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Accepté par la Rédaction le 27. 6. 1974

An annulus theorem for suspension spheres

by

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Abstract. A space X is called a *suspension* $(n-1)$*-sphere* if $S(X)$, the suspension of X, is homeomorphic to S^n. Kirby has shown [3] that any orientation preserving self homeomorphism of S^n is stable for $n\geqslant 5$. The author shows that Kirby's result implies the following. If $n\geqslant 5$ and X is a suspension $(n-1)$-sphere, then for any two embeddings $f_i\colon X\to S^n$, $i=1,2$, so that $f_1(X)$ and $f_2(X)$ are disjoint and bicollared in S^n then M, the closed region in S^n bounded by $f_1(X)$ and $f_2(X)$, is homeomorphic to $X\times I$.

1. Introduction. If X_1 and X_2 are compact metric generalized manifolds then we shall say that X_1 and X_2 are *h-cobordant* if there exists a compact metric space M so that (i) M is a generalized manifold with boundary (as in [5]); (ii) there is a homeomorphism f from the disjoint union X_1+X_2 onto ∂M, the boundary of M; (iii) the restrictions $f_i=f|X_i$ induce isomorphisms between the homotopy groups of X_i and those of M, $i=1,2$. In addition, for the objects we shall consider it will be necessary to impose two further conditions: (iv) $\mathrm{Int}M=M-\partial M$ is a manifold and (v) ∂M is collared in M, that is there is a homeomorphism h from $\partial M\times[0,1)$ onto an open set in M so that for each $x\in\partial M$, $f(x,0)=x$.

When conditions (i)-(v) are satisfied we call M an *h-cobordism* between X_1 and X_2. This terminology was suggested to the author by L. C. Siebenmann.

It should be noted that conditions (iv) and (v) require $X_i\times R$ to be a manifold. Thus X_1, X_2 and M are generalized manifolds with respect to homology and cohomology over any coefficient domain. If $X\times R$ is a manifold then $X\times I$ is an h-cobordism between $X\times\{0\}$ and $X\times\{1\}$.

For any space X, $S(X)=S^0*X$ will be the suspension of X, $C(X)=I^0*X$ will be the cone over X and $OC(X)=C(X)-X$ will be the open cone over X. If $S(X)\approx S^n$, that is $S(X)$ and S^n are homeomorphic, we call X a *suspension* $(n-1)$*-sphere*.

Consider the proposition

(HCB_n): *If X_1 and X_2 are suspension $(n-1)$-spheres then up to homeomorphism there is exactly one h-cobordism M between X_1 and X_2.*

The purpose of this note is to show that (HCB_n) is true for $n\geqslant 5$. A fairly elementary proof is given for $n\geqslant 6$; the case for $n=5$ was originally proved by us in a more complicated fashion. For the sake of brevity we merely outline the details for the latter case.

It is appropriate to recall here that there have been many constructions of nontrivial suspension m spheres topologically distinct from S^m for each dimension $m \geqslant 3$. The simplest class of such examples is due to Andrews–Curtis [1]. If J is an arc in S^m, $m \geqslant 3$, so that $\pi_1(S^m - J) \neq \{1\}$ then the quotient space S^m/J is a nontrivial suspension m-sphere. For other facts about suspension spheres, see [6, 7]. All known examples of nontrivial suspension spheres are neither manifolds nor polyhedra. Indeed an outstanding question in the topology of manifolds is whether a suspension sphere which is either a manifold or a polyhedron must be a sphere.

There is some possibility that the results in this paper may be useful in studying this conjecture. These results suggest strong analogues between the structure of embeddings of suspension $(n-1)$-spheres in S^n with what is known about embeddings of S^{n-1} in S^n. If even stronger analogues hold, the author can show that a weaker form of this conjecture has a positive answer. Since further technicalities need to be introduced to discuss this matter further, the author will concentrate on establishing (HCB_n) for $n \geqslant 5$.

2. h-cobordisms and embeddings in spheres. Consider the following propositions.

(SHC_n): *All orientation preserving self homeomorphisms of S^n are stable.*

(A_n): *If S_1 and S_2 are disjoint flat homeomorphs of S^{n-1} embedded in S^n, then the closed region they bound is homeomorphic to $S^{n-1} \times I$.*

(I_n): *If f is any orientation preserving self homeomorphism of S^n, then f is isotopic to the identity map.*

Kirby proved in [3] that (SHC_n) is true for all $n \geqslant 5$. (He actually proved the corresponding statement for R^n. A simple direct argument or application of the systematic work of Brown–Gluck [2] easily extends Kirby's theorem to S^n.)

In their series of papers on stable manifolds [2] Brown–Gluck exhibited the following logical relationship between the three propositions stated above.

$$(\mathrm{A}_n) + (\mathrm{I}_{n-1}) \Rightarrow (\mathrm{SHC}_n) \begin{array}{l} \Rightarrow (\mathrm{A}_n) \\ \Rightarrow (\mathrm{I}_n) \end{array}$$

Let X_1 and X_2 be two disjoint homology $(n-1)$-spheres embedded in S^n. We shall find it convenient to let $[X_1, X_2]$ denote the closed region in S^n which is bounded by X_1 and X_2.

A natural extension of (A_n) for the embeddings of a fixed suspension $(n-1)$-sphere X would be,

(X-A_n): *If X_1 and X_2 are flat disjoint homeomorphs of X embedded in S^n, then* $[X_1, X_2] \approx X \times I$.

An embedding $f\colon X \to S^n$ is called *flat* if we have a homeomorphism of pairs $(S(X), X) \approx (S^n, f(X))$.

LEMMA 1. (a) *If M is an h-cobordism between the suspension $(n-1)$-spheres X and Y and $n \geqslant 5$ then M may be embedded in S^n so that ∂M is bicollared.*

(b) *If X and Y are disjoint flat suspension $(n-1)$-spheres embedded in S^n then $[X, Y]$ is an h-cobordism.*

Proof. (a) Attach disjoint cones $p * X$ and $q * Y$ to M by adjunction via the identity maps on X and Y, after identifying $X+Y$ with ∂M. The adjunction space Z so formed is an n-manifold and a homotopy n-sphere. Since $n \geqslant 5$, by [4] $Z \approx S^n$.

(b) Since X and Y are bicollared in S^n, they are collared in $M = [X, Y]$. This allows us to show that M is of the same homotopy type as $M - X$, i.e. $M \sim M - X$. Since the components of $S^n - \mathrm{Int}\, M$ are cellular, it is easily seen that $M - X \approx Y \times [0, 1)$. Hence $M \sim Y$ and also $M \sim X$.

LEMMA 2. *If X and Y are suspension $(n-1)$-spheres there is a subset $M \subseteq S^n$ so that M is an h-cobordism between X and Y and ∂M is bicollared in S^n.*

Proof. Since $OC(X) \approx R^n$ (similary for Y) [7'; Corollary to Theorem 4] we pick disjoint open n-cells U and V in S^n. U contains a bicollared homeomorph X' of X and V contains a bicollared homeomorph Y' of Y. Obviously $X' \cap Y' = \emptyset$. By Lemma 1 of [6] X' and Y' are flat in S^n. By Lemma 1 (b), $[X', Y']$ is an h-cobordism.

3. h-cobordism theorems.

THEOREM 1. *Let X and Y be suspension $(n-1)$-spheres and M_i be h-cobordisms with $\partial M_i = X_i + Y_i$, $X_i \approx X$ and $Y_i \approx Y$, $i = 1, 2$. If (Y-A_n) is true and $n \geqslant 5$ then for each $f\colon X_1 \approx X_2$ there is an extension $F\colon M_1 \approx M_2$.*

Proof. From Lemma 1 (a), we may assume $M_1 \cup M_2 \subseteq S^n$. Let $X_i \subseteq N_i \subseteq M_i - Y_i$ so that $(N_i, X_i) \approx (X_i \times [0, 1], X_i \times \{0\})$, $i = 1, 2$, with $x \in X_i$ corresponding to $(x, 0)$. Via these homeomorphisms f may be extended to $f'\colon N_1 \approx N_2 \subseteq M_2 - Y_2$. From the argument used in Lemma 1 of [6], since the compact component of $S^n - (M_1 - X_1)$ is cellular in S^n, it follows that $\mathrm{Int}\, N_1$ contains a flat homeomorph Y_1' of Y which separates the components of the boundary of N_1. Let $Y_2' = f'(Y_1')$ and $P_i = [X_i, Y_i']$, $Q_i = [Y_i', Y_i]$, $i = 1, 2$. Note that $P_i \subseteq N_i$, $P_i \cup Q_i = M_i$ and $P_i \cap Q_i = Y_i'$. Clearly $f'|P_1\colon P_1 \approx P_2$. Inasmuch as Y_2' is bicollared and thus flat [8; Theorem 7], the assumption (Y-A_n) true implies that $Q_i \approx Y \times I$. Thus $f'|Y_1$ can be extended to give $g\colon Q_1 \approx Q_2$. Accordingly $F = (f'|P_1) \cup g$ is the desired extension of f.

COROLLARY. *If $n \geqslant 5$ and (Y-A_n) is true for every suspension $(n-1)$-sphere Y, then (HCB_n) is true.*

THEOREM 2. $(\mathrm{A}_n) + (\mathrm{I}_{n-1}) \Rightarrow (\mathrm{HCB}_n)$, $n \geqslant 5$.

Proof. Again let M_i be h-cobordisms with $\partial M_i = X_i + Y_i$, $M_i \subseteq S^n$, $X_i \approx X$ and $Y_i \approx Y$, $i = 1, 2$. By Lemma 1 (a) we may assume ∂M_i is bicollared in S^n. Because X_i is flat in S^n, by global homeomorphisms of S^n we may assume $X_1 = X_2 = X$ and that Y_1 and Y_2 are on the same side of X in S^n.

By collaring arguments we may consider $X \subseteq N \subseteq (M_1 - Y_1) \cap (M_2 - Y_2)$ and $Y_i \subseteq P_i \subseteq M_i - N$ so that $(N, X) \approx (X \times [0, 1], X \times \{0\})$ and

$$(P_i, Y_i) \approx (Y_i \times [0, 1], Y_i \times \{0\}), \quad i = 1, 2.$$

We note that $P_i \cap N = \emptyset$. Accordingly we may define $f\colon N \cup P_1 \approx N \cup P_2$ with $f|N = 1_N$ and $f|P_1$, orientation preserving.

Let S be a flat $(n-1)$-sphere in $\operatorname{Int} N$ and S_1, a flat $(n-1)$-sphere in $\operatorname{Int} P_1$, which separate N, respectively P_1, between components of their boundaries. (See proof of Lemma 1 in [5].) Set $S_2 = f(S_1)$. By (A_n), $([S, S_i], S) \approx (S \times [0, 1], S \times \{0\})$ by homeomorphisms H_i with the property that for $x \in S$, $H_i(x) = (x, 0)$. $H_2 H_1^{-1}|S \times \{0\} = 1_{S \times \{0\}}$. On the other hand we may suppose $g = H_2 f H_1^{-1}|S \times \{1\}$ is orientation preserving, regarded either with respect to the product orientation on $S \times I$ or by projection on S. By (I_{n-1}) there is a homeomorphism $G\colon S \times I \approx S \times I$ so that $G|S \times \{0\} = 1_{S \times \{0\}}$ and $G|S \times \{1\} = g$.

Consider $H_2^{-1} G H_1\colon [S, S_1] \approx [S, S_2]$. If $x \in S$, $H_2^{-1} G H_1(x) = H_2^{-1} G(x, 0) = H_2^{-1}(x, 0) = x$. If $x \in S_1$, then $H_1(x) \in S \times \{1\}$. $G|S \times \{1\} = H_2 f H_1^{-1}|S \times \{1\}$, so $H_2^{-1} G H_1(x) = H_2^{-1}(H_2 f H_1^{-1}) H_1(x) = f(x)$. Thus $f|([X, S] \cup [S_1, Y_1]) \cup H_2^{-1} G H_1$ is a homeomorphism of M_1 onto M_2.

COROLLARY. (HCB_n) *is true for* $n \geqslant 6$.

THEOREM 3. *If X is a suspension $(n-1)$-sphere and $n \geqslant 5$ then* $(\mathrm{X\text{-}A}_n)$ *is true.*

Proof. For $n \geqslant 6$ this follows from Theorem 2. Instead we originally showed that $(\mathrm{SHC}_n) \Rightarrow (\mathrm{X\text{-}A}_n)$. The method used was to imitate the construction of Brown–Gluck [2; pp. 2–8] thereby getting an annular equivalence relation for flat embeddings of X in S^n. Eventually one determines by use of (SHC_n) that there is only one such equivalence class. This construction is complicated by the fact that if $X \neq S^{n-1}$, there is no single canonic family of embeddings. Although this procedure is somewhat long and cumbersome, it is fairly straightforward. As mentioned in the introduction, we considered it reasonable to omit further details.

THEOREM 4. (HCB_n) *is true for* $n \geqslant 5$.

Proof. This follows from the Corollary to Theorem 1 and Theorem 3.

Addendum. After submitting this paper for publication the author noticed that (HCB_n) may be proved for $n = 5$ in a manner similar to the proof for $n = 6$ above (Corollary to Theorem 2). If f and g are self homeomorphisms of S^n, f is said to be *weakly isotopic to* g if there is a homeomorphism $H\colon S^n \times I \approx S^n \times I$ so that for all $x \in S^n$, $H(x, 0) = (f(x), 0)$ and $H(x, 1) = (g(x), 1)$.

Now let (WI_n) denote the proposition:

If f is an orientation preserving self homeomorphism of S^n, then f is weakly isotopic to the identity map of S^n.

In the papers of Brown–Gluck [2] a stronger result was established than the implications used by the author in Section 2 above. They actually showed that

$$(\mathrm{WI}_{n-1}) + (\mathrm{A}_n) \Leftrightarrow (\mathrm{SHC}_n).$$

It is fairly clear that Theorem 2 above can be sharpened, with virtually the same proof, to read:

$$(\mathrm{A}_n) + (\mathrm{WI}_{n-1}) \Rightarrow (\mathrm{HCB}_n), \quad n \geqslant 5.$$

Since (SHC_5) is true, this would establish (HCB_5).

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Accepté par la Rédaction le 1. 7. 1974