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Normality and paracompactness in subsets of product spaces

by

Teodor Przymusiński (Warszawa)

Abstract. Let M be a metric space and G an open subset of the product space $M\times Y$.

THEOREM 1. *If Y is hereditarily paracompact, then: G is normal $\Leftrightarrow$ G is paracompact $\Leftrightarrow$ G is countably paracompact.*

THEOREM 2. *If Y is hereditarily normal and M is dense-in-itself, then: G is normal $\Leftrightarrow$ G is countably paracompact.*

THEOREM 3. *If Y is hereditarily normal and hereditarily countably paracompact, then: G is normal $\Leftrightarrow$ G is countably paracompact.*

1. Introduction. Throughout this paper M is assumed to be a metric space and G is an open subset of the product space $M\times Y$.

The aim of this note is to study the relation between normality, paracompactness and countable paracompactness in open subsets of the product space $M\times Y$.

Results of Tamano [8], Morita [4] and Starbird and Rudin [7] imply the following facts:

A. *If Y is paracompact, then: $M\times Y$ is normal $\Leftrightarrow$ $M\times Y$ is paracompact $\Leftrightarrow$ $M\times Y$ is countably paracompact.*

B. *If Y is normal and M is non-discrete, then: $M\times Y$ is normal $\Leftrightarrow$ $M\times Y$ is countably paracompact.*

C. *If Y is normal and countably paracompact, then: $M\times Y$ is normal $\Leftrightarrow$ $M\times Y$ is countably paracompact.*

In this paper we prove analogous theorems for open subsets G of the product space $M\times Y$.

THEOREM 1. *If Y is hereditarily paracompact, then: G is normal $\Leftrightarrow$ G is paracompact $\Leftrightarrow$ G is countably paracompact.*

THEOREM 2. *If Y is hereditarily normal and M is dense-in-itself, then: G is normal $\Leftrightarrow$ G is countably paracompact.*

THEOREM 3. *If Y is hereditarily normal and hereditarily countably paracompact, then: G is normal $\Leftrightarrow$ G is countably paracompact.*

COROLLARY. *If Y is a generalized ordered space* [1], *then*: *G is normal* $\Leftrightarrow$ *G is countably paracompact.*

Modifying the proofs of Morita [4], Nagami proved in [5]

D. *Let Y be hereditarily normal (resp. hereditarily paracompact). If G is countably paracompact, then G is normal (resp. paracompact).*

In Section 2 we modify the proof of Starbird–Rudin's theorem ([8], Theorem 1) and obtain the following:

THEOREM 4. *Let M be dense-in-itself (or let Y be hereditarily countably paracompact). If G is normal, then G is countably paracompact.*

Theorems 1, 2 and 3 are immediate consequences of Theorem 4 and Nagami's D. In connection with Theorem 1, let us recall, that there exists a hereditarily paracompact (generalized ordered) space Y and a separable metric space M such that $M\times Y$ is not normal (Michael [3]). It is also consistent with the axioms of set theory to assume that there exists a hereditarily paracompact (generalized ordered) space X such that $Y\times Y$ is normal but not paracompact (Przymusiński [6]).

2. Generalization of Starbird–Rudin's theorem. In this section we shall prove Theorem 5, which is a generalization of Starbird–Rudin's theorem ([7], Theorem 1). Theorem 4 follows immediately from this result.

The proof of Theorem 5 is based on the idea used by Starbird and Rudin in [7], but its technical form has been essentially modified and complicated.

THEOREM 5. *Assume that for every isolated $x\in M$ the space*

$$G_x = \{y\in Y:\ (x,y)\in G\}$$

is countably paracompact. If G is normal, then G is countably paracompact.

Proof of Theorem 5. Let G be an open and normal subspace of $M\times Y$ and $\{F_n\}_{n\in\omega}$ a sequence of closed in G sets such that $\bigcap_{n\in\omega} F_n=\emptyset$ and $F_{n+1}\subset F_n$ for every $n\in\omega$. We have to prove that there exists a sequence $\{C_n\}_{n\in\omega}$ of closed in G sets satisfying $\bigcup_{n\in\omega} C_n = G$ and $C_n\cap F_n=\emptyset$ (cf. [1], Corollary 5.2.2).

Denote by M_0 the set of all non-isolated points of M and let $\mathcal{B} = \bigcup_{n\in\omega}\mathcal{B}_n$, where $\mathcal{B}_n$ is a locally finite open covering of M consisting of sets of diameter less than $1/2^n$. We may obviously assume that $\mathcal{B}'_n\cap\mathcal{B}'_m=\emptyset$, where $n\neq m$ and $\mathcal{B}'_n = \{B\in\mathcal{B}_n:\ B\cap M_0\neq\emptyset\}$. Let us choose from every $B\in\mathcal{B}' = \bigcup_{n\in\omega}\mathcal{B}'_n$ distinct points p_B and q_B in such a way that no point of M is selected twice (see [8], Lemma). The sets $P_n=\{p_B:\ B\in\mathcal{B}'_n\}$, $Q_n=\{q_B:\ B\in\mathcal{B}'_n\}$, $P=\bigcup_{n\in\omega}P_n$ and $Q=\bigcup_{n\in\omega}Q_n$ have the following properties:

[1] Subspaces of linearly ordered topological spaces are called *generalized ordered.* Any such space is hereditarily normal and hereditarily countably paracompact (cf. [2]).

(1) (i) if U is open in M and $U\cap M_0\neq\emptyset$, then there exist $p\in P$ and $q\in Q$ belonging to U,
(ii) P_n and Q_n are discrete and closed in M,
(iii) $P\cap Q=\emptyset$.

For every $n\in\omega$ and $B\in\mathcal{B}'$ define

$$U_{B,n} = \{y\in Y:\ \{x\in M:\ (x,y)\in G\}\cap B\cap P_n\neq\emptyset\},$$

$$V_{B,n} = \{y\in Y:\ \{x\in M:\ (x,y\in G\}\cap B\cap Q_n\neq\emptyset\},$$

$$D_{B,n} = (B\cap P_n)\times(Y\setminus\bigcup_{i=0}^{n-1}U_{B,i})\cap G,$$

$$E_{B,n} = (B\cap Q_n)\times(Y\setminus\bigcup_{i=0}^{n-1}V_{B,i})\cap G,$$

$$D_B = \bigcup_{n\in\omega}D_{B,n},\qquad E_B=\bigcup_{n\in\omega}E_{B,n}.$$

The sets $U_{B,n}$ and $V_{B,n}$ are obviously open in Y and consequently the sets $D_{B,n}$ and $E_{B,n}$ are closed in G.

We shall show that the sets D_B and E_B are also closed in G. Let $(x,y)\in G\setminus D_B$. If $x\in\overline{\bigcup_{n\in\omega}(B\cap P_n)}$, then there is an $n\in\omega$ and $p\in B\cap P_n$ such that

$$p\in\{x'\in M:\ (x',y)\in G\}.$$

It follows that $y\in U_{B,n}$, $(x,y)\in M\times U_{B,n}$ and for every $k>n$ the intersection $(M\times U_{B,n})\cap D_{B,k}$ is empty. Hence D_B is closed in G. In an analogous wav one can prove the closedness of E_B.

For every $B\in\mathcal{B}'$ there exists precisely one m such that $B\in\mathcal{B}'_m$. Let

$$A_B = \overline{\{y\in Y:\ ((B\cap M_0)\times\{y\})\cap F_m\neq\emptyset\}},$$

$$K_B = D_B\cap(X\times A_B),\qquad L_B=E_B\cap(X\times A_B),$$

$$K_n=\bigcup\{K_B:\ B\in\mathcal{B}'_n\},\quad L_n=\bigcup\{L_B:\ B\in\mathcal{B}'_n\},\quad K=\bigcup_{n\in\omega}K_n,\quad L=\bigcup_{n\in\omega}L_n.$$

For every $B\in\mathcal{B}'_m$ the following conditions are satisfied:

(2) (i) if $((B\cap M_0)\times\{y\})\cap F_m\neq\emptyset$, then there exist $p,q\in B$ such that $(p,y)\in K_B$ and $(q,y)\in L_B$,
(ii) $L_B\cup K_B\subset B\times A_B$,
(iii) the sets K and L are disjoint and closed in G.

Ad (i). There exists an $x\in B\cap M_0$ such that $(x,y)\in F_m\subset G$. The set $U = \{x'\in M:\ (x',y)\in G\}\cap B$ is open in M and $x\in U\cap M_0$. By (1) we have $U\cap P\neq\emptyset$ and $U\cap Q\neq\emptyset$. Let $n_0=\min\{n:\ U\cap P_n\neq\emptyset\}=\min\{n:\ y\in U_{B,n}\}$,

$k_0 = \min\{k: U \cap Q_k \neq \emptyset\} = \min\{k: y \in V_{B,k}\}$, $p \in U \cap P_{n_0}$ and $q \in U \cap Q_{k_0}$. Then $y \in A_B$, $(p, y) \in D_{B,n_0} \subset D_B$, $(q, y) \in E_{B,k_0} \subset E_B$ and consequently $(p, y) \in K_B$ and $(q, y) \in L_B$.

Ad (ii). Clear.

Ad (iii). The disjointness of K and L follows from (1). As the family $\mathscr{B}'_n$ is locally finite, the sets K_n and L_n are closed in G. Let $(x, y) \in G \setminus K$. There exist a $k \in \omega$ such that $(x, y) \notin F_k$, an open ball $B(x, 1/2^{n-1})$ in M and an open neighbourhood T of y in Y such that $n > k$ and $\big(B(x, 1/2^{n-1}) \times T\big) \cap F_k = \emptyset$. We shall prove that for every $m \geqslant n$ the intersection $\big(B(x, 1/2^n) \times T\big) \cap K_m = \emptyset$. By (ii) it suffices to prove that $\big(B(x, 1/2^n) \times T\big) \cap (B \times A_B) = \big(B(x, 1/2^n) \cap B\big) \times (T \cap A_B) = \emptyset$, for every $B \in \mathscr{B}'_m$. If $B(x, 1/2^n) \cap B \neq \emptyset$ then $B \subset B(x, 1/2^{n-1})$ and $(B \times T) \cap F_m \subset (B \times T) \cap F_k = \emptyset$ and consequently $T \cap A_B = \emptyset$. This shows that K is closed in G. In an analogous way one can prove the closedness of L.

By the normality of G let U and V be open subsets of G satisfying $K \subset U$, $L \subset V$ and $\overline{U}^G \cap \overline{V}^G = \emptyset$. For every $n \in \omega$ find a closed covering $\{S_B\}_{B \in \mathscr{B}'_n}$ of M_0 such that $S_B \subset B$ for every $B \in \mathscr{B}'_n$ (cf. [1], Theorem 1.5.18) and define $T_B = \{y \in Y: (B \times \{y\}) \cap U = \emptyset$ or $(B \times \{y\}) \cap V = \emptyset\}$ and $C'_n = \bigcup_{B \in \mathscr{B}'_n} (S_B \times T_B) \cap G$.

As the sets T_B are closed in Y and the family $\{S_B\}_{B \in \mathscr{B}'_n}$ is locally finite, we conclude that the sets C'_n are closed in G. Moreover, the following conditions are satisfied:

(3) (i) $C'_m \cap F_m = \emptyset$ for $m \in \omega$,

(ii) $\bigcup_{n \in \omega} C'_n = (M_0 \times Y) \cap G$.

Ad (i). If $(x, y) \in C'_m \cap F_m$, then there is a $B \in \mathscr{B}'_m$ such that $x \in S_B \subset B$ and $y \in T_B$, hence, for instance, $(B \times \{y\}) \cap U = \emptyset$. We infer from (2) and the inequality $\big((B \cap M_0) \times \{y\}\big) \cap F_m \neq \emptyset$, that there exists a $p \in B$ such that $(p, y) \in K_B \subset U$, which is a contradiction.

Ad (ii). If $(x, y) \in G$ and $x \in M_0$, then $(x, y) \notin \overline{U}$ or $(x, y) \notin \overline{V}$. Let, for instance, $(x, y) \notin \overline{U}$. There exists an $n \in \omega$, a $B \in \mathscr{B}'_n$ and an open neighbourhood T of $y \in Y$ such that $x \in S_B \subset B$ and $(B \times T) \cap \overline{U} = \emptyset$. It follows that $y \in T \subset T_B$ and consequently $(x, y) \in (S_B \times T_B) \cap G \subset C'_n$.

As M is metric, the open set $M \setminus M_0$ can be represented as a union $\bigcup_{n \in \omega} Z_n$ of closed subsets of M such that $Z_{n+1} \supset Z_n$ for $n \in \omega$. For every $x \in M \setminus M_0$ define $G_x = \{y \in Y: (x, y) \in G\}$ and $F_{x,n} = \{y \in Y: (x, y) \in F_n\}$. By the assumption, the space G_x is countably paracompact, so one can find closed subsets $C_{x,n}$ of G_x such that $F_{x,n} \cap C_{x,n} = \emptyset$, $\bigcup_{n \in \omega} C_{x,n} = G_x$ and $C_{x,n+1} \supset C_{x,n}$. The closed subset $C''_n = \bigcup_{x \in Z_n} (\{x\} \times C_{x,n})$ of G satisfies $C''_n \cap F_n = \emptyset$ for $n \in \omega$ and $\bigcup_{n \in \omega} C''_n = \big((M \setminus M_0) + Y\big) \cap G$. To complete the proof, it suffices to put $C_n = C'_n \cup C''_n$.

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