on f_1, we conclude that $f_2 f_1\colon A_1\to A_3$ is an epimorphism too. On the other hand, $f_2 f_1 = 0$ and therefore, A_3 must be a zero-object.

5.4. For a pointed topological space (X, x_0) we can define homotopy pro-groups, only up to isomorphic objects in $\mathrm{pro}(\mathscr{G})$, as $\pi_k(X, x)$, where (X, x) is any system in $\mathscr{W}_0$ associated with (X, x_0). Indeed, any two systems (X, x), (X', x') associated with (X, x_0) are isomorphic in $\mathrm{pro}(\mathscr{W}_0)$, because the identity shape map $1\colon (X, x_0)\to(X, x_0)$ determines a unique isomorphism $(X, x)\to(X', x')$. Similarly, one defines homotopy pro-groups for pointed pairs of spaces (X, A, x_0). One then obtains an exact homotopy sequence for pro-groups of spaces.

6. Shape deformation retraction

6.1. In this section we shall prove the following

THEOREM 1. *Let (X, A, x) and (Y, B, y) be inverse systems in $\mathscr{W}_0^2$. Let $(X, A, x)_\lambda$ be simplicial, $\dim X_\lambda\leqslant n<\infty$. Let $(Y, B, y) = ((Y, B, y)_\mu, q_{\mu\mu'}, M)$ be such that $\pi_k(Y, B, y) = 0$, for $1\leqslant k\leqslant n+1$, each Y_μ is connected and M is closure-finite. Then every morphism $f\colon (X, A, x)\to(Y, B, y)$ in $\mathrm{pro}(\mathscr{W}_0^2)$ admits a morphism $g\colon (X, x)\to(B, y)$ in $\mathrm{pro}(\mathscr{W}_0)$ such that*

$$jg = f\colon (X, x)\to(Y, y)\,,\quad g|(A, x) = f|(A, x)\colon (A, x)\to(B, y)\,,$$

where $j\colon (B, y)\to(Y, y)$ is given by the inclusions $j_\mu\colon (B, y)_\mu\to(Y, y)_\mu$.

If one applies Theorem 1 to the identity morphism $1\colon (X, A, x)\to(X, A, x)$, one obtains

THEOREM 2. *Let (X, A, x) be an inverse system in $\mathscr{W}_0^2$ over a closure-finite index set Λ. Let $(X, A, x)_\lambda$ be simplicial, X_λ connected and $\dim X_\lambda\leqslant n<\infty$. If $\pi_k(X, A, x) = 0$ for $1\leqslant k\leqslant n+1$, then there is a morphism $r\colon (X, x)\to(A, x)$ in $\mathrm{pro}(\mathscr{W}_0)$ such that $jr = 1$ and $rj = r|(A, x) = 1$. Consequently, the morphism $j\colon (A, x)\to(X, x)$ given by the inclusions $j_\lambda\colon (A, x)_\lambda\to(X, x)_\lambda$ is an isomorphism in $\mathrm{pro}(\mathscr{W}_0)$.*

The proof of Theorem 1 is based on two lemmas.

6.2. LEMMA 1. *There is an increasing function $\delta\colon M\to M$, $\delta(\mu) = \mu^*\geqslant\mu$, such that for any pair of simplicial complexes (P, Q, x_0), $\dim(P\setminus Q)\leqslant n+1$, and for any* *map* $\varphi\colon (P, Q, x_0)\to(Y, B, y)_{\mu^*}$ *there is a map* $\psi\colon (P, Q, x_0)\to(B, B, y)_\mu$ *such that*

$$(1)\qquad j_\mu\psi\simeq\varkappa_{\mu\mu^*}\varphi\colon (P, Q, x_0)\to(Y, B, y)_\mu ,$$

where $\varkappa_{\mu\mu'}$ *is any map from the class* $q_{\mu\mu'}$, $\mu\leqslant\mu'$, *and* $j_\mu\colon (B, B, y)_\mu\to(Y, B, y)_\mu$ *denotes inclusion.*

Proof. Let $\mu=\mu_0\leqslant\mu_1\leqslant\ldots\leqslant\mu_{n+1}=\mu^*$ be a chain in $(M, \leqslant)$ such that for every k, $1\leqslant k\leqslant n+1$, the homomorphism

$$(q_{\mu_j\mu_{j+1}})_k\colon \pi_k(Y, B, y)_{\mu_{j+1}}\to\pi_k(Y, B, y)_{\mu_j}$$

equals 0, $0\leqslant j\leqslant n$. Such a chain exists because $\pi_k(Y, B, y)=0$, $1\leqslant k\leqslant n+1$. We choose a map $\varkappa_{\mu_j\mu_{j+1}}$ from $q_{\mu_j\mu_{j+1}}$ and denote by $\varkappa_{\mu_j\mu_{j'}}$, $j<j'$, the corresponding composition of such maps; $\varkappa_{\mu_j\mu_j}=1$. We also choose a triangulation of (P, Q, x_0) such that Q is a full subcomplex of P (i.e. if all vertices of a simplex belong to Q, then so does the simplex). Let $L_k=(Q\cup P^k)\times I\cup(P\times 0)$, where P^k is the k-skeleton of P, $0\leqslant k\leqslant n+1$.

We shall define, by induction, maps $\chi_k\colon L_k\to Y_{\mu_{n+1-k}}$ such that

$$(2)\qquad \chi_k(x, t)=\varkappa_{\mu_{n+1-k}\mu_{n+1}}\varphi(x), \qquad (x, t)\in(Q\times I)\cup(P\times 0),$$

$$(3)\qquad \chi_k(x, 1)\in B_{\mu_{n+1-k}}, \qquad x\in P^k,$$

$$(4)\qquad \chi_k(x, 1)=y_{\mu_{n+1-k}}, \qquad x\in P^0\setminus Q .$$

For $k=0$ and $(x, t)\in(Q\times I)\cup(P\times 0)$ we put $\chi_0(x, t)=\varphi(x)$. If x is a vertex of $P\setminus Q$, we put $\chi_0(x, 1)=y_{\mu_{n+1}}$. Since $Y_{\mu_{n+1}}$ is pathwise connected, one can define χ_0 on $x\times I$ as a path connecting $\varphi(x)$ to $y_{\mu_{n+1}}$. Thus χ_0 has all the required properties.

Assume now that χ_{k-1} has already been defined. Then we put $\chi_k|L_{k-1}=\varkappa_{\mu_{n+1-k}\mu_{n+2-k}}\chi_{k-1}$. If E^k is a k-simplex of $P^k\setminus Q$, it has a vertex x not belonging to Q. $(E^k\times 0\cup\partial E^k\times I, \partial E^k\times 1, x\times 1)$ is homeomorphic to the standard k-cell, its boundary and a base-point and χ_{k-1} maps this pointed pair into $(Y, B, y)_{\mu_{n+2-k}}$. Since $\varkappa_{\mu_{n+1-k}\mu_{n+2-k}}$ induces the zero-homomorphism

$$\pi_k((Y, B, y)_{\mu_{n+2-k}})\to\pi_k((Y, B, y)_{\mu_{n+1-k}}),$$

we conclude that

$$\chi_k\colon (E^k\times 0\cup\partial E^k\times I, \partial E^k\times 1, x\times 1)\to(Y, B, y)_{\mu_{n+1-k}}$$

determines the zero element of $\pi_k((Y, B, y)_{\mu_{n+1-k}})$. Consequently, there is a homotopy rel $(\partial E^k\times 1)$ of $E^k\times 0\cup\partial E^k\times I$ into $Y_{\mu_{n+1-k}}$ connecting χ_k with some map $(E^k\times 0\cup\partial E^k\times I)\to B_{\mu_{n+1-k}}$ ([25], Theorem 1, p. 372). This homotopy yields an extension of χ_k to $E^k\times I$ such that $\chi_k(E^k\times 1)\subset B_{\mu_{n+1-k}}$. This completes the induction step.

Now observe that $L_{n+1}=P\times I$ and consider $\chi_{n+1}\colon P\times I\to Y_\mu$. By (2) $\chi_{n+1}(Q\times I)\subset B_\mu$, $\chi_{n+1}(x_0\times I)=\{y_\mu\}$ and $\chi_{n+1}(x, 0)=\varkappa_{\mu\mu^*}\varphi(x)$, $x\in P$. Consequently, putting $\psi(x)=\chi_{n+1}(x, 1)$, $x\in P$, we obtain a map $\psi\colon (P, Q, x_0)\to(B, B, y)_\mu$ such that (1) holds.

Finally, using 2.3, one can achieve that $\mu\leqslant\mu'$ implies $\mu^*\leqslant\mu'^*$.

6.3. LEMMA 2. *For every* $\mu\in M$ *let* $\mu^*\geqslant\mu$ *be chosen in accordance with Lemma* 1. *Let* (P, x_0) *be a simplicial complex,* $\dim P\leqslant n$. *Furthermore, let* $\varphi_0, \varphi_1\colon (P, x_0)\to(B, y)_{\mu^*}$ *be maps such that*

$$(5)\qquad j_{\mu^*}\varphi_0\simeq j_{\mu^*}\varphi_1\colon (P, x_0)\to(Y, y)_{\mu^*} .$$

Then

$$(6)\qquad \varkappa_{\mu\mu^*}\varphi_0\simeq\varkappa_{\mu\mu^*}\varphi_1\colon (P, x_0)\to(B, y)_\mu .$$

Proof. Consider the triple $(P\times I, P\times 0\cup P\times 1, x_0\times I)$ and shrink $x_0\times I$ to a point. Let

$$\Phi\colon (P\times I, P\times 0\cup P\times 1, x_0\times I)/(x_0\times I)\to(Y, B, y)_{\mu^*}$$

be a map given by a homotopy (5). Since $\dim(P\times I/x_0\times I)\leqslant n+1$, Lemma 1 yields a map

$$\Psi\colon (P\times I, x_0\times I)/(x_0\times I)\to(B, y)_\mu$$

such that

$$(7)\qquad \Psi|(P\times 0\cup P\times 1, x_0\times I)/(x_0\times I)\simeq\varkappa_{\mu\mu^*}\Phi|(P\times 0\cup P\times 1, x_0\times I)/(x_0\times I)$$

and the homotopy takes place in $(B, y)_\mu$.

Consequently,

$$(8)\qquad \varkappa_{\mu\mu^*}\varphi_0\simeq\Psi|(P\times 0, x_0\times 0)\simeq\Psi|(P\times 1, x_0\times 1)\simeq\varkappa_{\mu\mu^*}\varphi_1\colon (P, x_0)\to(B, y)_\mu .$$

6.4. Proof of Theorem 1. Let $f\colon (X, A, x)\to(Y, B, y)$ be given by f and by homotopy classes of maps $f_\mu\colon (X, A, x)_{f(\mu)}\to(Y, B, y)_\mu$ with representatives φ_μ. Since M is closure-finite, we can assume that f is increasing and that

$$(9)\qquad f_\mu p_{\mu\mu'}=q_{\mu\mu'}f_{\mu'}, \qquad \mu\leqslant\mu'.$$

For every μ' we choose μ^* according to Lemma 1. Then for each $\mu\in M$ there is a mapping $\psi_{\mu^*}\colon (X, x)_{f(\mu^{**})}\to(B, y)_{\mu^*}$ such that

$$(10)\qquad j_{\mu^*}\psi_{\mu^*}\simeq\varkappa_{\mu^*\mu^{**}}\varphi_{\mu^{**}}\colon (X, x)_{f(\mu^{**})}\to(Y, y)_{\mu^*}$$

$$(11)\qquad \psi_{\mu^*}|(A, x)_{f(\mu^{**})}\simeq\varkappa_{\mu^*\mu^{**}}\varphi_{\mu^{**}}|(A, x)_{f(\mu^{**})}\colon (A, x)_{f(\mu^{**})}\to(B, y)_{\mu^*} .$$

We now put $g(\mu)=f(\mu^{**})$ and

$$(12)\qquad g_\mu=q_{\mu\mu^*}[\psi_{\mu^*}]\colon (X, x)_{f(\mu^{**})}\to(B, y)_\mu ,$$

where square brackets denote homotopy classes. The map g and the homotopy classes g_μ determine a map of systems $\underline{g}\colon (X, \boldsymbol{x})\to(\boldsymbol{B}, \boldsymbol{y})$. Indeed, let $\mu\leqslant\mu'$. Then, by (10)

$$j_{\mu^*}\psi_{\mu^*}\simeq\varkappa_{\mu^*\mu^{**}}\varphi_{\mu^{**}}\simeq\varphi_{\mu^*}\varrho_{f(\mu^*)f(\mu^{**})}\colon (X, x)_{f(\mu^{**})}\to(Y, y)_{\mu^*}\,, \tag{13}$$

$$j_{\mu'^*}\psi_{\mu'^*}\simeq\varkappa_{\mu'^*\mu'^{**}}\varphi_{\mu'^{**}}\simeq\varphi_{\mu'^*}\varrho_{f(\mu'^*)f(\mu'^{**})}\colon (X, x)_{f(\mu'^{**})}\to(Y, y)_{\mu'^*}\,, \tag{14}$$

where $\varrho_{\lambda\lambda'}$ is any map from the class $p_{\lambda\lambda'}$. Since

$$\varphi_{\mu^*}\varrho_{f(\mu^*)f(\mu'^*)}\simeq\varkappa_{\mu^*\mu'^*}\varphi_{\mu'^*}\colon (X, x)_{f(\mu'^*)}\to(Y, y)_{\mu^*}\,, \tag{15}$$

(13) and (14) yield

$$j_{\mu^*}\psi_{\mu^*}\varrho_{f(\mu^{**})f(\mu'^{**})}\simeq j_{\mu^*}\varkappa_{\mu^*\mu'^*}\psi_{\mu'^*}\colon (X, x)_{f(\mu'^{**})}\to(Y, y)_{\mu^*}\,. \tag{16}$$

Applying Lemma 2 to (16), we now conclude that

$$\varkappa_{\mu\mu^*}\psi_{\mu^*}\varrho_{f(\mu^{**})f(\mu'^{**})}\simeq\varkappa_{\mu\mu'^*}\psi_{\mu'^*}\colon (X, x)_{f(\mu'^{**})}\to(B, y)_{\mu^*}\,, \tag{17}$$

so that

$$g_\mu p_{f(\mu^{**})f(\mu'^{**})} = q_{\mu\mu'}g_{\mu'}\colon (X, x)_{f(\mu'^{**})}\to(B, y)_\mu\,. \tag{18}$$

This proves that $\boldsymbol{g}$ is indeed a map of systems in $\mathscr{W}_0$.

From (12) and (13) we derive

$$[j_\mu]g_\mu = q_{\mu\mu^*}f_{\mu^*}p_{f(\mu^*)f(\mu^{**})} = f_\mu p_{f(\mu)f(\mu^{**})}\colon (X, x_0)_{f(\mu^{**})}\to(Y, y)_\mu\,, \tag{19}$$

which proves that

$$\boldsymbol{jg} = \boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x})\to(\boldsymbol{Y}, \boldsymbol{y})\,. \tag{20}$$

Finally, by (11),

$$\begin{aligned} g_\mu|(A, x)_{f(\mu^{**})} &= q_{\mu\mu^{**}}f_{\mu^{**}}|(A, x)_{f(\mu^{**})} \\ &= f_\mu p_{f(\mu)f(\mu^{**})}|(A, x)_{f(\mu^{**})}\colon (A, x)_{f(\mu^{**})}\to(B, y)_\mu\,, \end{aligned} \tag{21}$$

so that $\boldsymbol{g}|(\boldsymbol{A}, \boldsymbol{x}) = \boldsymbol{f}|(\boldsymbol{A}, \boldsymbol{x})\colon (\boldsymbol{A}, \boldsymbol{x})\to(\boldsymbol{B}, \boldsymbol{y})$. This completes the proof of Theorem 1.

6.5. From Theorems 1 and 2 follow analogous results for spaces. In particular we have

THEOREM 3. *Let (X, A, x_0) and (Y, B, y_0) be pointed pairs of topological spaces, let $\dim X\leqslant n<\infty$, let $\pi_k(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y}) = 0$, $1\leqslant k\leqslant n+1$ and let Y be connected. Then every shape map $\boldsymbol{f}\colon (X, A, x_0)\to(Y, B, y_0)$ admits a shape map $\boldsymbol{g}\colon (X, x_0)\to(B, y_0)$ such that $\boldsymbol{jg} = \boldsymbol{f}$, $\boldsymbol{g}|(A, x_0) = \boldsymbol{f}|(A, x_0)$, where $\boldsymbol{j}$ is the shape map induced by the inclusion $j\colon B\to Y$.*

Proof. According to 3.4, there exist inverse systems $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x})$, $(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y})$ in $\mathscr{W}_0^2$ associated with (X, A, x_0) and (Y, B, y_0) respectively and such that each $(X, A, x)_\lambda$ is simplicial, $\dim X_\lambda\leqslant n$ and each Y_μ is connected. Moreover, by 2.3

and 3.2, one can achieve that the index set M of $(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y})$ is closure-finite. Therefore, Theorem 1 yields the desired conclusion.

From Theorem 3 (or from Theorem 2) one derives

THEOREM 4. *Let (X, A, x_0) be a pair of pointed topological spaces, let X be connected and let $\dim X\leqslant n<\infty$. If $\pi_k(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = 0$ for $1\leqslant k\leqslant n+1$, then (A, x_0) is a shape deformation retract of (X, x_0), i.e. the inclusion $j\colon (A, x_0)\to(X, x_0)$ induces a shape equivalence.*

DEFINITION. We say that a space (X, x_0) is *shape-n-connected*, provided $\pi_k(\boldsymbol{X}, \boldsymbol{x}) = 0$ for $0\leqslant k\leqslant n$ ([5]).

THEOREM 5. *Let (X, x_0) be a pointed connected topological space, $\dim X\leqslant n<\infty$. If (X, x_0) is shape-$(n+1)$-connected, then (X, x_0) is of trivial shape.*

Proof. It suffices to apply Theorem 4 to (X, x_0, x_0).

Remark. D. S. Kahn [10] has defined a metric continuum (X, x_0) which is shape connected for all n but is not of trivial shape as shown by D. Handel and J. Segal [8]. However, $\dim X = \infty$.

7. The Whitehead theorem for maps

7.1. Let $f\colon (X, x_0)\to(Y, y_0)$ be a map of pointed topological spaces. The mapping cylinder (Z, z_0) of f is obtained from the topological sum $(X\times I)\cup Y$ by identifying $(x, 1)$ and $f(x)$, $x\in X$, and by shrinking $(x_0\times I)\cup\{y_0\}$ to a point z_0. The image of (x, t) and y under this identification will be denoted by $[(x, t)]$ and $[y]$ respectively. Along with the mapping cylinder, we consider embeddings $i\colon (X, x_0)\to(Z, z_0)$, $j\colon (Y, y_0)\to(Z, z_0)$, given by $i(x) = [(x, 0)]$ and $j(y) = [y]$ respectively, as well as the map $g\colon (Z, z_0)\to(Y, y_0)$ given by $g[(x, t)] = f(x)$, $g[y] = y$. Note that $gj = 1$ and $jg\simeq 1$ so that j and g are homotopy equivalences. Furthermore, $jf\simeq i$ and therefore, $f = gi$.

Since g is a homotopy equivalence and $f\simeq gi$, we conclude that f is a shape equivalence if and only if i is a shape equivalence. Similarly, f induces a monomorphism (epimorphism) of homotopy pro-groups $\pi_k(\boldsymbol{X}, \boldsymbol{x})\to\pi_k(\boldsymbol{Y}, \boldsymbol{y})$ if and only if i induces one.

Note that the dimension of the mapping cylinder (Z, z_0) of $f\colon (X, x_0)\to(Y, y_0)$ satisfies

$$\dim Z\leqslant\max\{1+\dim X, \dim Y\}\,. \tag{1}$$

Indeed, consider the subsets $Z_n = X\times I_n/x_0\times I_n\subset Z$, where $I_n = \left[\frac{n-1}{n}, \frac{n}{n+1}\right]$, $n = 1, 2, \ldots$ These subsets are C^*-embedded in Z, i.e. every map $f_n\colon Z_n\to I = [0, 1]$ admits a continuous extension to all of Z. This is easily seen by considering the mapping cone C of f and the mapping $Z\to C$ which identifies Y to a point. Since Y

([5]) K. Borsuk has introduced this notion in [3] under the name "approximative n-connectedness".

is a retract of Z, it is also C^*-embedded in Z. Now Lemma 5.9 of [15] yields the assertion because $Z = (\bigcup Z_n) \cup Y$ and $\dim Z_n = \dim X \times I \leqslant \dim X + 1$ (Theorem 5.5 of [15]).

7.2. We can now state and prove our main result.

THEOREM 6. *Let (X, x_0) and (Y, y_0) be pointed topological spaces, connected and finite-dimensional, and let $f: (X, x_0) \to (Y, y_0)$ be a map which induces bimorphisms $f_k: \pi_k(X, x) \to \pi_k(Y, y)$ of homotopy pro-groups for*

$$0 \leqslant k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1$$

and an epimorphism for $k = n_0 + 1$. Then f is a shape equivalence, i.e. there is a shape map $\boldsymbol{g}: (Y, y_0) \to (X, x_0)$ such that $\boldsymbol{fg} = \boldsymbol{1}$, $\boldsymbol{gf} = \boldsymbol{1}$.

Proof. Let (Z, z_0) be the mapping cylinder of f. Z is connected and $\dim Z \leqslant \max\{1 + \dim X, \dim Y\} = n_0 < \infty$. Furthermore, by 7.1, the inclusion induces a bimorphism $\boldsymbol{i}_k$ of homotopy pro-groups for $0 \leqslant k < n_0 + 1$ and an epimorphism for $k = n_0 + 1$. By 5.4, there is an exact sequence of homotopy pro-groups belonging to the pair (Z, X, x_0). We conclude, by 5.3, that $\pi_k(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}) = 0$ for $1 \leqslant k \leqslant n_0 + 1$. Consequently, by Theorem 4, the inclusion $i: (X, x_0) \to (Z, x_0)$ is a shape equivalence, which implies that f is a shape equivalence too.

8. The Whitehead theorem for shape maps

8.1. THEOREM 7 (⁶). *Let (X, x_0) be a compact Hausdorff space, (Y, y_0) a compact metric space and $\boldsymbol{f}: (X, x_0) \to (Y, y_0)$ a shape map. Then there is a compact Hausdorff space (Z, z_0) and there are embeddings $i: (X, x_0) \to (Z, z_0)$, $j: (Y, y_0) \to (Z, z_0)$ such that j admits a shape inverse $\boldsymbol{g}: (Z, z_0) \to (Y, y_0)$ and $\boldsymbol{f} = \boldsymbol{gi}$. If X is metric, then so is Z.*

Proof. There exist an inverse sequence $(\boldsymbol{Y}, \boldsymbol{y}) = ((Y, y)_n, q_{n,n+1}, N)$ in $\mathscr{W}_0$ of compact polyhedra $(Y, y)_n$ whose inverse limit is (Y, y_0). Then $\boldsymbol{f}$ is given by homotopy classes of maps $f_n: (X, x_0) \to (Y, y)_n$ such that $q_{n,n+1} f_{n+1} = f_n$, $n \in N$. For each $n \in N$ let $\varkappa_{n,n+1}$ and φ_n be maps from the classes $q_{n,n+1}$ and f_n respectively, let $(Z, z)_n$ be the mapping cylinder of φ_n and let $i_n: (X, x_0) \to (Z, z)_n$, $j_n: (Y, y)_n \to (Z, z)_n$ be inclusions as in 7.1. We choose for every $n \in N$ a homotopy

$$\Phi_n: (X \times I, x_0 \times I) \to (Y, y)_n$$

which connects φ_n and $\varkappa_{n,n+1}\varphi_{n+1}$. Then a map $\varrho_{n,n+1}: (Z, z)_{n+1} \to (Z, z)_n$ is defined by

$$\varrho_{n,n+1}[(x, t)] = \begin{cases} [x, 2t], & 0 \leqslant t \leqslant \frac{1}{2},\ x \in X, \\ [\Phi_n(x, 2t-1)], & \frac{1}{2} \leqslant t \leqslant 1,\ x \in X, \end{cases} \tag{1}$$

$$\varrho_{n,n+1}[y] = [\varkappa_{n,n+1}(y)], \quad y \in Y. \tag{2}$$

(⁶) According to a communication from M. Moszyńska the theorem has been already proved by W. Holsztyński (unpublished) if both X and Y are metric compacta (cf. also [17]).

We thus obtain an inverse sequence $(\boldsymbol{Z}, \boldsymbol{z}) = ((Z, z)_n, \varrho_{n,n+1}, N)$ and a space $(Z, z_0) = \varprojlim(\boldsymbol{Z}, \boldsymbol{z})$. Note that $(Z, z)_n$ has the homotopy type of $(Y, y)_n$ so that $(\boldsymbol{Z}, \boldsymbol{z})$ is a system in $\mathscr{W}_0$. Moreover, each Z_n is compact and therefore Z is compact too. Since i_n is an embedding and $i_n = \varrho_{n,n+1} i_{n+1}$, we obtain an embedding $i: (X, x_0) \to (Z, z_0)$. Similarly, the maps j_n define an embedding $j: (Y, y_0) \to (Z, z_0)$, because $\varrho_{n,n+1} j_{n+1} = j_n \varkappa_{n,n+1}$. Since j_{n+1} is a homotopy equivalence, $\boldsymbol{j}$ is a shape equivalence. Finally, $j_n \varphi_n \simeq i_n$ implies $\boldsymbol{jf} = \boldsymbol{i}$, i.e. $\boldsymbol{f} = \boldsymbol{gi}$, where $\boldsymbol{g}$ is the shape inverse of $\boldsymbol{j}$.

8.2. We now prove a slight generalization of Moszyńska's theorem stated in Introduction.

THEOREM 8. *Let (X, x_0) be a compact Hausdorff space and (Y, y_0) a compact metric space, let both be connected and finite-dimensional and let $\boldsymbol{f}: (X, x_0) \to (Y, y_0)$ be a shape map which induces bimorphisms of homotopy pro-groups $\pi_k(\boldsymbol{X}, \boldsymbol{x}) \to \pi_k(\boldsymbol{Y}, \boldsymbol{y})$ for $0 \leqslant k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1$ and an epimorphism for $k = n_0 + 1$. Then $\boldsymbol{f}$ is a shape equivalence.*

Proof. We apply Theorem 7 and obtain a factorization $\boldsymbol{f} = \boldsymbol{gi}$, where $i: (X, x_0) \to (Z, z_0)$ is an embedding. Since $\boldsymbol{g}$ is a shape equivalence, the assumption on $\boldsymbol{f}$ carries over to $\boldsymbol{i}$. As in the proof of Theorem 6 we conclude that $\pi_k(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}) = 0$ for $1 \leqslant k \leqslant n_0 + 1$, and therefore $\boldsymbol{i}: (X, x_0) \to (Z, x_0)$ is a shape equivalence. This implies that $\boldsymbol{f}$ is a shape equivalence too.

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Invariant uniformization

by

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Abstract. We show that the invariant version of the Kondo-Addison Uniformization Theorem fails. Several counterexamples of algebraic interest are presented.

Can one pick a point from each countable linear order? To make this problem model-theoretically interesting we identify isomorphic structures and to make it nontrivial in ZFC, set theory with choice, we require that the picking be done in a countable-ordinal-sequence-definable way. Roughly speaking, a set is definable from a countable sequence of ordinals iff for some ZF formula φ and some countable sequence α of ordinals, it is the unique solution of $\varphi(\underline{x}, \alpha)$. A set definable in any mathematically accepted way will be countable-ordinal-sequence-definable. Henceforth we shall interpret "one can pick a point (proper substructure, proper extension, etc.) from each linear order" as meaning that there is a countable-ordinal-sequence-definable function which assigns to each isomorphism type of a countable linear order the isomorphism type of a point (proper substructure, proper extension etc.) of the linear order, i.e., to the isomorphism type of $\langle A, <\rangle$ it assigns the isomorphism type of some structure $\langle A, <, a\rangle$ where $a \in A$ ($\langle A, B, <\rangle$ where $B \subseteq A$, $\langle B, A, <\rangle$ where $A \subseteq B$, etc.). "One cannot always pick..." shall be interpreted as meaning that it is relatively consistent with ZFC that there is no countable-ordinal-sequence-definable function which picks... All of the results below of the form "one cannot always pick a ..." have as consequences "it is relatively consistent with ZF and the principle of dependent choices that there is no function (definable or not) which selects a ...".

We first show that one cannot always pick points from certain structures called bireals and then show that: One cannot always pick a point from each countable linear order or from each countable semigroup and one cannot always pick a proper substructure for each countable algebra which has such. Although one can always pick a proper extension by adding a new point, it is relatively consistent with the existence of an inaccessible cardinal that one cannot pick a countable nonisomorphic extension of each countable structure which has such. It is also relatively consistent

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