# Representing real numbers in denumerable Boolean algebras

by

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**Abstract.** There is no set-theoretical definition which singles out a unique order type of an ordered basis for each isomorphism type of a denumerable Boolean algebra. The proof of this result is based on a theorem of Dale Myers concerning definability of selection functions for sets of real numbers. The coding method used in the proof is used also to establish a result concerning automorphisms of a denumerable Boolean algebra and the number of extensions of an atomless denumerable Boolean algebra.

Dale Myers in [5] has developed techniques for showing that various selection functions cannot be defined in set theory. We will use his method and one of his results to show that there is no definable way to choose an ordered basis for each denumerable Boolean algebra.

In the 1930's, Tarski and Mostowski studied various aspects of the theory of Boolean algebras. It was known that there were continuum many isomorphism types of denumerable Boolean algebras. A natural (although neccessarily vague) question arose as to how to characterize these isomorphism types by finding, for example, a set of isomorphism invarients. One of the tools developed was the notion of ordered basis (see [4]). Each denumerable Boolean algebra has an ordered basis and the order type of the ordered basis uniquely determines the isomorphism type of the Boolean algebra. However, many different order types may be associated with a given isomorphism type and attempts to place enough additional conditions on the order types (for example, along the lines suggested by Feiner, see Pierce [6]) to make them unique have failed. The result of this paper shows that no such conditions (definable in set theory) can succeed.

The specific result (Corollary 5 of Myers [5]) which we use is that there is no definable way to choose a real number from each denumerable set of real numbers. The method (similar to that used by Myers to obtain a number of other results) is to code each denumerable set of real numbers in a Boolean algebra in such a way that choosing an ordered basis for the Boolean algebra will enable us to choose a particular real number from the set.

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Although the entire proof could be carried out strictly in terms of Boolean algebras, it is more convenient to formulate the construction and proof in terms of topological spaces; the Stone spaces of the Boolean algebras. Corresponding to the ordered basis theorem for denumerable Boolean algebras is the theorem that every separable Boolean space can be linearly ordered in such a way that the topology matches the order topology. For the equivalence, see Mayer and Pierce [3], Theorem 2.5. Thus we will show that there is no definable way to choose such an ordering for every separable Boolean space.

DEFINITION 1. For each $n \in \omega$, let $C_n$ be the Boolean space formed by taking a Cantor set $C$ with a sequence of points of type $\omega^n$ approaching its left endpoint $c_n$.

The spaces $C_n$ were defined and used by Kinoshita in [2]. The corresponding Boolean algebras, which have ordered basis $\omega^n+\eta$, were used in Hanf [1]. Note that any neighborhood of the point $c_n$ contains a clopen neighborhood homeomorphic to $C_n$. Furthermore, $C_n$ is topologically distinguishable from $C_m$ for $m \neq n$ by the fact that the $n$th topological derivative of $C_n$ is a perfect set $C$. Thus any neighborhood of $c_n$ uniquely determines the integer $n$.

DEFINITION 2. For each real number $x$ $(0 \neq x \subseteq \omega)$, let $D_x$ be the one-point compactification, by a point $d_x$, of the disjoint union of denumerably many copies of each of the spaces $C_n$ where $n \in x$.

Note that each neighborhood of $d_x$ excludes only finitely many copies of each of a finite number of different $C_n$'s. Thus the neighborhood contains infinitely many points $c_n$ for each $n \in x$ but no point $c_m$ for $m \notin x$. Thus any neighborhood of $d_x$ uniquely determines the real number $x$.

THEOREM 3. *Given any denumerable set $X$ of real numbers, there exists a separable Boolean space $B(X)$ such that any linear ordering of $B(X)$ (which gives the topology on $B(X)$ as the order topology) determines a linear ordering of $X$ having a first or a last element.*

Proof. Let $B(X)$ be the one-point compactification, by a point $b$, of the disjoint union of spaces $D_x$ for $x \in X$. $B(X)$ is a separable Boolean space since each $C_n$ is and such spaces are closed to the operation of forming the one-point compactification of a denumerable disjoint union (this operation corresponds to the operation of weak direct product of the corresponding denumerable Boolean algebras or the operation of taking an ordinal sum (over $\omega$) of their ordered bases). $B(X)$ contains a point $d_x$ for each $x \in X$, so any ordering of $B(X)$ will order the points $d_x$. But the point $b$ is the only accumulation point of these points $d_x$. Thus, if there is no first point in the ordering of the points $d_x$, then $b$ must preceed all the $d_x$ in the ordering in order to be their limit. This means that there must be a last point $d_x$ in the ordering, for otherwise $b$ would also have to follow all the points in the ordering.

COROLLARY 4. *There is no definable way to pick a linear ordering for each separable Boolean space.*



Proof. If there were such a definition, then we could also pick a real number from every denumerable set of reals, contrary to Myers' theorem.

We now establish some cardinality results for denumerable Boolean algebras using this same method of representing sets of integers in the Boolean algebras. If $f$ is an automorphism of $\mathscr{L}$, we say that the element $b$ is an $n$th order fixed point of $f$ if $n$ is the least integer such that $f^n(b) = b$. The spectrum of $f$ is the set of all orders of fixed points of $f$. The following theorem shows that any denumerable Boolean algebra has continuum many structurally different automorphisms.

THEOREM 5. *If $\mathscr{L}$ is any denumerable Boolean algebra and $P$ is a set of prime numbers, then there is an automorphism $f$ of $\mathscr{L}$ such that $P$ is exactly the set of prime numbers in the spectrum of $f$.*

Proof. Consider the Stone space of $\mathscr{L}$ to be a subset of the real line (and hence ordered). If there are infinitely many isolated points, let $I_1, I_2, \ldots$ be either an increasing or decreasing sequence of such points. If there are only finitely many isolated points, then the remainder of the space is a Cantor set and we can take $I_1, I_2, \ldots$ to be an increasing disjoint sequence of clopen intervals each of which is homeomorphic to the Cantor set.

Now let $g$ be a permutation of the positive integers which has a cycle of length $p$ for each $p \in P$ and leaves all integers outside these cycles fixed. Let $h$ be the function which maps each $I_k$ homeomorphically onto $I_{g(k)}$ and leaves all other points of the space fixed. Since $I_1, I_2, \ldots$ has only one accumulation point (and it is left fixed by $h$), it is easily checked that $h$ is a homeomorphism and that $f$, the function determined by the action of $h$ on the clopen sets, is an automorphism of $\mathscr{L}$ with the desired properties.

If $\mathscr{L}$ is a subalgebra of $\mathscr{L}'$, then we define $E(\mathscr{L}, \mathscr{L}')$ to be the set of points $p$ in the Stone space of $\mathscr{L}$ whose neighborhood system is enriched by the extension, i.e. such that for every clopen neighborhood $N$ of $p$, the principal ideal Boolean algebra $\mathscr{L}'[N]$ contains elements not in $\mathscr{L}[N]$. Note that $E(\mathscr{L}, \mathscr{L}')$ is a closed subset of the Stone space of $\mathscr{L}$ and can be given the relative topology. The following theorem, which we state without proof, shows that there are continuum many different ways that the denumerable atomless Boolean algebra can be made an extension of itself.

THEOREM 6. *For any real number $x$, there exist denumerable atomless Boolean algebras $\mathscr{L}$ and $\mathscr{L}'$ such that $E(\mathscr{L}, \mathscr{L}')$ is homeomorphic to $D_x$. $\mathscr{L}'$ can be taken to be an extension of $\mathscr{L}$ by a single element which is a regular open set in the Stone space of $\mathscr{L}$.*

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# Extension of a valuation on a lattice

by

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**Abstract.** In a recent paper [2], Fox and Morales give necessary and sufficient conditions in order that a strongly additive (= valuation) set function from a lattice of sets $\mathfrak{L}$ into a complete metric group be uniquely extendable to the generated $(\sigma, \delta)$-lattice. It is shown in the present note that the same conditions are valid in a more general setting, i.e., when $L$ is an arbitrary lattice and $v$ is a valuation on $L$ with values in a sequentially complete Hausdorff topological group. The proof is accomplished by means of the elimination of Pettis' theorem ([3], Theorem 1.2), the basic lemma in the proof of Fox and Morales.

**1. Introduction.** Let $L$ be a lattice and $G$ an Abelian topological group. A function $v: L\to G$ is called a *valuation* [1], [3] if

$$v(x\vee y)+v(x\wedge y) = v(x)+v(y).$$

It is easy to show that ([1], p. 75) and ([4], p. 239) that if $L$ is a relatively complemented lattice (or, in particular, a Boolean ring), then (1.1) is equivalent to

$$v(x\vee y) = v(x)+v(y) \quad \text{for} \quad x\wedge y = 0.$$

We do not assume, however, that a null element belongs to $L$. A valuation $v$ on $L$ is said to be (*order*) $\sigma$-*continuous* ($\delta$-*continuous*) if, for every increasing (decreasing) sequence $(x_n)$ such that $x_n\in L$ $(n = 1, \ldots)$ with $\sup_n x_n\in L$ $(\inf_n x_n\in L)$, we have $v(x_n)\to v(\sup_n x_n)$ $(v(x_n)\to v(\inf_n x_n))$.

$v$ is $(\sigma, \delta)$-*continuous* if it is both $\sigma$-continuous and $\delta$-continuous.

A lattice $H$ is said to be $\sigma$-*continuous* if it is $(\sigma, \delta)$-complete (i.e. the limits of both increasing and decreasing countable sequences of elements of $H$ are in $H$) and the following condition holds: $y, x_n\in H$ $(n = 1, \ldots)$, $x_n\uparrow x \Rightarrow x_n\wedge y\uparrow x\wedge y$; and dually.

All lattices occurring in the present note are supposed to be contained in a fixed $\sigma$-continuous lattice $H$. The only lattice operations which will be considered are restrictions of those on $H$. Accordingly we shall use the word lattice to mean the subset of $H$ closed with respect to the restrictions of the lattice operations on $H$.

Let $v$: $L\to G$ be a $(\sigma, \delta)$-continuous valuation. The aim of this note is to establish necessary and sufficient conditions for the unique extension of $v$ to a con-