To show that (II) implies (I), it suffices to show that any ring satisfying (II) is one of the types (i), (ii), and (iii). According to Lemma 1, A has at most two proper ideals.

Case 1. If A has no proper ideals, then, by Theorem 1, R is either a division ring or is isomorphic to a 2×2 matrix ring over a division ring. Thus A is of type (i) or (iii).

Case 2. If A has exactly one proper ideal, namely I, then by Lemma 2, $I^2 = (0)$ and R/I is a division ring. Thus I is the Jacobson radical of R. From Theorems 3, 4, and 5, we can see easily that I is the only proper right ideal in A, so A is of type (ii).

Case 3. If A has two proper ideals, then, by Theorem 2, A is isomorphic to a direct sum of two division rings, so A is of type (iii).

This completes the proof.

As we pointed out earlier, in view of Theorem 8 our results provide a further classification for the rings studied by Koh.

References

[1] K. Koh, *Quasi-simple modules and other topics in ring theory*, Lecture Notes in Mathematics 246, Springer—Verlag, New York 1972.
[2] F. I. Perticani, *Commutative rings in which every proper ideal is maximal*, Fund. Math. 71 (1971), pp. 193–198.
[3] G. Thierrin, *Sur la structure d'une classe d'anneaux*, Canad. Math. Bull. 3 (1960), pp. 11–16.

NORTH CAROLINA STATE UNIVERSITY
Raleigh, North Carolina

Accepté par la Rédaction le 17. 6. 1974

On subspaces of separable first countable T_2-spaces

by

G. M. Reed (Athens, Ohio)

Abstract. It is the purpose of this paper to provide conditions under which certain first countable T_2-spaces (in particular, Moore spaces) of cardinality $\leqslant \mathfrak{c}$ can be embedded in spaces of the same type which are also separable. Related results deal with pseudo-compactness and point-countable separating open covers.

In this paper the author considers the following questions: (1) Can each first countable T_2-space (Moore space) of cardinality $\leqslant \mathfrak{c}$ be embedded in a separable first countable T_2-space (Moore space)? (2) Can each locally compact Moore space of cardinality $\leqslant \mathfrak{c}$ be embedded in a separable Moore space? (3) Can each locally compact, separable Moore space be embedded in a pseudo-compact Moore space? (4) Does each Moore space with the DCCC have cardinality $\leqslant \mathfrak{c}$? (5) Does each Moore space have a point-countable separating open cover?

In Section 1, significant partial answers of a positive nature are given to Question (1) from which it follows that the most obvious candidates for counter examples (i.e., certain CCC, nonseparable spaces, in particular Souslin spaces) will not suffice. In Section 2, Questions (2) and (3) are given positive answers. The answer to Question (2) is, however, obtained under the assumption of the continuum hypothesis, In Section 3, Question (4) is answered in the negative and significant progress is made on Question (5).

(i) Motivation. During Professor Steve Armentrout's talk at the 1967 Arizona State University Topology Conference, the question was raised as to whether there exists a separable, noncompletable Moore space. In 1970, J. Ott in [23] under the assumption of the continuum hypothesis, embedded a non-completable Moore space due to M. E. Rudin in a separable Moore space. Furthermore, Ott obtained the rather remarkable result that there exists a complete separable Moore space which contains a copy of every metric space of cardinality $\leqslant \mathfrak{c}$. And, in 1972, the author in [29] constructed a noncompletable Moore space which could be embedded in a separable Moore space without any set-theoretic assumptions other than the axiom of choice. Thus, the original question was answered completely. However, in attempting to answer this question, Ott raised the seemingly more difficult question as to whether each Moore space of cardinality $\leqslant \mathfrak{c}$ can be embedded in a separable Moore space. Recently, the latter question has been raised again by Professor Ben Fitzpatrick in his invited hour address at the November, 1973 meeting of the American Mathematical Society in Atlanta, Georgia.

Also, in [8] and [9], J. W. Green has defined a Moore space to be Moore-closed provided that it is a closed subset of each Moore space in which it is embedded. Green has shown that each Moore-closed space is complete. And, in [30], the author has shown that (1) each Moore-closed space is separable and (2) for completely regular Moore spaces, Moore-closure is equivalent to pseudo-compactness. Thus, the following question arose in [30] which relates both to Ott's question and to the considerable literature on completing Moore spaces: Can each completable Moore space of cardinality $\leqslant \mathfrak{c}$ be embedded in a Moore-closed space? Or, perhaps the more approachable question: Can each locally compact Moore space of cardinality $\leqslant \mathfrak{c}$ be embedded in a pseudo-compact Moore space?

Finally, due to the cardinality considerations in the above questions, the question arises as to the cardinality of Moore spaces which have the DCCC. A space has the DCCC (discrete countable chain condition) provided each discrete collection of mutually exclusive open sets is countable. Moore spaces which have the DCCC but not the CCC are given in [28] and [30]. It is well known (see [17]) that first countable T_2-spaces with the CCC must have cardinality $\leqslant \mathfrak{c}$.

(ii) Preliminaries.

Moore spaces. A development for a space S is a sequence $G_1, G_2, \ldots$ of open covers of S such that for each i, $G_{i+1} \subset G_i$ and for each point p in S and each open set D containing p, there exists an n such that each element of G_n containing p is contained in D. A regular T_2-space which has a development is a More space. A complete Moore space is one which has a complete development, i.e., a development $G_1, G_2, \ldots$ such that if for each i, M_i is a closed subset of some element of G_i and M_i contains M_{i+1}, then $\bigcap M_i \neq \emptyset$. The source book for Moore spaces is [21]. Note from [5], completely regular Moore spaces have a complete development if and only if they are Čech complete.

DEFINITION 1. [4] A T_2-space S is (continuously) *semi-metrizable* provided it admits a (continuous) distance function d from $S\times S$ into the nonnegative real numbers such that (1) if each of x and y is a point of S, then $d(x,y) = d(y,x) \geqslant 0$, (2) $d(x,y) = 0$ if and only if $x = y$, and (3) the topology of S is invariant with respect to d.

DEFINITION 2 [2]. A T_2-space is *submetrizable* provided it admits a one-to-one continuous map onto a metric space.

DEFINITION 3 [11]. A sequence $G_1, G_2, \ldots$ of open covers of a T_2-space S has the three link property provided that if p and q are two points of S, then there exists an n such that no element of G_n intersects both $\mathrm{St}(x, G_n)$ and $\mathrm{St}(y, G_n)$.

DEFINITION 4 [39]. A T_2-space S has a *regular G_δ-diagonal* provided there exists a sequence $D_1, D_2, \ldots$ of open sets in $S\times S$ such that

$$\Delta S = \{(x,x) \mid x\in S\} = \bigcap D_i = \bigcap \bar{D}_i .$$

THEOREM 5 [39]. *A T_2-space S has a regular G_δ-diagonal if and only if it has a sequence $G_1, G_2, \ldots$ of open covers satisfying the three link property.*

THEOREM 6 ([4] and [37]). *A T_2-space which is either continuously semi-metrizable or submetrizable has a regular G_δ-diagonal.*

Notation. The positive integers will be denoted by N and the letters i, j, m, n will be used exclusively to denote elements of N. If M is a subset of the space S, then $\mathrm{CL}_S(M)$ will denote the closure of M in S. If H is a collection of sets, then H^* will denote $\bigcup \{h \mid h\in H\}$.

1. First countable T_2-spaces.

In [30], [31], [32], [33] and [34] the author has extensively developed a technique which associates a Moore space to each first countable T_2-space. Hence, the author's original approach to Ott's question was an attempt to decide if each first countable T_2-space of cardinality $\leqslant \mathfrak{c}$ could be embedded in a separable first countable T_2-space. Hopefully, a solution to this question would then produce the corresponding solution for Moore spaces. However, although it is not difficult to show that each first countable T_1-space can be embedded in a separable first countable T_1-space [24], the valuity of the corresponding result for first countable T_2-spaces of cardinality $\leqslant \mathfrak{c}$ is apparently not known. The only positive result concerning this particular question in the literature seems to be the well-known theorem due to Tychonoff that each completely regular T_2-space of weight $\leqslant \mathfrak{c}$ can be embedded in a compact, separable T_2-space ($I^{\mathfrak{c}}$). But, as first countability of subspaces of $I^{\mathfrak{c}}$ is a very difficult property to establish, perhaps a search for a counterexample might be a better attack. The obvious candidate, or so it would seem to the author, would be nonseparable, CCC first countable T_2-spaces, in particular, Souslin spaces. However, in this section the author establishes positive results which show that this is not the case.

THEOREM 1.1. *Each first countable (developable) T_2-space S with the countable chain condition and a regular G_δ-diagonal can be embedded in a first countable (developable) separable T_2-space.*

Proof. Suppose S is first countable. Since S has a regular G_δ-diagonal, by Theorem 5, S has a sequence of open covers satisfying the three link property. Hence, for each $p\in S$, denote by $g_1(p), g_2(p), \ldots$ a nonincreasing local base for p such that $G_1, G_2, \ldots$, where for each i, $G_i = \{g_j(p) \mid p\in S \text{ and } j\geqslant i\}$, satisfies the three link property. Furthermore, let H_1 denote a maximal collection of mutually exclusive elements of G_1 such that H_1^* is dense in S and for each $i>1$, let H_i denote a maximal collection of mutually exclusive elements of G_i which refines H_{i-1} such that H_i^* is dense in S. Also, for each i, let $H_i' = \{h\in H_i \mid h \text{ is not a singleton}\}$. Now, let $H = \bigcup_{i=1}^{\infty} H_i$ and $H' = \bigcup_{i=1}^{\infty} H'_i$. Finally, let $X = S \cup H'$ and define a base B for the desired topology on X as follows:

(1) if $x \in H'$, let $\{x\} \in B$; and

(2) if $x \in S$, then for each i, let $b_i(x) \in B$, where $b_i(x) = g_i(x) \cup \{h|\ h \in \bigcup_{j=i}^{\infty} H'_j$ and $g_i(x) \cap h \neq \emptyset\}$.

It is easily seen that X is a first countable T_2-space and that S is embedded in X. Also, from the construction, it follows that $K = H' \cup \{p \in S|\ \{p\} \in H\}$ is a countable dense subset of S, hence X is separable.

If S is developable, let $G_1, G_2, \ldots$ denote a development for S satisfying the three link property. For each $p \in S$, denote by $g_1(p), g_2(p), \ldots$ a nonincreasing local base for p such that for each i, $g_i(p) \in G_i$. Now, the space X obtained by the same construction as above is easily seen to be developable also.

Example 1.2. There exists a hereditarily Lindelöf, nonseparable first countable T_2-space which can be embedded in a separable first countable T_2-space.

Proof. Consider the following well-known space (X, T). Let X denote a subset of the real line with cardinality $= \aleph_1$ and let Ω denote a well-ordering of X in which each initial segment is countable. For each point p in X and for each $i \in N$, let $r_i(p)$ denote the open segment $(p-1/i, p+1/i)$ of the real line topology and let $g_i(p) = r_i(p) \cap \{x \in X|\ x$ is p or x follows p in $\Omega\}$. Then $B = \{g_j(p)|\ p \in X$ and $j \in N\}$ is a base for the desired topology T on X and X is easily seen to be a hereditarily Lindelöf, nonseparable first countable T_2-space. However $G_1, G_2, \ldots$, where for each i, $G_i = \{g_i(p)|\ p \in X\}$, is a sequence of open covers of S which has the three link property. Hence, by Theorem 5, X has a regular G_δ-diagonal and, by Theorem 1.1, X can be embedded in a separable, first countable T_2-space.

Example 1.3. There exists a CCC, nonseparable Moore space which can be embedded is a separable Moore space.

Proof. Let S denote the set of all finite subsets of the real line R. For each $x \in R$ and $n \in N$, let $h_n(x) = (x-1/n, x+1/n)$. For each $p \in S$ and $n \in N$, let $u_n(p) = \bigcup \{h_n(x)|\ x \in p\}$ and let $g_n(p) = \{q \in S|\ p \subset q$ and $q \subset u_n(p)\}$. In [26], C. Pixley and P. Roy show that $G_1, G_2, \ldots$, where for each n, $G_n = \{g_i(p)|\ p \in S$ and $i \geqslant n\}$ is a development for a CCC, nonseparable Moore space topology on S. In [18], S. Kenton showed that S is continuously semi-metrizable. Hence, by Theorem 6 and Theorem 1.1, S can be embedded in a separable, developable T_2-space X. The author does not know if the construction technique in the proof of Theorem 1.1 will, in general, preserve regularity. However, by considering a certain subspace S' of S, we obtain the desired result.

Let I denote the irrationals with the inherited topology of the real line and let S' denote the set of all finite subsets of I. For each $x \in I$ and $n \in N$, let $h_n(x)$ denote an open set in I such that $(x-1/n+1, x+1/n+1) \cap I$ is contained in $h_n(x)$, $h_n(x)$ is contained in $(x-1/n, x+1/n) \cap I$, and $h_n(x)$ has no boundary in I. For each

point $p \in S'$ and each $n \in N$, let $u_n(p) = \bigcup \{h_n(x)|\ x \in p\}$ and let $g_n(p) = \{q \in S'|\ p \subset q$ and $q \subset u_n(p)\}$. It now follows (see [27]) that $G_1, G_2, \ldots$, where for each n, $G_n = \{g_i(p)|\ p \in S'$ and $i \geqslant n\}$, is a development for a CCC, nonseparable Moore space topology on S'.

Claim. For each $p \in S'$ and $n \in N$, if $q \in S' - \mathrm{CL}(g_n(p))$, then there exists $m \geqslant n$ such that no element of G_m intersects both $g_n(p)$ and $g_m(q)$. To see that this is true, suppose $q \notin \mathrm{CL}(g_n(p))$. Then either (1) $q \not\subset u_n(p)$ or (2) $p \not\subset q$. If (1), then there exists $x \in q$ such that $x \notin \mathrm{CL}_I(u_n(p))$. Thus there exists $m \geqslant n$ such that if $r \in S'$ and $r \subset u_n(p)$, then $x \notin u_m(r)$. Now, suppose that there exists $t \in S'$ such that $g_m(t) \cap g_n(p) \neq \emptyset$ and $g_m(t) \cap g_m(q) \neq \emptyset$. Then, it follows that $t \subset u_n(p)$ and $q \subset u_m(t)$. But this is a contradiction. If (2), then there exists $x \in p$ such that $x \notin q$. Thus there exists $m \geqslant n$ such that if $r \in S'$ and $r \subset u_m(q)$, then $x \notin u_m(r)$. Now, suppose that there exists $t \in S'$ such that $g_m(t) \cap g_n(p) \neq \emptyset$ and $g_m(t) \cap g_m(q) \neq \emptyset$. Then, it follows that $t \subset u_m(q)$ and $p \subset u_m(t)$. But, again this is a contradiction. It is easily seen that ths "strengthened" version of the three link property produced by the above claim ie exactly what is needed in the construction technique of Theorem 1.1 to preserve regularity. The desired result now follows.

Remarks on Example 1.3. Note, it follows from the work of B. Fitzpatrick in [5] that S' is not completable. Hence, Example 1.3 yields perhaps the most simple example yet of a separable, noncompletable Moore space. However, the techniques used by the author in [33] will yield such examples that are easier to visualize.

Also, from the work of T. Przymusiński and F. Tall in [27], it follows that if we had based the space S' on a subset of the irrationals of card. $= \aleph_1$ then it is consistent that S' be normal. Hence, it is consistent that there exists a normal, CCC, nonseparable Moore space which can be embedded in a separable Moore space.

Finally, the author does not know if there exists a Moore space with a sequence of open covers satisfying the three link property but which has no such sequence satisfying the stronger version of the property needed to ensure regularity in Example 1.3. It would be helpful to know if continuously semi-metrizable or submetrizable Moore spaces have the "stronger" property.

Theorem 1.4. *Each submetrizable first countable (developable) T_2-space S of cardinality $\leqslant \mathfrak{c}$ can be embedded in a first countable (developable) separable T_2-space.*

Proof. Suppose that S is first countable. Let M denote a metric space and let f denote a one-to-one continuous map from S onto M. Now, M has cardinality $\leqslant \mathfrak{c}$, hence by [23] there exists a separable Moore space Y such that M is embedded in Y. Furthermore, without loss of generality, let us assume that $S \cap Y = \emptyset$, $Y = M \cup K$ where $K = \{k_1, k_2, \ldots\}$ is countable and dense in Y, and $M \cap K = \emptyset$. Now, denote by $H_1, H_2, \ldots$ a development for Y. For each $p \in S$, let $h_1(f(p)), h_2(f(p)) \ldots$ denote a nonincreasing local base for $f(p)$ in Y such that for each i, $h_i(f(p)) \in H_i$. Then, for each $p \in S$, let $g_1(p), g_2(p), \ldots$ denote a nonincreasing local base for p in S such that for each i, $\{f(q) \in M|\ q \in g_i(p)\} \subset h_i(f(p))$.

Finally, let $X = S \cup K$ and construct a base B for the desired topology on X as follows: (1) if $x \in K$, let $\{x\} \in B$; and (2) if $x \in S$, then for each i, let $b_i(x) \in B$, where $b_i = g_i(x) \cup \{k_j \in K \mid j \geqslant i \text{ and } k_j \in h_i(f(p))\}$. It is easily seen that X is a first countable separable T_2-space and that S is embedded in X.

If S is developable, let $G_1, G_2, \ldots$ denote a development for S. The same construction as above with the added restriction that for each $p \in S$ and each i, $g_i(p) \in G_i$ will now yield a space X that is also developable.

THEOREM 1.5. *Each normal Moore space of cardinality $\leqslant \mathfrak{c}$ can be embedded in a separable developable T_2-space.*

Proof. In [35], the author and Phil Zenor show that each normal Moore space of cardinality $\leqslant \mathfrak{c}$ is submetrizable. Hence, the desired result follows from Theorem 1.4.

In [25], I. I. Parovičenko showed that each compact T_2-space of weight $\leqslant \aleph_1$ is the continuous image of $\beta N - N$. In [20], K. D. Magill showed that if S is a T_2-space which is the continuous image of $\beta X - X$ for a locally compact T_2-space X, then there exists a T_2-compactification $\alpha(X)$ of X such that $\alpha(X) - X$ is homeomorphic to S. Based on these two results, S. P. Franklin and M. Rajagopalan observed in [7] the very useful Theorem 1.6 given below. Theorems 1.7, 1.8, and 1.9, are now obtained using the techniques which were established in [7].

THEOREM 1.6 [7]. *For each compact T_2-space S of weight $\leqslant \aleph_1$, there exists a T_2 compactification $\alpha(N)$ of N such that $\alpha(N) - N$ is homeomorphic to S.*

THEOREM 1.7. *Each compact first countable T_2-space S of weight $\leqslant \aleph_1$ can be embedded in compact, first countable separable T_2-space.*

Proof. By Theorem 1.6, there exists a T_2 compactification $Z(N)$ of N such that $Z(N) - N$ is homeomorphic to S. Hence, S is embedded in $Z(N)$, $Z(N)$ is a compact T_2-space, and N is an open, countable, dense subset of $Z(N)$. It remains only to show that $Z(N)$ is first countable. But each point p of $Z(N)$ is a G_δ-set in $Z(N)$ since S is first countable and N is open in $Z(N)$. Thus, since $Z(N)$ is locally compact, it follows that $Z(N)$ is first countable.

THEOREM 1.8. *Each locally compact, first countable T_2-space S of weight $\leqslant \aleph_1$ can be embedded in a locally compact, separable first countable T_2-space.*

Proof. Let $X = S \cup \{p\}$ be the one-point T_2-compactification of S. Then, by Theorem 1.6, there exists a T_2 compactification $Z(N)$ of N such that $Z(N) - N$ is homeomorphic to X. Thus, $Y = Z(N) - \{p'\}$, where p' is the element of $Z(N)$ identified with p, is a locally compact, separable T_2-space. Furthermore, S is embedded in Y, and, as in the proof of Theorem 1.7, Y is first countable.

DEFINITION. A Souslin space is a *nonseparable linearly ordered space* which satisfies the CCC. The existence of such space is known to be independent of the usual axioms of set theory.

THEOREM 1.9 (CH). *Each Souslin space S can be embedded in a compact, separable first countable T_2-space.*

Proof. Each Souslin space S can be embedded in a compact Souslin space [19]. And, since X is first countable and perfectly normal, X has cardinality $\leqslant \mathfrak{c}$ [1]. Hence, with the assumption of the continuum hypothesis, the desired result now follows from Theorem 1.7.

Remark. Each *connected* Souslin space S can be embedded in a compact, separable T_2-space without the assumption of the continuum hypothesis. Each such space S can be embedded in a compact, connected Souslin space X [19]. And, from [17], X has weight = density = $\aleph_1$.

Finally, Theorem 1.10 gives another approach to the embedding question. It follows from the work of J. Silver, F. Tall in [38], and T. Shinoda in [36] that, under the assumption of Martin's Axiom, each subset H of the x-axis with cardinality $\leqslant \mathfrak{c}$ has the property that each of its subsets is an F_σ-set in the relative topology. R. H. Bing has shown in [3] that the tangent disk topology defined on such a set H yields a normal, separable Moore space S in which H is a discrete subset.

THEOREM 1.10 (M. A.). *Each T_2-space X of cardinality $< \mathfrak{c}$ can be embedded in a separable T_2-space.*

Proof. If X has cardinality $= \mathfrak{m}$ where $\aleph_0 < \mathfrak{m} < \mathfrak{c}$, identify in a one-to-one fashion the points of X with the points of the subset H of a Moore space S as above. Let (Y, T) denote the space such that $Y = S$ and $u \in T$ if and only if u is open in the space S and $u \cap H$, if nonempty, is identified with an open set in X. It is easily seen that X is embedded in Y and that Y is separable. That Y is a T_2-space follows from the normality of S.

Remark. Thus far, the author has been unable to use the technique involved in Theorem 1.10 to embed first countable T_2-spaces of cardinality $< \mathfrak{c}$ in separable first countable T_2-spaces. However, such a result seems resonable for spaces X with cardinality $= \aleph_1 < \mathfrak{c}$.

Also, Teodor Przymusiński has pointed out to the author that, under the assumption of Martin's Axiom, each completely regular T_2-space of cardinality $< \mathfrak{c}$ can be embedded in $I^{\mathfrak{c}}$. This is true since, under Martin's Axiom, if $m < c$, then $2^{\mathfrak{m}} \leqslant 2^{\aleph_0}$. Hence, X has weight $\leqslant \mathfrak{c}$.

2. Locally compact and pseudo-compact Moore spaces.

THEOREM 2.1. *Each locally compact Moore space S of weight $\leqslant \aleph_1$ can be embedded in a locally compact, separable Moore space.*

Proof. By the proof of Theorem 1.8, there exists a locally compact first countable T_2-space Y such that $Y = S \cup N$ and N is open and dense in Y. It remains only to show that Y is a Moore space. However, it follows directly from the work of Ott in [24] that any such space is semi-metrizable. From [24], a semi-metric d for S can be defined as follows: Denote by $G_1, G_2, \ldots$ a development for S. For each point p in S, let $g_1(p), g_2(p), \ldots$ denote a nonincreasing local base for p in Y such that for each i, $g_i(p) \cap S \in G_i$ and $g_i(p) \cap \{1, 2, \ldots, i\} = \emptyset$. Also for each i, let $H_i = \{g_i(p)|\ p \in S\} \cup \{\{j\}|\ j \in N\}$. Now, define the function d from $Y \times Y$ into the nonnegative real numbers such that if each of p and q is in Y, then:

(1) $d(p, q) =$ g.l.b. $\{1/n|\ n = 1$ or there is an element of H_n which contains both p and $q\}$ if $\{p, q\} \subset S$ or $\{p, q\} \subset N$;

(2) $d(p, q) =$ g.l.b. $\{1/n|\ n = 1$ or $q \in g_n(p)\}$ if $p \in S$ and $q \in N$; or

(3) $d(p, q) =$ g.l.b. $\{1/n|\ n = 1$ or $p \in g_n(q)\}$ if $q \in S$ and $p \in N$.

Thus, Y is a locally compact semi-metrizable T_2-space. And, R. W. Heath shows in [10] that each such space is a Moore space. This completes the proof.

LEMMA 2.2. *Each locally compact separable Moore space S can be embedded in a locally compact separable Moore space X such that $X = S \cup K$ where K is countable and dense in X, $S \cap K = \emptyset$, and each point of K is isolated.*

Proof. Let $G_1, G_2, \ldots$ denote a development for S and for each $p \in S$, let $g_1(p), g_2(p), \ldots$ denote a nonincreasing local base for p such that for each i, $g_i(p) \in G_i$ and $\mathrm{CL}(g_i(p))$ is compact. Also, denote by $(k_1, k_2, \ldots\}$ a countable dense subset of S. Now, for each i and each $j \geqslant i$, let k_{ij} denote an object that is not in S such that if $(i_1, j_1) \neq (i_2, j_2)$ then $k_{i_1j_1} \neq k_{i_2j_2}$. Finally, let $K = \{k_{ij}|\ j \geqslant i\}$ and let $X = S \cup K$. Construct a base B for the desired topology on X as follows:

(1) if $x \in K$, then for each n, let $b_n(x) \in B$ where $b_n(x) = \{x\}$;

(2) if $x \in S$, then for each n, let $b_n(x) \in B$ where $b_n(x) = g_n(x) \cup \{k_{ij}|\ k_i \in g_n(x)$ and $j \geqslant \max\{i, n\}\}$.

It is easily seen that X is a regular first countable T_2-space in which S is embedded and that K is a countable subset of X with the required properties. Furthermore, $B_1, B_2, \ldots$, where for each n, $B_n = \{b_m(x)|\ x \in X$ and $m \geqslant n\}$, is a development for X.

To see that X is locally compact, recall that a closed subset H of a Moore space is compact if and only if it is countably compact [21]. Hence, it is sufficient to show that if $x \in X$, $n \in N$, and H is an infinite subset of $\mathrm{CL}(b_n(x))$, then H has a limit point.

Case 1. If $x \in K$, then $\mathrm{CL}(b_n(x)) = \{x\}$ and we are finished.

Case 2. If $x \in S$ and $H \cap S$ is infinite, then $H \cap S \subset \mathrm{CL}_S(g_n(x))$ and $H \cap S$ has a limit point in S.

Case 3. If $x \in S$ and $H \cap S$ is finite, then $H \cap K$ is infinite. But, by the construction of X, for each j, $\{i|\ k_{ij} \in H \cap K\}$ is at most finite. Thus, either (1) there exists an infinite subset T of $H \cap K$ and an $m \in N$ such that if $k_{ij} \in T$, then $i = m$, or (2) there exists an infinite subset T of $H \cap K$ such that if $k_{i_1j_1}$ and $k_{i_2j_2}$ are two elements of T, then $i_1 \neq i_2$. If (1), then k_m is a limit point of T and, hence, of H. If (2), then $S(T) = \{k_i|\ k_{ij} \in T$ for some $j \in N\}$ is an infinite subset of $\mathrm{CL}_S(g_n(x))$ and, therefore, has a limit point y in S. But y is also a limit point of T. For suppose that $m \in N$, then $b_m(y) \cap S(T)$ is infinite. Also, since $\{k_i|\ k_{ij} \in T$ and $j < m\}$ is finite, $b_m(y) \cap T$ must be infinite. Hence y is a limit point of H. This completes the proof.

LEMMA 2.3. *Each locally compact, separable Moore space X which is zero-dimensional at each point of a countable dense subset can be densely embedded in a locally compact, pseudo-compact Moore space Y.*

Proof. In [30], the author noted that for completely regular Moore spaces, pseudo-compactness is equivalent to Moore-closure. Also, in [30], it was observed that a Moore space is Moore-closed if and only if each discrete collection of mutually exclusive open sets in the space is finite. Hence, suppose X is not pseudo-compact, then it is not Moore-closed. Thus, there exists an infinite discrete collection of mutually exclusive open sets in X.

Let $K = \{p_1, p_2, \ldots\}$ denote a countable dense subset of X such that X is zero-dimensional at each point of K. For each i, let $r(p_i)$ denote an open set containing p_i such that $r(p_i)$ has no boundary in X and $r(p_i)$ is compact. Denote by Y_1 the collection of all sequences $p = p_{n1}, p_{n2}, \ldots$ of elements of K such that there exists a discrete collection $\{u_1(p), u_2(p), \ldots\}$ of open sets in X such that for each i, $p_{ni} \in u_i(p) \subset r(p_{ni})$ and $u_i(p)$ has no boundary, and for each $i > 1$, $u_i(p) \cap \cap (\bigcup_{j=1}^{i-1} r(p_j)) = \emptyset$.

Now, denote by Ω a well-ordering of Y_1. Define the subset Y_1' of Y_1 as follows:

(1) the first element of Y_1' is the first element p of Y_1 and for each i, let $h_i(p) = u_i(p)$;

(2) if I is an initial segment of Y_1', then the first element of $Y_1' - I$ is the first element $q = p_{m1}, p_{m2}, \ldots$ of $Y_1 - I$ such that there exist a discrete collection $\{h_1(q), h_2(q), \ldots\}$ of open sets in X such that for each i, $p_{mi} \in h_i(q) \subset u_i(q)$ and $h_i(q)$ has no boundary, and for each $t = p_{n1}, p_{n2}, \ldots$ in I,

$$\{\{h_j(t), h_i(q)\}|\ j \in N,\ i \in N \text{ and } h_j(t) \cap h_i(q) \neq \emptyset\}$$

is finite; and

(3) if Y_1' is a subset of Y_1 satisfying (1) and (2) then either Y_1'' is Y_1' or Y_1'' is an initial segment of Y_1'.

Finally, let $G_1, G_2, \ldots$ denote a development for X such that for each i, if $g \in G_1$, then $\mathrm{CL}(g)$ is compact. Also, for each $p \in Y_1'$ and for each i, let $g_i(p) = \{p\} \cup (\bigcup \{h_j(p)|\ j \geqslant i\})$. Then, let $Y = Y_1' \cup X$. It now follows that for each n, $H_n = G_n \cup \{g_i(p)|\ p \in Y_1'$ and $i \geqslant n\}$, is a base for the desired topology on Y. Furthermore, $H_1, H_2, \ldots$ is a development for the locally compact, pseudo-compact Moore space Y.

To verify that Y is a locally compact Moore space, consider the following:

Claim 1. If $h \in G_n$ for some n, then $\mathrm{CL}_Y(h) \subset X$. For, if there were to exist $p \in Y_1'$ such that $p \in \mathrm{CL}_Y(h)$, then $\{h_j(p)|\ j \in N$ and $h_j(p) \cap h \neq \emptyset\}$ would be infinite. But, since $\mathrm{CL}_X(h)$ is compact and $\{h_j(p)|\ j \in N\}$ is discrete in X, this would involve a contradiction.

Claim 2. If $x \in X$, then there exists an n such that if $x \in h \in H_n$, then $\mathrm{CL}_Y(h) \subset X$. For, if $x \notin r(p_j)$ for $j \in N$, then $x \notin g_i(p)$ for $p \in Y_1'$ and $i \in N$. And if $x \in r(p_j)$ for some $j \in N$, then $x \notin g_i(p)$ for $p \notin Y_1'$ and $i \geqslant j$. Hence, there exists an n such that if $p \in h \in H_n$, then $h \in G_n$ and $\mathrm{CL}_Y(h) \subset X$.

Claim 3. If $p \in Y_1'$ and for some m, $p \in h \in H_m$, then $h = g_n(p)$ for some $n \geqslant m$ and $g_n(p)$ is compact. That $h = g_n(p)$ for some $n \geqslant m$ is obvious. Also, by the construction of Y_1', it follows that $g_n(p)$ is closed in Y. Hence, since each infinite subset of $g_n(p)$ has a limit point, $g_n(p)$ is compact.

To verify that Y is pseudo-compact, suppose that Y is not Moore-closed. Then there exists an infinite discrete collection $\{r_1, r_2, \ldots\}$ of mutually exclusive open sets in Y such that for each i, $\mathrm{CL}_Y(r_i)$ is compact. Since K is dense in Y, for each i, let $p_{mi} \in K \cap r_i$. Let $T = \{p_{m1}, p_{m2}, \ldots\}$. Then there exists a sequence $p = p_{n1}, p_{n2}, \ldots$ of elements of T such that $p \in Y_1$. But, either $p \in Y_1'$, or there exists $q \in Y_1'$ such that $\{\{r_k, h_j(q)\} |\ i \in N, j \in N, p_{ni} = p_{mk}, \text{ and } r_k \cap h_j(q) \neq \emptyset\}$ is infinite. In either case, $\{r_1, r_2, \ldots\}$ fails to be discrete. Thus, Y is Moore-closed and completely regular, and therefore, is also pseudo-compact.

THEOREM 2.4. *Each locally compact separable Moore space can be embedded in a locally compact pseudo-compact Moore space.*

Proof. The theorem follows immediately from Lemma 2.2 and Lemma 2.3.

Remark. The proof of Lemma 2.3 could be made much easier by requiring that X have a countable dense subset each point of which is isolated. This weakened version would still give Theorem 2.4. However, the author does not know if each locally compact separable Moore space can be *densely* embedded in a locally compact, pseudo-compact Moore space. Hence, as Lemma 2.3 in its stated form represents the best result on this question, the complication seems warranted.

THEOREM 2.5. *Each locally compact Moore space of weight $\leqslant \aleph_1$ can be embedded in a locally compact, pseudo-compact Moore space.*

Proof. The theorem follows from Theorem 2.1 and Theorem 2.4.

THEOREM 2.6 (CH). *Each locally compact Moore space of cardinality $\leqslant \mathfrak{c}$ can be embedded in a locally compact, pseudo-compact Moore space.*

3. The DCCC and separating open covers in Moore spaces.

THEOREM 3.1. *For each cardinal $\alpha \geqslant \aleph_0$, there exists a* DCCC, *locally separable Moore space of cardinality $\alpha^{\aleph_0}$.*

Proof. Suppose α is a cardinal $\geqslant \aleph_0$. Let X denote a discrete space of cardinality α. Denote by F the set of all countably infinite subsets of X. Now, denote by F' a maximal subcollection of F such that each two members of F' have at most finitely many elements in common. Note that since there exist first countable T_2-spaces of cardinality $\alpha^{\aleph_0}$ which have dense subspaces of cardinality α (for example $X^{\aleph_0}$), F' can be so chosen as to have cardinality $\alpha^{\aleph_0}$. Thus, let $Y = X \cup F'$ and define a base for the topology on Y as follows:

(1) if $p \in X$, let $\{p\} \in B$; and

(2) if $p = \{x_1, x_2, \ldots\} \in F'$ and $i \in N$, let $g_i(p) = \{p\} \cup \{x_j |\ x_j \in p$ and $j \geqslant i\} \in B$.

It is easily seen that Y is a regular, locally separable, first countable T_2-space of cardinality $\alpha^{\aleph_0}$. Also, X is a dense subset of Y such that each infinite subset of X has a limit point in Y. Hence, there does not exist an infinite discrete collection of mutually exclusive open sets in Y. Finally, the author has described a technique in [30], [31], and [32] which associates to each regular, locally separable DCCC, first countable T_2-space such a space that is, in addition, a Moore space. Furthermore, the Moore space so associated to Y will also have cardinality $\alpha^{\aleph_0}$. This completes the proof.

DEFINITION. The statement that an open cover H of the space S is a *separating open cover* of S means that if p and q are two points of S then there is an element of H which contains p but not q.

In [22], J. Nagata introduced the concept of point-countable separating open covers, i.e., a separating open cover in which each point is contained in at most countably many elements of the cover. This concept has proved to be extremely useful in the theory of metrization ([12], [13], [14], [15], and [16] for example). It follows from this work that if each Moore space could be shown to have a point-countable separating open cover, then several well known metrization theorems could be given a unified structure. Toward this goal, R. E. Hodel, during a small topology conference held at the University of Pittsburgh in December, 1973, raised the question to the author as to whether each Moore space does have such a cover. Noting that each Moore space which can be embedded in a separable Moore space trivially has a *countable* separating open cover, the author, being aware of the results in Sections 1 and 2 of this paper, was led to suspect that each Moore space of cardinality $\leqslant \mathfrak{c}$ might have such a cover. Indeed, this is the case and this somewhat surprising result is used by the author and Phil Zenor in [35] to show that each locally compact, locally connected, normal Moore space is metrizable. However, the question still remains as to whether each Moore space of cardinality $> \mathfrak{c}$ has a point-countable separating open cover.

Since the usual analogy to a countable structure for Moore spaces is σ-discrete, the fact that each Moore space of cardinality $\leqslant \mathfrak{c}$ has a countable separating open cover suggests that each Moore space might, in fact, have a σ-discrete separating open cover. However, Theorem 3.2 shows that this is not the case.

THEOREM 3.2. *For each cardinal $\alpha > \mathfrak{c}$, there exists a* DCCC, *locally separable Moore space of cardinality $\alpha^{\aleph_0}$ which does not have a σ-discrete separating open cover.*

Proof. Suppose α is a cardinal $> \mathfrak{c}$. Consider the DCCC, locally separable Moore space Y of cardinality $\alpha^{\aleph_0}$ provided by Theorem 3.1. Since Y has the DCCC, a σ-discrete separating open cover of Y would be countable. Hence, suppose that Y has a countable separating open cover H. For each point p in Y, let $f(p) = \{h \in H |\ p \in Y\}$. If p and q are two points of Y, then $f(p) \neq f(q)$. Thus f is a one-to-one function from Y into the set of all subsets of a countable set. But Y has cardinality $> \mathfrak{c}$ and this is a contradiction from which the theorem follows.

Remark. The author conjectures that the Moore space provided by Theorem 3.1 where $\alpha = \sup\{2^{\mathfrak{c}}, 2^{2^{\mathfrak{c}}}, \ldots\}$ will not have a point countable separating open cover.

4. Questions.

(1) Can each first countable T_2-space of cardinality $\leqslant \mathfrak{c}$ be embedded in a separable first countable T_2-space?

(2) Can each Moore space of cardinality $\leqslant \mathfrak{c}$ be embedded in a separable Moore space?

(3) If a Moore space can be embedded in a separable developable T_2-space, can it also be embedded in a separable Moore space?

(4) If local compactness is added to the hypothesis of (1) and (2), can positive answers be obtained without the assumption of the continuum hypothesis?

(5) Does there exist a universal separable Moore space?

(6) Is it consistent that each normal Moore space of cardinality $\leqslant \aleph_1$ can be embedded in a separable, normal Moore space?

(7) Can each locally compact, separable Moore space of cardinality $\leqslant \mathfrak{c}$ be densely embedded in a pseudo-compact Moore space?

(8) Can each completable, separable Moore space of cardinality $\leqslant \mathfrak{c}$ be embedded in a More-closed space?

(9) Can each metric space of cardinality $\leqslant \mathfrak{c}$ be embedded in a Moore-closed (pseudo-compact) Moore space?

(10) Does each Moore space have a point-countable separating open cover?

Acknowledgment. The author would like to thank Teodor Przymusiński for several helpful remarks during the final revision of this paper. This paper was written while the author was visiting the Institute of Mathematics of the Polish Academy of Sciences in Warsaw under exchange program between the U. S. and Polish Academies.

Added in proof. The author now has a submetrizable counterexample to the question raised in the remarks on Example 1.3. In addition: the answer to Question (2) is independent of the usual axioms of set theory (E. van Douwen and T. Przymusiński); the answer to Question (9) is in the affirmative (E. van Douwen and G. M. Reed); and the answer to Question (10) in the negative (D. K. Burke).

References

[1] P. Alexandroff and P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam 14 (1929), pp. 1–96.

[2] B. A. Anderson, *Topologies comparable to metric topologies*, Proc. 1967 Arizona State Univ. Top. Conf. (Tempe, Arizona 1968), pp. 15–21.

[3] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), pp. 175–186.

[4] H. Cook, *Cartesian products and continuous semi-metrics*, Proc. 1967 Arizona State Univ. Top. Conf. (Tempe, Arizona 1968), pp. 58–63.

[5] G. D. Creede, *Embedding of complete Moore spaces*, Proc. Amer. Math. Soc. 28 (1971), pp. 609–612.

[6] B. Fitzpatrick, *On dense subspaces of Moore spaces*, Proc. Amer. Math. Soc. 16 (1965), pp. 1324–1328.

[7] S. P. Franklin and M. Rajagopalan, *Some examples in topology*, Trans. Amer. Math. Soc. 155 (1971), pp. 305–314.

[8] J. W. Green, *Moore-closure and completeness*, to appear.

[9] J. W. Green, *Completion and semicompletion of Moore spaces*, to appear.

[10] R. W. Heath, *Arc-wise connectedness in semi-metric spaces*, Pacific J. Math. 12 (1962), pp. 1301–1319.

[11] — *Metrizability, compactness and paracompactness in Moore spaces* (Abstract), Notices Amer. Math. Soc. 10 (1963), p. 105.

[12] R. E. Hodel, *Some results in metrization theory*, pp. 1950–1972, Proc. 1973 V. P. I. Top. Conf., to appear.

[13] — *Extensions of metrization theorems to higher cardinality*, Fund. Math. 87 (1975), pp. 219–229.

[14] — *On the weight of a topological space*, Proc. Amer. Math. Soc., to appear.

[15] — *Metrizability of topological spaces*, to appear.

[16] — *On a theorem of Arhangel'skiĭ concerning Lindelöf p-spaces*, to appear.

[17] I. Juhász, *Cardinal functions in topology*, Mathematical Centre (Amsterdam 1971).

[18] S. Kenton, *Cauchy completions of neo-metric spaces*, to appear.

[19] D. J. Lutzer and H. R. Bennett, *Separability, the countable chain condition and the Lindelöf property in linearly orderable spaces*, Proc. Amer. Math. Soc. 23 (1969), pp. 664–667.

[20] K. D. Magill, *A note on compactifications*, Math. Z. 94 (1966), pp. 322–325.

[21] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ. 13 (Revised Edition, Providence, R. I., 1962).

[22] J. Nagata, *A note on Filipov's theorem*, Proc. Japan Akad. 45 (1969), pp. 30–33.

[23] J. W. Ott, *Subsets of separable spaces*, Proc. 1969. Auburn Univ. Top. Conf. (Auburn, Alabama 1969).

[24] — *Subsets of separable spaces*, Doctoral Dissertation (Auburn University 1969).

[25] I. I. Parovičenko, *On a universal biocompactum of weight* $\aleph_1$, Sov. Math. Dokl. 4 (1963), pp. 592–595.

[26] C. Pixley and P. Roy, *Uncompletable Moore spaces*, Proc. 1969 Auburn Univ. Top. Conf. (Auburn, Alabama 1969), pp. 75–85.

[27] T. Przymusiński and F. D. Tall, *The undecidability of the existence of a non-separable Moore space satisfying the countable chain condition*, Fund. Math. 85 (1974), pp. 291–297.

[28] G. M. Reed, *Concerning normality, metrizability and the Souslin property in subspaces of Moore spaces*, Gen. Top. Appl. 1 (1971), pp. 223–246.

[29] — *Concerning completable Moore spaces*, Proc. Amer. Math. Soc. 36 (1972), pp. 591–596.

[30] — *On chain conditions in Moore spaces*, Gen. Top. Appl., to appear.

[31] — *On continuous images of Moore spaces*, Canad. J. Math., to appear.

[32] — *On the existence of point countable bases in Moore spaces*, Proc. Amer. Math. Soc., to appear.

[33] — *Concerning first countable spaces*, III, Trans. Amer. Math. Soc., to appear.

[34] — *On countable paracompactness and normality*, to appear.

[35] — and P. L. Zenor, *Metrization of Moore spaces and generalized manifolds*, Fund. Math., this volume, pp. 203–210.

[36] T. Shinoda, *Some consequences of Martin's Axiom and the negation of the continuum hypothesis*, Nagoya Math. J. 49 (1973), 117–125.

[37] F. Slaughter, *Submetrizable spaces*, Proc. 1973 V. P. I. Top. Conf., to appear.

[38] F. Tall, *Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems*, Doctoral Dissertation (University of Wisconsin 1969).

[39] P. L. Zenor, *On spaces with regular G_δ-diagonals*, Pacific J. Math. 40 (1972), pp. 759–763.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

Accepté par la Rédaction le 17. 6. 1974

Metrization of Moore spaces and generalized manifolds

by

G. M. Reed (Athens, O.) and **P. L. Zenor** (Auburn, Ala.)

Abstract. Through the investigation of new mapping conditions, the authors are able to establish metrization theorems for certain locally compact, locally connected spaces. In particular, it is shown that: (1) each normal locally compact, locally connected Moore space is metrizable; and (2) each perfectly normal, subparacompact generalized manifold is metrizable.

The authors would like to dedicate these results to their teachers, Ben Fitzpatrick and D. R. Traylor.

1. Introduction. In this paper, the authors introduce new mapping conditions and investigate spaces which are the preimages of metric spaces under maps satisfying these conditions. As a consequence of this investigation, significant progress is made on two long outstanding questions in general topology concerning the metrization of locally compact, locally connected spaces.

In 1937, F. B. Jones showed in [13] that under the assumption of the continuum hypothesis each normal, separable Moore space (1) is metrizable. Since that time, Jones' "normal Moore space conjecture", i.e., the conjecture that each normal Moore space is metrizable, has been one of the most tantalizing open questions in general topology. Furthermore, other than R. H. Bing's result of 1951 in [6] that each collectionwise normal Moore space is metrizable, the only positive results on this particular problem have depended on various set theoretic assumptions. In fact, the work of Bing in [6] and [7], R. W. Heath in [12], J. H. Silver, and F. D. Tall in [24] and [25] has shown that the metrizability of normal, separable Moore spaces, as well as several related conjectures, are actually independent of set theory.

In Section 3, a positive result concerning the metrization of normal, locally compact Moore spaces is given which requires no set theoretic assumptions. B. Fitzpatrick and D. R. Traylor showed in [10] that if there exists a normal, separable, nonmetrizable Moore space, then there exists one that is also locally compact. Also, W. G. Fleissner has recently shown in [11] that it is consistent that each normal, locally compact Moore space be metrizable. Thus, it is now known that

(1) A Moore space is a developable T_3-space.