On the Whitehead theorem in shape theory I

by

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Abstract. Recently M. Moszyńska [17] has proved for finite-dimensional metric compact spaces and shape maps \( f: (X, x_0) \rightarrow (Y, y_0) \) an analogue of the classical Whitehead theorem. In this paper another analogue of the Whitehead theorem in shape theory is established. \((X, x_0)\) and \((Y, y_0)\) are allowed to be arbitrary topological spaces of finite covering dimension. However, \( f: (X, x_0) \rightarrow (Y, y_0) \) is required to be a continuous map. If \( f \) induces isomorphisms of the homotopy pro-groups, then \( f \) is shown to be a shape equivalence (in the sense of [12]). At the same time a shorter proof of Moszyńska's original theorem is given.

1. Introduction

Of great importance in homotopy theory is the classical theorem of J. H. C. Whitehead: Let \( f: (X, x_0) \rightarrow (Y, y_0) \) be a map of connected (pointed) CW-complexes and let \( f_i: \pi_i(X, x_0) \rightarrow \pi_i(Y, y_0) \) be an isomorphism for \( i = n_0 = \max\{1 + \dim X, \dim Y\} \) and an epimorphism for \( i = n_0 \). Then \( f \) is a homotopy equivalence. An analogue of this theorem in shape theory has been recently proved by M. Moszyńska [20](\(^1\)). Her result reads as follows:

**Theorem (M. Moszyńska).** Let \( f: (X, x_0) \rightarrow (Y, y_0) \) be a shape map of connected (pointed) metric finite-dimensional compacta and let \( f_k: \pi_k(X, x) \rightarrow \pi_k(Y, y) \) be the induced maps of homotopy pro-groups. If \( f_k \) is a bimorphism for \( 0 \leq k < n_0 + 1 = \max\{1 + \dim X, \dim Y\} + 1 \) and an epimorphism for \( k = n_0 + 1 \), then \( f \) is a shape equivalence.

In this paper we extend Moszyńska's theorem to the case of arbitrary topological spaces and continuous maps \( f: (X, x_0) \rightarrow (Y, y_0) \) (see Theorem 6). At the same time we obtain a simpler proof of the original result, which consists entirely of steps analogous to corresponding steps in the proof of the classical Whitehead theorem.

The notion of shape used in this paper is that described in [12]. It has been shown by K. Morita [16] that also in the case of topological spaces shape can be treated essentially as in [14], i.e. that shape reduces to pro-homotopy category of CW-complexes. This made the present generalization to topological spaces possible.

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\(^1\) In [20] Moszyńska also exhibited a simpler theorem for the case when \((X, x_0)\) and \((Y, y_0)\) are assumed to be movable. However, the proof contains a gap. Moszyńska has recently modified this second theorem in a note of correction to appear in Fund. Math.
The question of whether Moishezon's theorem holds for topological spaces and shape maps remains open. The difficulty consists in setting up a satisfactory mapping cylinder. It should also be said that in this paper we do not discuss the case of movable spaces.

2. Category \( pro(X) \)

2.1. With every category \( X \) one can associate a new category \( pro(X) \) introduced in [7] (see also [1]). One first considers a category \( X \) whose objects are inverse systems \( X = (X_{i}, p_{ij}, A) \), where \( (A, \leq) \) is a directed set \( Y \) and \( X_{i} = \text{morphisms from } A \) respectively. A morphism \( f: X \to Y = (Y_{i}, q_{ij}, M) \) in \( X \), called a mapping of systems, consists of a map \( f: M \to A \) and of a collection of morphisms \( f_{i}: X_{i} \to Y_{f(i)} \) such that for \( i \leq j \) there exists a \( \lambda \geq f(i), f(j) \) for which \( f_{i} f_{j} p_{ij} = q_{i j} f_{i} p_{ij} \). If \( g: Y \to Z = (Z_{i}, r_{ij}, N) \), then the composition \( h = g f \) is \( X \to Z \) is given by \( h = g f: N \to A \) and \( h_{i} = g_{i} f_{i} X_{i} \to Z_{i} \), the identity \( i: X \to X \) is given by \( 1: A \to A = A \) and \( 1_{i} = 1: X_{i} \to X_{i} \). Two mappings of systems \( f, g: X \to Y \) are considered to be equivalent, \( f \sim g \), provided for every \( \mu \in M \) there is \( \lambda \in A, \mu \geq f(\lambda), g(\mu) \), such that \( f_{\mu} p_{\mu} = g_{\mu} p_{\mu} \). This is an equivalence relation on every set \( X \times Y \). Moreover, \( f \sim f' \) and \( g \sim g' \) implies \( f g \sim f' g \). We thus obtain a quotient category \( X/\sim \) called \( pro(X) \). It has the same objects \( X, Y \), etc. as \( X \) and its morphisms are equivalence classes of morphisms \( f: X \to Y \); the class containing \( f \) will be denoted by \( f \) (cf. [18], [20], [12]).

2.2. Especially simple are maps of systems \( f: X \to Y \) where both \( X \) and \( Y \) are indexed by the same set \( A, f = 1: A \to A \) and \( f_{\mu} p_{\mu} = q_{\mu} p_{\mu} \) for \( \mu \leq \lambda \). We refer to such maps as to special maps of systems.

For every map of systems \( f: X \to Y \) there exist systems \( X, Y \) and \( Y \)'s maps of systems \( i: X \to X, j: Y \to Y, f': X \to Y \)'s such that \( f' = i f j \) and \( f' \) is a special map of systems. In order to see this, one considers the set \( N \) of all morphisms \( n_{\mu} \in X_{\mu} \), each admitting a \( \lambda \geq f(\mu) \), such that \( n_{\mu} p_{\mu} = f_{\mu} p_{\mu} \). Note that \( \lambda \) and \( \mu \) run through cofinal subsets of \( A \) and \( M \) respectively as \( n_{\mu} \) runs through \( X_{\mu} \). Taking \( n_{\mu} \leq n_{\mu} \) providing \( \mu \leq \lambda \) and \( f_{\mu} p_{\mu} = q_{\mu} p_{\mu} \), \( N \) becomes a directed set. Now one defines \( X = (X, p_{\mu} n_{\mu}, N) \), \( Y = (Y, q_{\mu} n_{\mu}, N) \) and \( f = i f j \). Furthermore, \( i \) consists of a map \( i: N \to A \), where \( i(\mu) = \lambda \) and \( j \) is a special map \( N \times M \), where \( j(\nu_{\mu}) = \mu \) and \( j(\nu_{\mu}) = 1 \). Hence \( f_{\mu} p_{\mu} = q_{\mu} p_{\mu} \) (cf. Corollary 3.2, p. 160 of [1]).

2.3. \( (M, \leq) \) is said to be closure-finite provided every \( \mu \in M \) has only finitely many predecessors. In this case every function \( f: M \to A \) into an ordered set admits an increasing function \( f': M \to A \) such that \( f \leq f' \) (Lemma 5, [14]).

Every system \( X = (X_{\lambda}, p_{\lambda \mu}, A) \) admits a system \( Y = (Y_{\mu}, q_{\mu \nu}, M) \) isomorphic to \( X \) in \( pro(X) \) and such that \( M \) is closure-finite. Moreover, every \( X_{\lambda} \) is some \( X_{\lambda} \), and every \( q_{\mu \nu} = X_{\lambda} \) is some \( p_{\lambda \mu} = X_{\lambda} \) (cf. Theorem 10 of [12]). Indeed, one can take for \( (M, \leq) \) the set of all finite subsets \( \mu = \{ \lambda_{1}, \ldots, \lambda_{n} \} \) of \( A \) ordered by inclusion. One defines then an increasing function \( f: M \to A \) such that \( f(\lambda) = \lambda \) for every \( \lambda \in A \). Let \( Y_{\mu} = X_{\nu} \) and \( q_{\mu \nu} = p_{\mu \nu} \), \( \mu \leq \mu \), \( Y = (Y_{\mu}, q_{\mu \nu}, M) \). One defines maps of systems \( f: X \to Y \) and \( g: X \to Y = f_{\mu} p_{\mu} = q_{\mu \nu} p_{\mu \nu} \).

3. Pro-homotopy category and shapes

3.1. Let \( \mathcal{W} \) denote the category whose objects are topological spaces having the homotopy type of a CW-complex and whose morphisms are homotopy classes of maps. Note that a topological space has the homotopy type of a CW-complex if and only if it has the homotopy type of a simplicial CW-complex (with the weak or with the metric topology) or equivalently of an ANR for metric spaces (see e.g. [12]).

Following K. Morita [16], we say that an inverse system \( X = (X_{\lambda}, p_{\lambda \mu}, A) \) in \( \mathcal{W} \) is associated with a topological space \( X \) provided there are homotopy classes \( p_{\lambda} = X_{\lambda} = X_{\lambda} \) for each homotopy class \( m = X_{\lambda} \), \( P = X_{\lambda} \), where \( m \in \mathcal{W} \), \( X_{\lambda} \), and \( P \) is an object of \( \mathcal{W} \), admits a factorization \( m = m_{\lambda} p_{\lambda} \), \( m_{\lambda} = X_{\lambda} \), \( m_{\lambda} = \mathcal{W} \), \( X_{\lambda} \), and \( m_{\lambda} = \mathcal{W} \). Then there is a \( \lambda \geq \lambda \) such that \( m_{\lambda} = m_{\lambda} \). (compare with conditions (i), (ii) in Theorem 5.2 of [12] and with [9]).

3.2. If \( Y \) is isomorphic with \( X \) in \( pro(\mathcal{W}) \) and \( X \) is associated with a space \( X \) then so is \( Y \). Indeed, let \( f: X \to Y, g: Y \to X \) be inverse isomorphisms. We define homotopy classes \( g_{X} = X_{Y} \) by \( g_{X} = f \). Clearly, \( \mu \leq \mu \) implies \( q_{\mu} = g_{\mu} \). If \( m = X_{\lambda} \) is a homotopy class of maps, \( P \in Ob \mathcal{W} \), then a factorization exists \( m = m_{\lambda} p_{\lambda} \), \( m_{\lambda} = X_{\lambda} \), \( P \). Since \( g \) is isomorphic to \( f \), \( f_{X} = f_{X} \) implies \( g_{X} = f_{X} \). Clearly, \( m_{\lambda} = m_{\lambda} \), \( m_{\lambda} = \mathcal{W} \), \( X_{\lambda} \), and \( m_{\lambda} = \mathcal{W} \). Every \( m_{\lambda} = m_{\lambda} \). Consequently, \( m_{\lambda} = m_{\lambda} \). A similar argument shows that \( m_{\lambda} = m_{\lambda} \). Consequently, \( m_{\lambda} = m_{\lambda} \). If \( m_{\lambda} = m_{\lambda} \) then \( m_{\lambda} = m_{\lambda} \). Consequently, \( m_{\lambda} = m_{\lambda} \). If \( m_{\lambda} = m_{\lambda} \) then \( m_{\lambda} = m_{\lambda} \). Consequently, \( m_{\lambda} = m_{\lambda} \). If \( m_{\lambda} = m_{\lambda} \) then \( m_{\lambda} = m_{\lambda} \).

3.3. Morita has shown [16] that every topological space \( X \) is associated with the inverse system \( X \) in \( \mathcal{W} \) formed by the nerves of all open locally-finite normal (2)

(2) By definition, an open covering \( \lambda_{n} \) of \( X \) is normal provided there is a sequence of open coverings \( \lambda_{n}, n \in \mathcal{N} \), such that \( \lambda_{n} = \lambda_{n} \). The existence of canonical mappings shows that open locally-finite normal coverings coincide with open locally-finite numerable coverings as defined in [4].
covering of $X$. For $f_1: X \to X_1$ one takes (unique) homotopy classes determined by canonical mappings, i.e., mappings $\varphi_1: X \to X_1$ such that $(\varphi_1)^{-1}(U_1(U_2)) = U$ for every element $U$ of the covering $\lambda$.

3.4. Following Morita [15] we say for a topological space $X$ that $\dim X \leq n < \infty$ provided every finite open normal covering admits a finite open normal refinement of order $\leq n + 1$. Notice that $\dim X \leq n$ implies that every locally finite open normal covering $\mathcal{U} = \{ U_\alpha, \alpha \in A \}$ admits a locally finite open normal refinement of order $\leq n + 1$. Indeed, let $\varphi: X \to N(\mathcal{U})$ be a canonical map, where $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ provided with the metric topology. Then there is a metric space $M$, $\dim M \leq n$, and a factorization $\varphi = \psi\chi$ through $M$ (15). If $\chi = \chi_{\text{metric}}(\mathcal{U}_1, N(\mathcal{U}))$, then $\varphi = (V_\alpha, \alpha \in A)$ is an open covering of $M$ such that $\varphi^{-1}(V_\alpha) \subseteq U_\alpha$. Since $M$ is metric and $\dim M \leq n$, $\varphi$ admits an open locally finite normal refinement $\varphi' = Q$ of order $\leq n + 1$ (21), Definition II. 6, p. 22). $\varphi'$ is normal because every open covering of a metric space is normal. Now $\psi^{-1}(\varphi')$ is an open covering of $X$ with all the desired properties.

Since the nerve of an open covering of a connected space is connected, we see that every connected space $X$ with $\dim X \leq n$ admits an associated system $X$ in $\mathcal{K}$ all of whose members are connected simplicial complexes of dimension $\leq n$.

3.5. In this paper shape theory for topological spaces is understood in the sense of [12]. It thus coincides with Borsuk’s theory [2] on metric compacta (see [12]) with Fox’s theory [6] on metric spaces (see [13], [16]) and with the ANR-system approach [14] on compact Hausdorff spaces (see [12], [16]).

Generalizing results from [14] and [12] Morita has shown (16), Theorem 2.4) that there is a functorial bijection between shape maps $\varphi: X \to Y$ of topological spaces and morphisms $f: X \to Y$ from $\text{pro}(\mathcal{K})$, where $X$ and $Y$ are systems associated with $X$ and $Y$ respectively. $\varphi$ and $f$ correspond to each other provided $g(\varphi) = f_1 \circ f_2$ for every $\varphi \in \mathcal{K}$. This fact enables us to use the same notation $f$ for morphisms $X \to Y$ from $\text{pro}(\mathcal{K})$ and for the corresponding shape maps $X \to Y$. A similar approach to shape was studied by T. Porter [22], [23].

3.6. In this paper we also consider the homotopy category $\mathcal{K}$ of pointed spaces $(X, x_0)$ having the homotopy type of a pointed CW-complex (the base-point is a 0-cell of the CW-decomposition). We also deal with the category $\mathcal{K}$ of pairs of pointed spaces $(X, A, x_0)$, $x_0 \in A \subseteq X$, having the homotopy type of a pointed CW-complex and a subcomplex. The shape of pointed spaces and pairs of pointed spaces can be described in terms of $\text{pro}(\mathcal{K})$ and $\text{pro}(\mathcal{K})$ following the same pattern as in the absolute case (16).

4. Pro-category of groups

4.1. Let $\mathcal{K}$ be the category of groups and homomorphisms. Objects of the corresponding pro-category $\text{pro}(\mathcal{K})$ are called pro-groups. In this section we recall some facts about pro-$\mathcal{K}$ established essentially already by Morzyńska in [20].

$\text{pro}(\mathcal{K})$ is a category with zero objects. Indeed, a system $\mathcal{K}$ consisting of a single trivial group is obviously a zero-object in $\text{pro}(\mathcal{K})$. A pro-group $G = (G_1, \rho_{G_1})$ is a zero-object of $\text{pro}(\mathcal{K})$ if and only if it is isomorphic with $0$. This is the case if and only if every $\lambda$ admits a $\lambda' \geq \lambda$ such that $\rho_{G_1} = 0$. Indeed, if $f: G \to \mathcal{K}$ and $g: 0 \to G$ are maps of systems and $g_{\lambda} = 1$, then every $\lambda$ admits a $\lambda' \geq \lambda$ such that $0 = g_{\lambda'} f_{\lambda'} \rho_{G_1} f_0 = \rho_{G_1} f_0$.

4.2. Let $G = (G_1, \rho_{G_1})$, $H = (H_1, \rho_{H_1})$ be pro-groups and let $f: G \to H$ be a special map of pro-groups consisting of homomorphisms $f_\lambda: G_\lambda \to H_\lambda$. If $N_\lambda = (f_\lambda)^{-1}(0)$, then $\rho_{G_\lambda} N_\lambda = N_\lambda$ for $\lambda' \geq \lambda$ so that $N = (N_\lambda, \rho_{N_\lambda})$ is a pro-group. Let $i: N \to G$ consist of inclusions $i_\lambda: N_\lambda \to G_\lambda$. Then $i$ is the kernel of $f$ in $\text{pro}(\mathcal{K})$ (20), § 1, 3.3). Indeed, $i_{\lambda'} = 0$ so that $fi = 0$. Moreover, if $M = (M_\lambda, \rho_{M_\lambda})$, and $m: M \to G$ is a morphism in $\text{pro}(\mathcal{K})$, $fm = 0$, then there is no loss of generality in assuming that $m$ is determined by $m: A \to A$ and by homomorphisms $m_\lambda: M_{\rho_{M_\lambda}} \to G_\lambda$ such that $m_\lambda m_\lambda = 0$ for every $\lambda \in A$. Consequently, $m_\lambda$ factors uniquely through $N_\lambda$ and we obtain a unique morphism of pro-groups $m': M \to N$ such that $im' = m$. Now it follows by 2.2, that every morphism in $\text{pro}(\mathcal{K})$ has a kernel, i.e., that $\text{pro}(\mathcal{K})$ is a category with zero-objects and kernels.

4.3. Notice that for a special map of pro-groups $f: G \to H$, $f$ is an epimorphism in $\text{pro}(\mathcal{K})$ if all $f_\lambda: G_\lambda \to H_\lambda$ are epimorphisms (20), § 1, 3.1). Indeed, let $g, g': H \to K$ be morphisms in $\text{pro}(\mathcal{K})$ and let $gf = g'f$.

There is no loss of generality in assuming that $g = g': M \to K$, $K = (K_\lambda, \rho_{K_\lambda})$. Then there is a $\lambda' \geq \lambda$ such that $g_{\lambda'} f_{\lambda'} = g'_{\lambda'} f_{\lambda'}$. Consequently, $g_{\lambda'} f_{\lambda'} = g'_{\lambda'} f_{\lambda'}$, which implies $g_{\lambda'} f_{\lambda'} = g'_{\lambda'} f_{\lambda'}$, because $f_{\lambda'}$ is an epimorphism. This proves that $g = g'$, and $f$ is indeed an epimorphism in $\text{pro}(\mathcal{K})$.

4.4. In general in a category $\mathcal{K}$ with zero-objects and kernels one can define exactness of sequences (24), p. 114). A sequence

$$G \to H \to K$$

in $\mathcal{K}$ is said to be exact at $H$ provided: (i) $gf = 0$, (ii) in the unique factorization $f = jf'$, where $i: N \to H$ is the kernel of $g$, the morphism $f'$ is an epimorphism.

Now let us establish the following fact (20), § 1, Corollary 3.6): Let $f: G \to H$, $g: H \to K$ be special maps of pro-groups with the property that

$$G^{f_1} \to H^{f_1} \to K$$

is exact at $H$ in $\mathcal{K}$. Then the sequence

$$G \to H \to K$$

is exact in $\text{pro}(\mathcal{K})$. Indeed, $g_{f_1} f_1 = 0$ implies $gf = 0$. Let $(N_\lambda, i_\lambda) = Ker g_{\lambda}$ so that $N = (N_\lambda, i_\lambda)$ is a kernel of $g$. By the exactness assumption, $f_1$ admits a unique factorization $f_1 = i_\lambda f_1$ where $i_\lambda: G_\lambda \to N_\lambda$ is an epimorphism. The homomorphisms $f_1$ define a morphism.
This is an immediate consequence of the following assertion:
Let \( \mathcal{X} \) be a category with zero-objects and kernels and let
\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} A_5
\]
be an exact sequence. If \( f_1 \) is an epimorphism and \( f_4 \) is a monomorphism, then \( A_3 \) is a zero-object.

Proof. Since \( f_2 f_3 = f_1 f_2 \) factors through \( \text{Ker} f_4 \). However, \( f_4 \) being a monomorphism, \( \text{Ker} f_4 = 0 \) and thus \( f_2 = 0 \). Let \( \text{Ker} f_3 = (K, \delta) \). By exactness at \( A_3 \), there is an epimorphism \( g : A_2 \to K \) such that \( kg = f_2 \). Now \( f_2 = 0 \) implies \( k = 1 \), so that \( f_3 = g \) is an epimorphism. By assumption on \( f_1 \), we conclude that \( f_2 f_3 f_1 : A_1 \to A_3 \) is an epimorphism too. On the other hand, \( f_2 f_3 f_1 = 0 \) and therefore, \( A_3 \) must be a zero-object.

5.4. For a pointed topological space \( (X, x_0) \) we can define homotopy pro-groups, only up to isomorphic objects in \( \text{pro}(\mathcal{Y}) \), as \( \pi_n(X, x_0) \), where \( (X, x_0) \) is any system in \( \mathcal{Y} \) associated with \( (X, x_0) \). Indeed, any two systems \( (X, x_0) \), \( (X', x_0') \) associated with \( (X, x_0) \) are isomorphic in \( \text{pro}(\mathcal{Y}) \), because the identity shape map \( 1 : (X, x_0) \to (X, x_0) \) determines a unique isomorphism \( (X, x_0) \to (X', x_0') \). Similarly, one defines homotopy pro-groups for pointed pairs of spaces \( (X, x_0) \). One then obtains an exact homotopy sequence for pro-groups of spaces.

6. Shape deformation retraction
6.1. In this section we shall prove the following

Theorem 1. Let \( (X, A, x) \) and \( (Y, B, y) \) be inverse systems in \( \mathcal{W} \). Let \( (X, A, x) \) be simplicial, \( \dim X_n \leq n < \infty \). Let \( (Y, B, y) = ((Y, B, y_0), \Delta, \Delta') \) be such that \( \pi_n(Y, B, y) = 0 \), for \( 1 \leq n \leq n+1 \), each \( Y_0 \) is connected and \( M \) is closure-finite. Then every morphism \( f : (X, A, x) \to (Y, B, y) \) in \( \text{pro}(\mathcal{W}) \) admits a morphism \( g : (X, x) \to (B, y) \) in \( \text{pro}(\mathcal{W}) \) such that
\[
g(f(x)) = (f(A)) : (A, x) \to (B, y),
\]
where \( (B, y) \to (Y, y) \) is given by the inclusions \( j_1 : (B, y) \to (Y, y) \).

If one applies Theorem 1 to the identity morphism \( 1 : (X, A, x) \to (X, A, x) \), one obtains

Theorem 2. Let \( (X, A, x) \) be an inverse system in \( \mathcal{W} \) over a closure-finite index set \( A \). Let \( (X, A, x) \) be simplicial, \( X_0 \) connected and \( \dim X_n \leq n < \infty \). If \( \pi_n(X, A, x) = 0 \) for \( 1 \leq n \leq n+1 \), then there is a morphism \( x : (X, A, x) \to (A, x) \) in \( \text{pro}(\mathcal{W}) \) such that \( f_1 x = f_2 x = 0 \). Consequently, the morphism \( f : (X, A, x) \to (X, x) \) given by the inclusions \( j_1 : (A, x) \to (X, x) \), is an isomorphism in \( \text{pro}(\mathcal{W}) \).

The proof of Theorem 1 is based on two lemmas.

6.2. Lemma 1. There is an increasing function \( \delta : M \to N \), \( \delta(y) = n^* \geq m^* \) such that for any path of simplicial complexes \( (P, Q, x), \dim(P, Q) \leq n-1, \) and for any
map \( \phi: (P, Q, x_0) \to (Y, B, y) \), there is a map \( \psi: (P, Q, x_0) \to (B, y) \) such that

\[
j^* \phi \sim \psi_M \phi: (P, Q, x_0) \to (Y, B, y),
\]

where \( \psi_M \) is any map from the class \( \psi_M \), \( \mu \leq \mu' \), and \( j^* \) is a chain in \( (M, \leq) \) such that for every \( k, 1 \leq k \leq n+1 \), the homomorphism

\[
\eta_{\mu, x_0}(\mu_1, \ldots, \mu_k, x_0) \vdash \eta_{\mu_1} \eta_{\mu_2} \cdots \eta_{\mu_k}(x_0, y_0)
\]
equals 0, \( 0 \leq j \leq n \). Such a chain exists because \( \eta_{\mu, x_0}(Y, B, y) = 0 \), \( 1 \leq k \leq n+1 \). We choose a map \( \eta_{\mu, x_0} \), from \( \eta_{\mu, x_0} \), and denote by \( \eta_{\mu, x_0, j} \), \( j = 0 \), the corresponding composition of such maps; \( \eta_{\mu, x_0} \), is 1. We also choose a triangulation of \((P, Q, x_0)\) such that \( Q \) is a full subcomplex of \( P \) (i.e. if all vertices of a simplex belong to \( Q \), then so does the simplex). Let \( L_k = (Q \cup P^k) \times I \cup (P \times 0) \), where \( P^k \) is the \( k \)-skeleton of \( P \), \( 0 \leq k \leq n+1 \).

We shall define, by induction, maps \( X_k: L_k \to Y_{n+1} \) such that

\[
x_k(x, t) = \eta_{\mu, x_0, (x_0, y_0, t)} \psi(x), \quad (x, t) \in (Q \times I) \cup (P \times 0),
\]

\[
x_k(x, t) \in B_{n+1}, \quad x \in P^k.
\]

\[
x_k(x, t) = y_{\mu, x_0, n+1}, \quad x \in P^k \setminus Q.
\]

For \( k = 0 \) and \((x, t) \in (Q \times I) \cup (P \times 0)\) we put \( x_k(x, t) = \phi(x) \). If \( x \) is a vertex of \( P \cup Q \), we put \( x_k(x, t) = y_{\mu, x_0} \). Since \( Y_{n+1} \) is pathwise connected, one can define \( x_k \) on \( x \times I \) as a path connecting \( \phi(x) \) to \( y_{\mu, x_0} \). Thus \( x_k \) has all the required properties.

Assume now that \( X_{k+1} \) has already been defined. Then we put \( x_{k+1} = \eta_{\mu, x_0, n+1} X_k \bullet x \). If \( E^k \) is a \( k \)-simplex of \( P^k \), \( Q \), it has a vertex \( x \) not belonging to \( Q \). \((E^k \times 0) \cup (E^k \times I) \cup (E^k \times 0) \cup (E^k \times x) \times I \) is homeomorphic to \( P \) \( k \)-dimensional cell, its boundary and a base-point and \( X_{k+1} \) maps this pointed pair into \((Y, B, y)_{n+1} \). Since \( \eta_{\mu, x_0, n+1} X_k \bullet x \), induces the zero-homomorphism

\[
\eta_{\mu, x_0}(Y, B, y)_{n+1} \vdash \eta_{\mu, x_0}(Y, B, y)_{n+1}
\]
we conclude that

\[
X_k: (E^k \times 0) \cup (E^k \times I) \cup (E^k \times 0) \cup (E^k \times x) \times I \to (Y, B, y)_{n+1}
\]
determines the zero element of \( \eta_{\mu, x_0}(Y, B, y)_{n+1} \). Consequently, there is a homotopy rel \((E^k \times 0) \cup (E^k \times I) \cup (E^k \times 0) \cup (E^k \times x) \times I \) into \( Y_{n+1} \) connecting \( x_k \) with some map \((E^k \times 0) \cup (E^k \times I) \to Y_{n+1} \). This homotopy yields an extension \( x_{k+1} \) from \( E^k \times 0 \) to \( E^k \) such that \( x_{k+1} = x_k \). This completes the induction step.

Now observe that \( L_{n+1} = P \times I \) and consider \( x_{n+1} = P \times I \to Y_{n+1} \). By \((2) \) \( x_{n+1} = (Q \times I) \cup B_{n+1} \), \( x \in P \). Consequently, putting

\[
\psi(x) = x_{n+1}(x, 1), \quad x \in P,
\]
we obtain a map \( \psi: (P, Q, x_0) \to (B, y) \), such that \((1) \) holds.

Finally, using 2.3, one can achieve that \( \mu \leq \mu' \) implies \( \mu \leq \mu' \).

6.3. Lemma 2. For every \( \mu \in M \) let \( \mu' \geq \mu \) be chosen in accordance with Lemma 1. Let \( (P, Q, x_0) \) be a simplicial complex, \( \text{dim} P \leq n \). Furthermore, let \( \phi_0, \phi_1: (P, Q) \to (B, y) \) be maps such that

\[
j_0^* \phi_0 = j_{n+1}^* \phi_1: (P, Q, x_0) \to (Y, B, y).
\]

Then

\[
\phi_0 = \phi_1: (P, Q, x_0) \to (B, y).
\]

Proof. Consider the triple \( (P \times I, P \times 0 \cup P \times 1, x_0 \times I) \) and shrink \( x_0 \times I \) to a point. Let

\[
\phi: (P \times I, P \times 0 \cup P \times 1, x_0 \times I)/(x_0 \times I) \to (Y, B, y),
\]
be a map given by a homotopy \((5) \). Since \( \text{dim}(P \times I, x_0 \times I) \leq n \), Lemma 1 yields a map

\[
\Psi: (P \times I, x_0 \times I)/(x_0 \times I) \to (B, y)
\]
such that

\[
\Psi(P \times 0 \cup P \times 1, x_0 \times I)/(x_0 \times I) = x_0, \quad \text{for} \quad \phi_0, \phi_1
\]

and the homotopy takes place in \((B, y)\). Consequently,

\[
x_0 \circ \phi_0 = \Psi(P \times 0, x_0 \times I)/(x_0 \times I) = \Psi(P \times 1, x_0 \times I)/(x_0 \times I)
\]

\[
= x_0 \circ \phi_1: (P, x_0) \to (B, y).
\]

6.4. Proof of Theorem 1. Let \( f: (X, A, x) \to (Y, B, y) \) be given by \( f \) and by homotopy classes of maps \( f_\mu: (X, A, x)_f \to (Y, B, y) \) with representatives \( \phi_\mu \).

Since \( M \) is closure-finite, we can assume that \( f \) is increasing and that

\[
f_{\mu} \phi_{\mu} = \phi_{\mu} f_{\mu} \quad \mu \leq \mu'.
\]

For every \( \mu' \) we choose \( \mu' \) according to Lemma 1. Then for every \( \mu \in M \) there is a mapping \( \psi_{\mu}: (X, A, x)_{f, \mu} \to (Y, B, y) \) such that

\[
f_{\mu} \psi_{\mu} \mu \phi_{\mu} \phi_{\mu} = (X, A, x)_{f, \mu} \to (Y, B, y)
\]

\[
\psi_{\mu} (A, x)_{f, \mu} \phi_{\mu} = \phi_{\mu} \phi_{\mu} (A, x)_{f, \mu} \to (A, x)_{f, \mu} \to (B, y)
\]

We now put \( g_{\mu} = f_{\mu}(\mu') \) and

\[
g_{\mu} = \psi_{\mu}(\psi_{\mu} f_{\mu} \phi_{\mu} (x, x)_{f, \mu} \to (B, y)).
\]
where square brackets denote homotopy classes. The map $g$ and the homotopy classes $g_{\mu}$ determine a map of systems of $(X, x) \to (B, y)$. Indeed, let $\mu \leq \mu'$. Then, by (10)

$$j_{\mu} \phi_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

$$= (X, x)_{f_{\mu}} \to (Y, y)_{f_{\mu}},$$

(13)

and hence $g_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$.

(14)

where $g_{\mu}$ is any map from the class $p_{\mu}$. Since $g_{\mu}$ is any map from the class $p_{\mu}$, then

$$g_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(15)

and (14) yield

$$g_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(16)

Applying Lemma 2 to (16), we now conclude that

$$g_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(17)

so that

$$g_{\mu} \cong [x_{\mu}] = [x_{\mu}'] \phi_{\mu} = [x_{\mu}'] \phi_{\mu} \circ f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(18)

This proves that $g$ is indeed a map of systems in $\mathcal{W}^0$.

From (12) and (13) we derive

$$[l_0] g_{\mu} = q_{\mu} \ast f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(19)

which proves that

$$j g = f : (X, x) \to (Y, y).$$

(20)

Finally, by (11),

$$g_{\mu} \ast [l_0] g_{\mu} = q_{\mu} \ast f_{\mu} \circ (f_{\mu}^{-1} \circ g_{\mu})$$

(21)

so that $g(A, x) = f(A, x)$.

Therefore, we have

$$g(A, x) = f(A, x)$$

(22)

which completes the proof of Theorem 1.

6.5. From Theorems 1 and 2 follow analogous results for spaces. In particular we have

Theorem 3. Let $(X, A, x_0)$ and $(Y, B, y_0)$ be pointed pairs of topological spaces, let $\dim X \leq n < \infty$, let $\pi_n(X, y, B) = 0$, and let $Y$ be connected. Then every shape map $f : (X, A, x_0) \to (Y, B, y_0)$ admits a shape map $g : (X, x_0) \to (B, y_0)$ such that $j g = f$, $g(A, x_0) = f(A, x_0)$, where $j$ is the shape map induced by the inclusion $j : B \to Y$.

Proof. According to 3.4, there exist inverse systems $(X, A, x_0), (Y, B, y_0)$ in $\mathcal{W}^0$ associated with $(X, A, x_0)$ and $(Y, B, y_0)$ respectively and such that each $(X, A, x_0)$ is simplicial, $\dim X \leq n$ and each $Y_0$ is connected. Moreover, by 2.3 and 3.2, one can achieve that the index set $M$ of $(Y, B, y_0)$ is closure-finite. Therefore, Theorem 1 yields the desired conclusion.

From Theorem 3 (or from Theorem 2) one derives

Theorem 4. Let $(X, A, x_0)$ be a pair of pointed topological spaces, let $X$ be connected and let $\dim X \leq n < \infty$. If $\pi_n(X, A, x) = 0$ for $1 \leq k \leq n+1$, then $(A, x_0)$ is a shape deformation retract of $(X, x_0)$, i.e., the inclusion $j : (A, x_0) \to (X, x_0)$ induces a shape equivalence.

Definition. We say that a space $(X, x_0)$ is shape-connected, provided $\pi_n(X, x_0) = 0$ for $0 \leq k \leq n$.

Theorem 5. Let $(X, x_0)$ be a pointed connected topological space, dim $X \leq n < \infty$.

If $(X, x_0)$ is shape-connected, then $(X, x_0)$ is of trivial shape.

Proof. It suffices to apply Theorem 4 to $(X, x_0, x_0)$.

Remark. D. S. Kahn [10] has defined a metric continuum $(X, x_0)$ which is shape-connected for all $n$ but is not of trivial shape as shown by D. Handel and J. Segal [8]. However, dim $X = \infty$.

7. The Whitehead theorem for maps

7.1. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed topological spaces. The mapping cylinder $(Z, z_0)$ of $f$ is obtained from the topological sum $(X \times I) \cup Y$ by identifying $(x, 1)$ and $f(x)$, $x \in X$, and by shrinking $(x_0 \times I) \cup (y_0)$ to a point $z_0$. The image of $(x, t)$ and $y$ under this identification will be denoted by $[(x, 0)]$ and $[y]$ respectively. Along with the mapping cylinder, we consider embeddings $i : (X, x_0) \to (Z, z_0)$, $j : (Y, y_0) \to (Z, z_0)$, given by $i(x) = [(x, 0)]$ and $j(y) = [y]$ respectively, as well as the map $g : (Z, z_0) \to (Y, y_0)$ given by $g[(x, 0)] = f(x)$, $g[y] = y$. Note that $g i = 1$ and $g z_0 = 1$ so that $f$ and $g$ are homotopy equivalences. Furthermore, $j f z_0 = 1$ and therefore, $f = g i$.

Since $g$ is a homotopy equivalence and $f = g i$, we conclude that $f$ is a shape equivalence if and only if $i$ is a shape equivalence. Similarly, $f$ induces a monomorphism (epimorphism) of homotopy pro-groups $\pi_n(X, x_0) \to \pi_n(Y, y_0)$ if and only if $i$ induces one. Note that the dimension of the mapping cylinder $(Z, z_0)$ of $f : (X, x_0) \to (Y, y_0)$ satisfies

$$\dim Z \leq \max [\dim X, \dim Y] + 1.$$

Indeed, consider the subsets $Z_0 = X \times I_0 / x_0 \times I_0 \subset Z$, where $I_0 = \left[0, \frac{n+1}{2}, n \right]$. These subsets are $C^n$-embedded in $Z$; i.e., every map $f : Z_0 \to I$ admits a continuous extension to all of $Z$. This is easily seen by considering the mapping cone $C$ of $f$ and the mapping $Z \to C$ which identifies $Y$ to a point. Since $Y$...
is a retract of $Z$. It is also $C^*$-embedded in $Z$. Now Lemma 5.9 of [15] yields the assertion because $Z = (U Z) \cup Y$ and $\dim Z = \dim X \times \ell_1 \times \dim X + 1$ (Theorem 5.5 of [15]).

7.2. We can now state and prove our main result.

THEOREM 6. Let $(X, x_0)$ and $(Y, y_0)$ be pointed topological spaces, connected and finite-dimensional, and let $f: (X, x_0) \to (Y, y_0)$ be a map which induces homomorphisms

$$f_*: \pi_k(X, x) \to \pi_k(Y, y)$$

of homotopy pro-groups for

$$0 \leq k < n_0 + 1 = \max(1 + \dim X, \dim Y) + 1$$

and an epimorphism for $k = n_0 + 1$. Then $f$ is a shape equivalence, i.e., there is a shape map $g: (Y, y_0) \to (X, x_0)$ such that $f g = 1$, $g f = 1$.

Proof. Let $(Z, z_0)$ be the mapping cylinder of $f$. $Z$ is connected and $\dim Z \leq \max(1 + \dim X, \dim Y) = n_0 < \infty$. Furthermore, by 7.1, the inclusion induces a bijection from homotopy pro-groups for $0 \leq k < n_0 + 1$ and an epimorphism for $k = n_0 + 1$. By 5.4, there is an exact sequence of homotopy pro-groups belonging to the pair $(Z, x_0)$. We conclude, by 5.3, that $\pi_k(Z, x_0) = 0$ for $0 \leq k < n_0 + 1$. Consequently, by Theorem 4, the inclusion $i: (X, x_0) \to (Z, z_0)$ is a shape equivalence, which implies that $f$ is a shape equivalence too.

8. The Whitehead theorem for shape maps

8.1. THEOREM 7. Let $(X, x_0)$ be a compact Hausdorff space, $(Y, y_0)$ a compact metric space and $f: (X, x_0) \to (Y, y_0)$ a shape map. Then there is a compact Hausdorff space $(Z, z_0)$ and embeddings $i: (X, x_0) \to (Z, z_0)$ and $j: (Y, y_0) \to (Z, z_0)$ such that $j$ admits a shape inverse $g: (Z, z_0) \to (Y, y_0)$ and $f = g i$. If $X$ is metric, then so is $Z$.

Proof. There exist an inverse sequence $(Y, y_n) = ((Y, y_n), \varphi_{n+1}, N)$ in $\pi_0$ of compact polyhedra $(Y, y_n)$ whose inverse limits is $(Y, y_0)$. Then $f$ is given by homotopy classes of maps $f_n: (X, x_0) \to (Y, y_n)$ such that $\varphi_{n+1} f_n = f_n$, $n \in N$. For each $n \in N$ let $\varphi_n$ be a map from the classes $\varphi_{n+1}$ and $f_n$ respectively, let $(Z, z_0)$ be the mapping cylinder of $\varphi_n$ and let $i_n: (X, x_0) \to (Z, z_0)$ and $j_n: (Y, y_n) \to (Z, z_0)$ be inclusions as in 7.1. We choose for every $n \in N$ a homotopy

$$\Phi_n: (X \times I, x_0 \times I) \to (Y, y_n),$$

which connects $\varphi_n$ and $\varphi_{n+1} \varphi_n$. Then a map $\varphi_{n+1} = (Z, z_0) \to (Z, z_0)$ is defined by

$$\varphi_{n+1}(x, t) = \begin{cases} (x, 2t), & 0 \leq t \leq \frac{1}{2}, \ x \in X, \\ (\Phi_n(x, 2t-1)), & \frac{1}{2} \leq t \leq 1, \ x \in X, \end{cases}$$

$$\varphi_{n+1}(y) = y \in Y, y 

(\ast)\text{ According to a communication from M. Mozyńska the theorem has been already proved by W. Holzatyński (unpublished) if both } X \text{ and } Y \text{ are metric compacts (cf. also [17]).}
Invariant uniformization

by

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Abstract. We show that the invariant version of the Kondo-Addison Uniformization Theorem fails. Several counterexamples of algebraic interest are presented.

Can one pick a point from each countable linear order? To make this problem model-theoretically interesting we identify isomorphic structures and to make it nontrivial in ZFC, set theory with choice, we require that the picking be done in a countable-ordinal-sequence-definable way. Roughly speaking, a set is definable from a countable sequence of ordinals if for some ZF formula $\varphi$ and some countable sequence $\alpha$ of ordinals, it is the unique solution of $\varphi(x, \alpha)$. A set definable in any mathematically accepted way will be countable-ordinal-sequence-definable. Henceforth we shall interpret “one can pick a point (proper substructure, proper extension, etc.) from each linear order” as meaning that there is a countable-ordinal-sequence-definable function which assigns to each isomorphism type of a countable linear order the isomorphism type of a point (proper substructure, proper extension etc.) of the linear order, i.e., to the isomorphism type of a $<\alpha, \lambda> \in \mathcal{A}$ assigns the isomorphism type of some structure $\langle A, B, \lambda >$ where $a \in A \cap \mathcal{A}$ and $B \subseteq A$. “One cannot always pick...” shall be interpreted as meaning that it is relatively consistent with ZFC that there is no countable-ordinal-sequence-definable function which picks... All of the results below of the form “one cannot always pick a...” have as consequences “it is relatively consistent with ZF and the principle of dependent choices that there is no function (definable or not) which selects a...”.

We first show that one cannot always pick points from certain structures called bireals and then show that: One cannot always pick a point from each countable linear order or from each countable semigroup and one cannot always pick a proper substructure for each countable algebra which has such. Although one can always pick a proper extension by adding a new point, it is relatively consistent with the existence of an inaccessible cardinal that one cannot pick a countable nonisomorphic extension of each countable structure which has such. It is also relatively consistent

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