Evidently, an arcwise continuum Y consisting of a compact subarc of A and finitely many A_i is a retract of X (i.e. Y consists of finitely many "$\sin(1/x)$ circles" each intersecting a common subarc of A in an arc which contains their respective limit intervals). The $\sin(1/x)$ circle has the fixed point property [1, p. 123] and a tedious but elementary argument can be used to show that Y has the fixed point property. If D is a dendrite in X such that $D \cap Y$ consists of a single point, then $D \cup Y$ must have the fixed point property [1, p. 121]. Hence retracts of X which are obtained from Y in this manner also have the fixed point property and this completes (iii).

PROBLEM 1. The following question posed in [4] still remains open. Namely, can a planar example be found?

PROBLEM 2. In [3] J. M. Łysko gives an example of a contractible continuum of dimension 3 which does not have the fixed point property for homeomorphisms. Does there exist a simply connected 1-dimensional continuum which does not have the fixed point property with respect to homeomorphisms?

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Rings in which every proper right ideal is maximal

by

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Abstract. We study the structure of rings in which every proper right ideal is maximal. We generalize some results of Perticani for non-commutative rings.

Recently, Perticani [2] has studied the structure of commutative rings with a unit element in which every proper ideal is maximal. In this paper we shall follow his line to discuss some generalizations for non-commutative rings.

A right ideal (or an ideal) of a ring R is said to be *proper* if it is different from (0) and R. Throughout this paper R will denote a ring (not necessarily commutative) with $R^2 = R \neq (0)$ in which every proper right ideal is maximal. We shall prove that R must be one and only one of the following types:

(1) R is a division ring;
(2) R is isomorphic to a 2×2 matrix ring over a division ring;
(3) R is isomorphic to the direct sum of two division rings;
(4) R is a left pseudo field over a division ring in the sense of Thierrin [3];
(5) R is a right pseudo field over a Galois field $GF(p)$ in the sense of Thierrin;
(6) R is a local ring (i.e., with unit and unique maximal ideal I) such that R/I is a division ring and $I^2 = (0)$.

Finally we shall show that in a ring A with $A^2 = A \neq (0)$, every proper right ideal is almost maximal if and only if every proper right ideal is maximal. Thus, this paper also provides a further classification for rings in which every proper right ideal is almost maximal given by Koh [1, Prop. 5.28].

We begin with

LEMMA 1. *R has at most two proper ideals.*

Proof. Suppose that I, J, K are distinct proper ideals in R. Then I, J, K are maximal right ideals and $I+J = R$. If $K \cap I \neq (0)$, then $K \cap I$ would be a maximal right ideal contained properly in K, a contradiction. Hence $K \cap I = (0)$ and $KI = (0)$. Similarly, $KJ = (0)$. It follows that $KR = K(I+J) \subseteq KI+KJ = (0)$ and $K \subseteq R^l$, the left annihilator of R. Since $R^l \neq R$ is an ideal containing K, $R^l = K$. Using a similar argument, we can show that $R^l = I$. This contradicts the fact that $I \neq K$.

THEOREM 1. *If R has no proper ideals, then either R is a division ring or R is isomorphic to a 2×2 matrix ring over a division ring.*

Proof. Clearly, R is right artinian. By the Wedderburn–Artin Theorem, R is isomorphic to an $n\times n$ matrix ring D_n over a division ring D. For $n\geqslant 3$, D_n does have proper right ideals which are not maximal. Hence n must be 1 or 2.

THEOREM 2. *If R has exactly two proper ideals, then R is isomorphic to a direct sum of two division rings.*

Proof. Let I and J be the two proper ideals in R. Clearly, $I+J = R$ and $I\cap J = (0)$, so $IJ = JI = (0)$. Consequently, $R = R^2 = (I+J)^2 = I^2+J^2$. Since $I^2 \neq (0)$ is an ideal contained in I, $I^2 = I$. We claim now that I is a division ring. To see this, it suffices to show that I considered as a ring has no proper right ideals. In fact, if I_0 is a non-zero right ideal in I, then since $I_0R = I_0(I+J) = I_0I\subseteq I_0$, I_0 is a right ideal in R and hence $I_0 = I$. Thus I is a division ring. Similarly, J is a division ring.

LEMMA 2. *If R has exactly one proper ideal, namely I, then $I^2 = (0)$ and R/I is a division ring.*

Proof. Consider the left annihilator I^l of I. Suppose $I^l = (0)$. Then, for any non-zero element $a\in I$, $aI \neq (0)$ is a right ideal contained in I and hence $aI = I$. This shows that the ring I has no proper right ideals and I is a division ring. Let e be the unit of I and let $\varphi\colon R\to I$ be the mapping defined by $\varphi(r) = re$ for every $r\in R$. It is easy to see that φ is a ring homomorphism of R onto I. Since $\varphi(e) = e\neq 0$ and $e\in I$, the kernel K of φ does not contain I. But I is the only proper ideal in R and $K\neq R$, so $K = (0)$. That is, φ is an isomorphism and $R\cong I$ is a division ring, a contradiction. Therefore, $I^l\neq (0)$. Since I^l is an ideal in R and every non-zero ideal contains I, $I^l\supseteq I$. Thus $I^2 = (0)$. Since R/I has no proper right ideals and $(R/I)^2 = R/I$, R/I is a division ring. This completes the proof.

Following Thierrin [3], a ring A is said to be *right bipotent* if $aA = a^2A$ for every $a\in A$. Let D be a division ring and $A = D\times D$, the Cartesian product of D and itself. Define addition and multiplication in A by the following way:

$$(a, b)+(c, d) = (a+c, b+d),$$

$$(a, b)(c, d) = (ac, ad).$$

Then A forms a right bipotent ring. A ring which is isomorphic to such a ring A is called a *right pseudo-field* over D. We note that

$$A\cong\left\{\begin{bmatrix} a & b\\ 0 & 0\end{bmatrix}\,\middle|\, a, b\in D\right\},$$

a subring of the 2×2 matrix ring over D.

We can define left bipotent rings and left pseudo-fields over a division ring by a similar manner and also we can see that a ring is a left pseudo-field over a division ring D if and only if it is isomorphic to the matrix ring

$$\left\{\begin{bmatrix} a & 0\\ b & 0\end{bmatrix}\,\middle|\, a, b\in D\right\}.$$

Thierrin has shown that a subdirect irreducible ring is right (left) bipotent if and only if it is a zero ring (i.e., its square is zero), a division ring, or isomorphic to a right (left) pseudo-field over a division ring.

THEOREM 3. *If R has exactly one proper ideal, namely I, and if $RI = (0)$, then R is isomorphic to a left pseudo-field over a division ring.*

Proof. We note first that R is subdirectly irreducible. In view of the Thierrin's result, we need only to show that R is a left bipotent ring. Let $a\in R$. If $a\in I$, then $Ra = Ra^2 = (0)$. Now assume $a\notin I$. By Lemma 2, $a^2\notin I$ and there is $b\in R$ such that $a-ba^2\in I$. Thus for any $r\in R$, $ra-rba^2 = r(a-ba^2)\in RI = (0)$ and hence $ra = rba^2\in Ra^2$. This shows that $Ra = Ra^2$ and R is left bipotent.

THEOREM 4. *If R has exactly one proper ideal, namely I, and if $IR = (0)$, then R is isomorphic to a right pseudo-field over a Galois field* GF(p).

Proof. Using a similar argument in the proof of Theorem 3, we can show that R is isomorphic to a right pseudo-field over a division ring D. We may assume that

$$R = \left\{\begin{bmatrix} a & b\\ 0 & 0\end{bmatrix}\,\middle|\, a, b\in D\right\}.$$

Then

$$I = \left\{\begin{bmatrix} 0 & b\\ 0 & 0\end{bmatrix}\,\middle|\, b\in D\right\}$$

is a proper ideal in R. For every additive subgroup D_0 of D,

$$I_0 = \left\{\begin{bmatrix} 0 & b\\ 0 & 0\end{bmatrix}\,\middle|\, b\in D_0\right\}\subseteq I$$

is a right ideal contained in I. Since R has no non-zero right ideals properly contained in I, the additive group D must be simple. Thus D must be a Galois field GF(p), where p is a prime number.

Remark. By considering the matrix ring $\left\{\begin{bmatrix} a & 0\\ b & 0\end{bmatrix}\,\middle|\, a, b\in D\right\}$, where D is a division ring, we can see easily that every left pseudo-field over a division ring satisfies the hypothesis in Theorem 3. Likewise every right pseudo-field over a Galois field GF(p) satisfies the hypothesis of Theorem 4.

THEOREM 5. *If R has exactly one proper ideal, namely I, and if $IR\neq (0)$, $RI\neq (0)$, then R has a unit element, I is the only proper right ideal in R, and $I\cong R/I$ as additive groups. Moreover, for $a\in I$, $\bar r = r+I\in R/I$ if we define $a\circ\bar r = ar$, then I is a one-dimensional right vector space over the division ring R/I.*

Proof. Let $r \in R \setminus I$. Since R/I is a division ring, $rR \nsubseteq I$. We claim first that $(rR)^2 \neq (0)$. For if $(rR)^2 = (0)$ then $(rR)^r$, the right annihilator of rR, is an ideal containing rR. This would imply that $(rR)^r = R$ and $rR = rR^2 = (0)$, a contradiction. Therefore, $(rR)^2$ is a non-zero right ideal contained in rR.

Next we shall show that $rR = R$. Suppose $rR \neq R$. Let $J = rR$. Then $J^2 = J$ and there exists $b \in J$ such that $bJ = J$. Thus there is $e \in J$ with $be = b$. It follows that $b(e-e^2) = 0$ and $e-e^2 \in \{b\}^r \cap J$. Since $\{b\}^r \cap J$ is a right ideal contained in J, $\{b\}^r \cap J = (0)$. Therefore, $e-e^2 = 0$, $eR = J$, and $R = eR+I$. Now, consider the set $S = \{ex-x \mid x \in I\}$. S is a right ideal contained in I. If $S \neq (0)$, then $S = I$ and $eI = eS = (0)$. It would follow that $e \in I^l = I$ and $J = eR \subseteq I$, a contradiction. Hence $S = (0)$ and $ex = x$ for every $x \in I$. Since $R = eR+I$, e is a left unit of R. It implies that $J = eR = R$, again a contradiction. Therefore $rR = R$ for every $r \in R \setminus I$, and I is the only proper right ideal in R.

Now we let $r \in R \setminus I$. Since $rR = R$, there exists $g \in R$ such that $rg = r$, so $r(g-g^2) = 0$ and $g-g^2 \in \{r\}^r$. We claim that $\{r\}^r = 0$. Suppose $\{r\}^r \neq 0$. If $\{r\}^r \subseteq I$ then $\{r\}^r = I$ and $r \in I^l = I$, a contradiction. Hence $\{r\}^r \nsubseteq I$ and $R = \{r\}^r + I$. This implies that $R = rR = r(\{r\}^r + I) = rI \subseteq I$, again a contradiction. Therefore $\{r\}^r = (0)$ and $g = g^2$.

Let $U = (gx-x \mid x \in R\}$. Since $rU = (0)$, $U \subseteq \{r\}^r$ and hence $U = (0)$. Thus $gx = x$ for every $x \in R$, i.e., g is a left unit in R.

Let $S = \{xg-x \mid x \in R\}$. Since g is a left unit, $SR = (0)$ and S is an ideal in R. Note that $g+I$ is the unit of the division ring R/I, so $xg-x \in I$ for every $x \in R$. Hence $S \subseteq I$. If $S \neq (0)$, then $S = I$ and $IR = SR = (0)$, a contradiction. Thus, $S = (0)$, i.e., $xg = x$ for every $x \in R$. Therefore g is a unit element of R.

For $a \in I$ and $\bar{r} = r+I \in R/I$, we define $a \circ \bar{r} = ar$. We can see readily that I is a right vector space over the division ring R/I. That I is one dimensional is obvious since every subspace of I is a right ideal contained in I. Since I and R/I both are one dimensional vector spaces over R/I, $I \cong R/I$ as additive groups.

From Lemma 2 and Theorem 5, we can see that if R has exactly one proper ideal I, and if $IR \neq (0)$, $RI \neq (0)$, then R has a unit element and the sequence

$$0 \to I \xrightarrow{j} R \xrightarrow{\pi} R/I \to 0$$

is exact, where j is the injection mapping, π is the canonical mapping, $I^2 = (0)$, R/I is a division ring and I is one dimensional right vector space over R/I (described as in Theorem 5).

By considering the converse of this result we have

THEOREM 6. *Let $0 \to I \xrightarrow{j} A \xrightarrow{\pi} D \to 0$ be an exact sequence of rings such that*

(1) $I^2 = (0)$,

(2) *D is a division ring,*

(3) *$j(I)$ is a one-dimensional right vector space over D by defining $j(a) \circ d = j(a)r$ for every $a \in I$ and $d \in D$, where r is an element in A with $\pi(r) = d$.*

Then A has only one proper right ideal, namely $j(I)$, and $j(I)A \neq (0)$. Moreover, if $Aj(I) \neq (0)$, then A has a unit element.

Proof. For convenience, we identify I and $j(I)$. By (2), I is a maximal left ideal in A and by (3) $IA \neq (0)$. Let $e \in A$ with $\pi(e) = 1$, the unit element in D. Then, for every $x \in I$, $x = x \circ 1 = x \circ \pi(e) = xe \in Ae$, so $I \subseteq Ae$. Moreover, since $\pi(e^2) = \pi(e)^2 = 1^2 = 1$, $e^2 \notin I$ and hence $Ae \neq I$. By the maximality of I, we have $Ae = A$. Now suppose there were a proper ideal J in A other than I. Then $I \cap J$ would be a subspace of the right vector space I over D and $I \cap J \neq I$. Since I is one-dimensional, $I \cap J = (0)$ and $A = I \oplus J$ as right A-modules. Let $e = f+g$ where $f \in I$, $g \in J$. Since fg, $gf \in I \cap J$, $fg = gf = 0$ and $e^2 = (f+g)^2 = f^2+g^2 = g^2$. Thus

$$f = f \circ 1 = f \circ \pi(e) = fe = f(f+g) = 0 .$$

Hence $e = g$. Since $e^2-e = g^2-g \in I \cap J = (0)$, $e^2 = e$. It would follow that $I = Ie = Ig \subseteq J$, a contradiction. Thus, I is the only proper ideal in A.

The last part of the theorem is an immediate consequence of Theorem 5.

Koh recently classified the rings A in which every proper right ideal is almost maximal. He proved that if $A^2 = A \neq (0)$ then A must be one of the following types:

(i) A is a division ring.

(ii) The Jacobson radical $J(A)$ is the only proper right ideal of A and $A/J(A)$ is a division ring.

(iii) A has unity element and A is a direct sum of two minimal right ideals (see [1, Prop. 5.28]).

We should note that, for an arbitrary ring, a maximal right ideal need not be almost maximal and *vice versa*. Now by examining all rings of type (i), (ii), or (iii), we can readily see that in each of these rings A every proper right ideal is maximal, and hence, particularly, each non-zero almost maximal right ideal is maximal. Moreover, if (0) is an almost maximal right ideal in A, then A is a division ring (see [1, Prop. 5.26]) and (0) is again a maximal right ideal in A. Hence, we have

THEOREM 7. *Let A be a ring with $A^2 = A \neq (0)$ such that every proper right ideal is almost maximal. Then a right ideal in A is almost maximal if and only if it is maximal.*

In conclusion, we shall prove the following.

THEOREM 8. *Let A be a ring with $A^2 = A \neq (0)$. Then the following two conditions are equivalent:*

(I) *Every proper right ideal in A is almost maximal;*

(II) *Every proper right ideal in A is maximal.*

Proof. That (I) implies (II) follows from Theorem 7.

To show that (II) implies (I), it suffices to show that any ring satisfying (II) is one of the types (i), (ii), and (iii). According to Lemma 1, A has at most two proper ideals.

Case 1. If A has no proper ideals, then, by Theorem 1, R is either a division ring or is isomorphic to a 2×2 matrix ring over a division ring. Thus A is of type (i) or (iii).

Case 2. If A has exactly one proper ideal, namely I, then by Lemma 2, $I^2 = (0)$ and R/I is a division ring. Thus I is the Jacobson radical of R. From Theorems 3, 4, and 5, we can see easily that I is the only proper right ideal in A, so A is of type (ii).

Case 3. If A has two proper ideals, then, by Theorem 2, A is isomorphic to a direct sum of two division rings, so A is of type (iii).

This completes the proof.

As we pointed out earlier, in view of Theorem 8 our results provide a further classification for the rings studied by Koh.

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On subspaces of separable first countable T_2-spaces

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Abstract. It is the purpose of this paper to provide conditions under which certain first countable T_2-spaces (in particular, Moore spaces) of cardinality $\leqslant\mathfrak{c}$ can be embedded in spaces of the same type which are also separable. Related results deal with pseudo-compactness and point-countable separating open covers.

In this paper the author considers the following questions: (1) Can each first countable T_2-space (Moore space) of cardinality $\leqslant\mathfrak{c}$ be embedded in a separable first countable T_2-space (Moore space)? (2) Can each locally compact Moore space of cardinality $\leqslant\mathfrak{c}$ be embedded in a separable Moore space? (3) Can each locally compact, separable Moore space be embedded in a pseudo-compact Moore space? (4) Does each Moore space with the DCCC have cardinality $\leqslant\mathfrak{c}$? (5) Does each Moore space have a point-countable separating open cover?

In Section 1, significant partial answers of a positive nature are given to Question (1) from which it follows that the most obvious candidates for counter examples (i.e., certain CCC, nonseparable spaces, in particular Souslin spaces) will not suffice. In Section 2, Questions (2) and (3) are given positive answers. The answer to Question (2) is, however, obtained under the assumption of the continuum hypothesis, In Section 3, Question (4) is answered in the negative and significant progress is made on Question (5).

(i) **Motivation.** During Professor Steve Armentrout's talk at the 1967 Arizona State University Topology Conference, the question was raised as to whether there exists a separable, noncompletable Moore space. In 1970, J. Ott in [23] under the assumption of the continuum hypothesis, embedded a non-completable Moore space due to M. E. Rudin in a separable Moore space. Furthermore, Ott obtained the rather remarkable result that there exists a complete separable Moore space which contains a copy of every metric space of cardinality $\leqslant\mathfrak{c}$. And, in 1972, the author in [29] constructed a noncompletable Moore space which could be embedded in a separable Moore space without any set-theoretic assumptions other than the axiom of choice. Thus, the original question was answered completely. However, in attempting to answer this question, Ott raised the seemingly more