On three types of simplicial objects

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Abstract. Let $A, A_n, A_\alpha$ denote categories with the same set of objects consisting of sets $[n] = \{0, 1, \ldots, n\}$, for $n = 0, 1, \ldots$ and the sets of morphisms consisting of increasing, strictly increasing or all functions, respectively. We consider three categories of simplicial objects $(A^n_\bullet, A)$, $(A^\bullet, A)$ and $(A^\bullet_\bullet, A)$ of all contravariant functors with values in an Abelian category $A$ and we present some results concerning functors between these categories and the category of complexes over $A$. Moreover we study the homotopy of maps of simplicial objects of three types and prove several theorems on a preservation of the homotopy by some functors.

Let us denote by $A$ (resp., $A_n$, resp., $A_\alpha$) a category with the set of objects consisting of sets $[n] = \{0, 1, \ldots, n\}$, $n = 0, 1, \ldots$ and sets of maps $x: [m] \to [n]$ consisting of all weakly increasing functions (resp., all strictly increasing functions, resp., all functions). Thus we have $A_n \subset A \subset A_\alpha$. All contravariant functors defined on $A_n$, $A$ or $A_\alpha$ with values in a fixed category $M$ and with natural transformations as maps form categories $(A^n_\bullet, M)$, $(A^\bullet, M)$, $(A^\bullet_\bullet, M)$. Categories $(A^\bullet, M)$ and especially $(A^\bullet, \mathbf{Set})$ play an important role in algebra and topology. Categories $(A^n_\bullet, M)$ and $(A^\bullet_\bullet, M)$ are considered rather seldom (see [5], [7], [4]).

In the first part of this paper we present some results concerning functors (defined below) in the following diagram:

\[
\begin{array}{ccc}
(A^n_\bullet, A) & \xrightarrow{z_2} & (A^\bullet, A) \\
\downarrow{z_1} & \downarrow{\eta_2} & \downarrow{\eta_3} \\
(A_\alpha, A) & \xrightarrow{z_3} & (A^\bullet_\bullet, A)
\end{array}
\]

$z_1, z_2$ and $z: (A^n_\bullet, A) \to (A^\bullet_\bullet, A)$ are forgetful functors induced by inclusions $A_n \subset A$, $A \subset A$, $A_\alpha \subset A$; thus $z = z_1 z_2 z_3$. If a category $A$ has finite colimits, there exist left adjoint functors $^*z_1$, $^*z_2$, $^*z_3$ of functors $z_1, z_2, z$. For each integer $n \geq 0$ we have functions $e^n_1: [n+1] \to [n]$, $e^n_i = \eta^n_i: [n+1] \to [n]$, $i = 0, 1, \ldots, n$, defined as follows: $e^n_i(j) = j$ for $j < i$ (resp., $j+1$ for $j \geq i$), $\eta^n_i(j) = j$ for $j < i$ (resp., $j-1$ for $j > i$).

If $X$ is an object in $(A^n_\bullet, A)$, then we write $X_n = X([n])$, $d^i_n = d^i_n = X(e^n_i)$ (face operator), and if $X$ is in $(A^\bullet_\bullet, A)$, then we write $s_i = s_i = X(\eta^n_i)$ (degeneracy operator).

Let $A$ be an Abelian category, then we denote by $(\mathbf{Ch}, A)$ the category of all left
chain complexes over $A$. $N_{n}$, $N_{i}$ denote normalization functors which associate with an object $X$ a chain complex with the $n$th component equal to $\bigcap_{i \geq n} \operatorname{Ker}(d_{i}: X_{i} \to X_{i-1})$ and the $n$th differential is induced by $d_{n}: X_{n} \to X_{n-1}$. Functions $k_{a}, k_{b}$, $k_{c}$ associate with $X$ a chain complex with the $n$th component $X_{n}$ and with the $n$th differential equal to $\sum_{i=0}^{n-1} (-1)^{i}d_{i}$. Let $X$ be an object in $(A_{*}, A)$; then we denote by $D_{n}(X), n = 0, 1, \ldots$ a subobject $\sum_{x}^{y} \text{Im}(X_{n} \to \text{Im}(X_{n}))$ of $X_{n}$, where $\pi$ runs over $S_{n} = \text{Aut}_{B}(n)$ and $\gamma$ runs over all such maps $\gamma : [n] \to [p]$ in $D_{n}$ that $p - n$. The subobjects $D_{n}(X)$ determine a subcomplex $D(X)$ of $\mathcal{K}_{n}(X)$ and we denote by $N(X)$ the complex $\text{Coker}(D(X) \to \mathcal{K}_{n}(X))$ (see [4]). Let $C$ be a chain complex in $(A*, A)$ with differentials $\delta_{n}: C_{n} \to C_{n-1}$; then the $n$th component of an object $J(C)$ in $(A_{*}, A)$ is equal to $C_{n}$, face operators $d_{i}$ are zero for $i > 0$ and $d_{0} = \delta_{n}$. The Kan-Dold functor $K$ (see [2]) associates with a chain complex $C$ in $(A, A)$ an object $K\mathcal{C}$ in $(A_{*}, A)$ such that $(K\mathcal{C})_{n} = \bigsqcup_{\gamma} C_{n}$ where $n: [n] \to [n]$ runs over all epimorphic maps in $A$, $q = 0, 1, \ldots, n$. We denote by $w_{n}: C_{n} \to (K\mathcal{C})_{n}$ the corresponding morphism. 

In the second part we study the homotopy of maps of simplicial objects of three types. In the third part we prove several theorems on a preservation of the homotopy of maps of simplicial objects of three types by some functors.

We give only sketchy proofs and omit all computations; details may be found in [1].

\section{Three types of simplicial objects and chain complexes}

\subsection{Adjoints of forgetful functors}

It is clear that the values of a left adjoint $*_{1}$ of $\mathcal{C}$ on an object $X$: $A_{*} \to A$, i.e., $*_{1}(X) : A_{*} \to A$, is a Kan extension of $X$ along $A_{*} \to A$ and similarly for $*_{2}$ and $*$. Using the well-known method of computing Kan extensions (see [6]), we get

**Theorem 1.** Suppose that a category $A$ has finite colimits.

1. If $X$ is an object of $(A_{*}, A)$, then

$$
(*_{2}(X))_{n} = \bigsqcup_{x} X_{n}, \quad n = 0, 1, \ldots,
$$

where $\eta: [n] \to [q]$ runs over epimorphisms in $\mathcal{A}$ and for a map $X: [m] \to [n]$ in $\mathcal{A}$ the induced map $\tilde{\eta}$ is defined by the formula $\tilde{\eta} = w_{\eta} = w_{\eta} X(\eta)$, if $\eta \neq e_{1}$ and $\eta$ is an epimorphic map in $\mathcal{A}$, $\varepsilon$ is a map in $\mathcal{A}_{*}$.

2. If $X$ is an object in $(A_{*}, A)$, then

$$
(*_{3}(X))_{n} = \text{Coker}(\bigsqcup_{x} X_{n} \to \bigsqcup_{x} X_{n}), \quad n = 0, 1, \ldots,
$$

where $\beta: [n] \to [q]$ runs over all maps such that $\beta$ is a map in $\mathcal{A}$, $\pi_{1}, \pi_{2} \in S_{n}$ and $\beta \pi_{1} = \beta \pi_{2}$. Moreover, $\lambda_{w_{n}}$, $\pi_{n}$ are defined by the formula $\lambda_{w_{n}} w_{n} = w_{n} X(\beta)$ and let $v_{n}: \bigsqcup_{x} X_{n} \to (*_{3}(X))_{n}$ denote the natural map. For a map $X: [m] \to [n]$ in $\mathcal{A}$ we define a map $\tilde{\beta} : \bigsqcup_{x} X_{n} \to \bigsqcup_{x} X_{n}$ by the formula $\tilde{\beta} w_{n} = w_{n} X(\beta)$. If $\eta \neq \beta_{q}$, $\beta$ is a map in $\mathcal{A}$, $\eta$ (not unique) is in $S_{n}$. Thus $v_{n} \tilde{\beta} w_{n} = v_{n} \tilde{\beta} w_{n}$, hence $\tilde{\eta}$ induces a unique map $\hat{\eta}: (*_{3}(X))_{n} \to (*_{3}(X))_{n}$. For example, let $B$ be a simplicial complex with $\eta$. Then the sets $L_{n} = \{\varepsilon_{0}, \ldots, \varepsilon_{n}\}$, where $\varepsilon_{0}, \ldots, \varepsilon_{n}$ are faces of a simplex in $L_{n}$ and $\varepsilon_{n} < \varepsilon_{n}$, with obvious face operators $\beta_{i}$, $L_{n} \to L_{n-1}$, determine an object $L \in (A_{*}, A)$. Similarly we define objects $L_{n}^{*}$ in $(A_{*}, A)$ and $L_{n}^{*}$ in $(A_{*}, A)$ in a natural way.

2. Direct decompositions of objects in $(A_{*}, A)$. In the sequel we denote by $A$ an Abelian category. It is known (see [2]) that the functor $X \mapsto X$ is equivalent to the identity functor; thus there exist natural isomorphisms $X_{k} \cong (X_{1})_{k}$ where $n: [n] \to [q]$ runs over epimorphisms in $\mathcal{A}$ and $X$ is an object in $(A_{*}, A)$. We give an effective method of decomposing for this purpose. We denote

$$
\rho_{q} = \left(1 - \varepsilon_{q} d_{q} \right) \ldots \left(1 - \varepsilon_{1} d_{1} \right): \quad X_{q} \to X_{q},
$$

$$
s_{q} = \left(1 - \varepsilon_{q} d_{q} \right) \ldots \left(1 - \varepsilon_{1} d_{1} \right): \quad X_{q} \to X_{q-1},
$$

for $q = 0, 1, \ldots$, $i = 0, 1, \ldots, q-1$. It is known and easy to see that

$$
\rho_{q} \rho_{q} = 1, \quad \text{Im} \rho_{q} = (X_{1})_{q}, \quad \operatorname{Ker} \rho_{q} = \sum_{j=i}^{q} \text{Im}(s_{j}: \quad X_{j} \to X_{j-1});
$$

thus there exists a decomposition

$$
\rho_{q}: \quad X_{q} \cong \left(\rho_{q}\rho_{q}\right): \quad X_{q} \cong (X_{1})_{q}. \quad X_{q} \cong (X_{1})_{q}. \quad X_{q} \cong (X_{1})_{q}.
$$
with \( p^1 \) epimorphic and \( w_1 \) monomorphic. Each epimorphic map \( \eta : [n] \to [q] \) in \( A \) may be uniquely represented as \( \eta = n_{1,1}^{1} \cdots n_{1,q}^{1} \) with \( 0 \leq j_1 < \cdots < j_q < n \).\[i + q = n.\]

For such \( \eta \) we define maps
\[
i_{\eta} = X(\eta) w_{\eta}; \quad (N X)_{\eta}, X_{\eta} \to X_{\eta} = X_{\eta},
\]
\[
p_{\eta} = p_{1}^{1} p_{3}^{1} \cdots p_{3(q)}^{1}: X_{\eta} \to X_{\eta} = X_{\eta} = (N X)_{\eta}.
\]

**Theorem 2.** Let \( A \) be an Abelian category and let \( X \) be an object in \((A^*, A)\). For each \( n = 0, 1, \ldots \) the family of maps \( \{ i_{\eta}, p_{\eta} \} \), where \( \eta \) runs over all epimorphic maps \( \eta : [n] \to [n] \) in \( A \), represents \( X \) as a direct sum \( X_{\eta} \approx \bigoplus_{\eta} X_{\eta} \).

Proof. We prove the formulas
\[
1_{X_{\eta}} = \sum_{\eta} i_{\eta} p_{\eta} = 0 \quad \text{for} \eta \neq \eta', \quad p_{\eta} 1_{X_{\eta}} = 1_{(N X)_{\eta}},
\]
using the following relations:
\[
i_{\eta} = \begin{cases} 0 & \text{for} \ i \neq \eta, \\ 1_{i} & \text{for} \ i = \eta. \end{cases}
\]
\[
p_{\eta} = \begin{cases} (1 - s_{j} d_{j+1}) \cdots (1 - s_{n-1} d_{n-1}) & \text{for} \ k = j, \\ 0 & \text{for} \ k > j, \\ s_{k} f_{j-1, \eta}^{j-1} & \text{for} \ k < j, \end{cases}
\]
\[
1_{X_{\eta}} = p_{\eta} + \sum_{\eta} i_{\eta} p_{\eta} = 0.
\]

3. **Functors on \( A_* \) and chain complexes.** It is known that functors \( N, k \) are homotopically equivalent, i.e., for each object \( X \) in \((A^*, A)\) the chain complexes \( N(X), k(X) \) are naturally homotopically equivalent (see [2]). The functors \( N, k \) are not homotopically equivalent. To show this let \( A \) be the category \( Z\text{-Mod} \) and let \( i \) be a fixed integer different from 0 and \( \pm 1 \). We define an object \( X \) in \((A^*, Z\text{-Mod})\) as follows: \( X_{n} = Z \) for \( n = 0, 1, \ldots \) and \( d_{i}(x) = i x \) for all \( x \in Z_i \). Then \( H_{i}(X) = Z/IZ \) for all \( m \) but \( (N X)_{n} = 0 \) for all \( n > 0 \); thus \( H_{i}(N X) = 0 \) for all \( n > 0 \), whence \( N(X) \) and \( k(X) \) are not homotopically equivalent.

**Theorem 3.** Let \( A \) be an Abelian category.
(i) The functors \( k_{\eta} \) and \( N_{\eta} * z_{1} \) are equivalent.
(ii) The functors \( k_{\eta} + J = N_{\eta} + J = 1_{(C_{\eta}, \eta)} \).
(iii) The functors \( * z_{1} + J = N + 1_{(A, A)} \) are equivalent.
(iv) \( k_{\eta} = * z_{1} + J \).
(v) (Kan-Dold Theorem) The functors \( N + K, K * N \) are equivalent to the identity functors.
(vi) Functors \( K * k_{\eta} \) and \( * z_{1} \) are equivalent.

Proof. (i) Let \( X \) be an object in \((A^*, A)\); then we define the equivalences \( \psi \) and \( \phi \) as compositions:
\[
\phi(X): (k_{\eta}(X))_{\eta} \overset{\eta}{\longrightarrow} X_{\eta} = (N z_{1})(X)_{\eta}, \quad \psi(X): (N z_{1})(X)_{\eta} \overset{\eta}{\longrightarrow} (k_{\eta}(X))_{\eta}.
\]
where \( \eta : [n] \to [q] \) runs over epimorphisms in \( A \) and \( q < n \); thus we get the isomorphisms:
\[
(k_{\eta}(X))_{\eta} \approx \text{Coker}(D_{\eta}(z_{1})(X))_{\eta} = (N_{\eta} z_{1})(X)_{\eta},
\]
and it is easy to check that they are natural and commute with differentials.
(ii) Let \( X \) be an object in \((A^*, A)\); then we define maps of chain complexes
\[
f(X): k_{\eta}(X) \overset{\eta}{\longrightarrow} (k_{\eta} * z_{1})(X), \quad g(X): (k_{\eta} * z_{1})(X) \overset{\eta}{\longrightarrow} k_{\eta}(X)
\]
as follows:
\[
f(X): k_{\eta}(X)_{\eta} = X_{\eta} \overset{\eta}{\longrightarrow} \bigoplus_{\eta} X_{\eta} = (N z_{1})(X)_{\eta} \approx (k_{\eta} * z_{1})(X)_{\eta},
\]
\[
g(X): (k_{\eta} * z_{1})(X)_{\eta} = \bigoplus_{\eta} X_{\eta} \overset{\eta}{\longrightarrow} X_{\eta} = (k_{\eta}(X))_{\eta},
\]
where $\sum_{s \in S_n} \text{sgn}(s) p_s^\ast$ denotes a projection onto a direct summand corresponding to $\pi$ and $\eta$: $[n] \to [q]$ runs over epimorphisms in $A_n$. A standard but lengthy computation shows that $f(x)$ and $g(x)$ are, in fact, maps of complexes. It is clear that $g(x)f(x) = 1_{A_n}$.

We prove that $f(x)g(x)$ is homotopic to the identity map of the complex $(k_n \circ *)^n(X)$ by means of acyclic models. In the categories $(\mathcal{A}_\pi, \mathcal{Z} \text{-} \text{Mod})$, $(\mathcal{A}_\pi, \mathcal{Z} \text{-} \text{Mod})$ the objects $C(p)$, $\mathcal{C}(p)$ corresponding to a standard $p$-dimensional simplex are defined as follows:

$$C(p)_n = \bigcup_i Z_i, \quad \mathcal{C}(p)_n = \bigcup_i Z_i, \quad n = 0, 1, \ldots$$

where $e: [n] \to [p]$ runs over maps in $A_n$, $F: [n] \to [p]$ runs over maps in $A_n$ and $Z_i$, $Z_i$ are free $\mathbb{Z}$-modules on free generators $e$, $f$. Maps induced by a map $[n] \to [n]$ in $A_n$, or $A_n$ are obvious. We know that $\mathcal{C}(p) = \ast(C(p))$ and the chain complex $k_n \mathcal{C}(p)$ is homotopically trivial.

For each object $X$ in $(\mathcal{A}_\pi, \mathcal{A})$ and for each epimorphism $\eta: [n] \to [p]$ in $A_n$ we denote by $w_{x \to y}X_{\eta}$ the imbedding $w_{x \to y}: X_{\eta} \to \ast(C(p))$. We define by induction such natural maps $h(x): (\ast(C(p))_p \to \ast(C(p))_{n+1}$, $n = 0, 1, \ldots$ that

$$f_{x}(y) g(x) = \frac{d_n^x + d_{n+1}^x}{(d_n^x + d_{n+1}^x)} + \delta_{n+1}^x - h_{x(y)}(x) \delta_n^x,$$

where $d_n^x$ denotes the $n$th differential of a complex $k_n \mathcal{C}(p)$. We put $h_0(x) = 0$ and let us assume that the maps $h_n(x), \ldots, h_{n-1}(x)$ are defined for all objects $X$ in $(\mathcal{A}_\pi, \mathcal{A})$ and an arbitrary Abelian category $A$ and that they satisfy $0, \ldots, (n-1)$. Then we have

$$d_n^{x(p)}(f(x)g(x))_{x(p)} = (d_{n+1}^x - h_{x(y)}(x)) + \delta_{n+1}^x - h_{x(y)}(x) \delta_n^x = 0$$

and for each epimorphic map $\eta: [n] \to [p]$ in $A_n$ there exists such an element $b_{n} \in C(p)_{x(p)} = \bigcup_i Z_i$ (where $\eta: [n+1] \to [p]$)

$$d_n^{x(p)}(\delta_{n+1}^x - h_{x(y)}(x)) \delta_n^x = 0$$

and for each epimorphic map $\eta: [n] \to [p]$ in $A_n$ there exists such an element $b_{n} \in C(p)_{x(p)} = \bigcup_i Z_i$ (where $\eta: [n+1] \to [p]$).

It is easy to verify that $h_X(X)$ is natural and the formula (n) follows by a standard, but lengthy, computation.

\section{Homotopy categories of simplicial objects}

1. A standard triangulation of prisms. We denote the vertices of a standard $n$-dimensional simplex $A_n$ by $0, 1, \ldots, n$. For any map $\alpha: [n] \to [n]$ in the category $A_\pi$ we denote by $\alpha: A_n \to A_n$ such an affine map that $x(0) = x(i), i = 0, 1, \ldots, n$ and by $\alpha: A_n \to A_n$ such a simplicial map of barycentric subdivisions that a barycenter of a face $\alpha(o)$ of $A_n$ is mapped onto a barycenter of a face $\alpha(o)$ of $A_n$. If $\alpha$ is mono-

monic, then $\alpha = i$. We denote by $\alpha: [n], \alpha: [n], \alpha: [n]$ a contravariant functor represented by an object $[n]$ and defined on the category $A_n$ (resp., $A_n$, $A_n$).

Let $\alpha: A_n \to A_n$ for $n = 0, 1, \ldots, j = 0, 1, \ldots, n$, be such affine maps that $\alpha_j(i) = 0$ for $i \in [j]$ and $\alpha_j(i) = 0$, $j = 0, 1, \ldots, n$, determine a standard triangulation of a prism $A_n \times I_j$. To this triangulation (with the usual ordering of vertices) correspond objects $A_n \times I_j$ in the category $A_n \times I_j$. It is easy to see that similar formulas do not hold for $A_n \times I_j$ and $A_n \times I_j$. Essential properties of standard triangulations of prisms are collected in the following proposition:

**Proposition 1.** Maps $\varphi_{x}: A_n \times I_j \to A_n \times I_j$, $x = 0, 1, \ldots, j = 0, 1, \ldots, n$, satisfy the following relations:

\begin{align*}
(1) & \quad \varphi_{x(0)} = |e^{x(0)}| = i_0, \quad \varphi_{x} = |e^{x}| = i_1, \\
(2) & \quad \varphi_{x, i} = |e^{x}| = \langle x, e \rangle, \quad \text{for } x \in A_n, \\
(3) & \quad \varphi_{x, i} = |e^{x}| = \langle x, e \rangle, \quad \text{for } i < j, \\
(4) & \quad \varphi_{x, i} = |e^{x}| = \langle x, e \rangle, \quad \text{for } i = 0, 1, \ldots, n - 1, \\
(5) & \quad \varphi_{x, i} = |e^{x}| = \langle x, e \rangle, \quad \text{for } i = j + 1, \\
(6) & \quad \varphi_{x, i} = |e^{x}| = \langle x, e \rangle, \quad \text{for } i = j. \\
\end{align*}
Each isomorphism $\varphi: [n] \to [n]$ in the category $\mathcal{A}_n$ induces the simplicial map $[n]: \Delta_n \to \Delta_n$ but the maps $[n] \times I: \Delta_n \times I \to \Delta_n \times I$ are not simplicial unless $\pi = 1$.

Thus to define a homotopy in a category $(\mathcal{A}_n^*, M)$ which is "consistent with models" we have to consider another triangulation of prisms.

2. Homotopy in categories $(\mathcal{A}_n^*, M)$. Let $M$ be an arbitrary category. We recall a well-known definition of homotopy in $(\mathcal{A}_n^*, M)$.

**Definition 2.** A homotopy of maps of an object $X$ in $Y$ in a category $(\mathcal{A}_n^*, M)$ is a family of maps $\{h_{j,n}\}, n = 0, 1, \ldots, I = 0, 1, \ldots, n$, where $h_{j,n}: \Delta_n \to Y$ which satisfy the relations

\begin{align*}
(7) & \quad d_{i}h_{j,n} = h_{j-1,n-1}d_{i} \quad \text{for} \quad i < j, \\
(8) & \quad d_{i+1}h_{j+1,n} = d_{i+1}h_{j,n} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \\
(9) & \quad s_{i}h_{j,n} = h_{j+1,n+1}s_{i} \quad \text{for} \quad i = 0, 1, \ldots, n, \\
(10) & \quad s_{i}h_{j,n} = h_{j+1,n+1}s_{i} \quad \text{for} \quad i < j, \\
(11) & \quad s_{i}h_{j,n} = h_{j+1,n+1}s_{i} \quad \text{for} \quad i > j.
\end{align*}

Homotopy $\{h_{j,n}\}$ joins maps $f_0, f_1: X \to Y$ where

$$
(f_0)_n = d_{n+1}h_{n,n}, \quad (f_1)_n = d_{n}h_{n,n}. $$

It is well known that there exists a natural one-to-one correspondence between the set of all homotopies of maps of object $X$ in $Y$ and the set of all maps of $X \times \Delta [1]$ in $Y$ (see [2]).

Let us suppose for a moment that $M = Set$. Then each element $x_\alpha \in X_\alpha$ determines a unique map $\tilde{x}_\alpha: \Delta [1] \to X$ such that $\tilde{x}_\alpha(1)_\alpha = x_\alpha$; thus for a map $h: X \times \Delta [1] \to Y$ corresponding to the homotopy $h_{j,n}$ we have

$$
h_{j,n}(x_\alpha) = h_{j+1,n}(xy, x_\sigma) = h_{j+1,n}(x_\beta, x_\sigma) = h_{j+1,n}(x_\beta, x_\sigma),
$$

where $\varphi_{j,n}: \Delta [1] \to \Delta [1]$ denotes such a map that $\varphi_{j,n} = \varphi_{j,n}$ and $\varphi_{j,n} = [n] \to [n]$ satisfies $\varphi_{j,n}$ for $i < j$ (resp., for $i > j$). Now it is easy to check that all the formulas (7)-(11) follow from Proposition 1; for instance, we obtain formula (7) as follows:

$$
d_{i}h_{j,n}(x_\alpha) = d_{i}(h_{j,n}(x_\alpha)) + \varphi_{j,n}d_{i}(h_{j,n}(x_\alpha)) = d_{i}(h_{j,n}(x_\alpha)) + \varphi_{j,n}d_{i}(h_{j,n}(x_\alpha)).
$$

3. Homotopy in categories $(\mathcal{A}_n^*, M)$.

**Definition 3.** A homotopy of maps of an object $X$ in $Y$ in a category $(\mathcal{A}_n^*, M)$ is a family of maps $\{h_{j,n}\}, n = 0, 1, \ldots, I = 0, 1, \ldots, n$, where $h_{j,n}: \Delta_n \to Y_{j,n}$, which satisfy relations (7)-(11) and hold properties of the standard triangulation of prisms.

Each map $f: X \to X$ induces a map $f: X \times \Delta [1] \to X \times \Delta [1]$ and maps $f_0, f_1: X \to X \times \Delta [1]$ are defined by $f_0 \circ h = f_0$ and $f_1 \circ h = f_1$.

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It is easy to see that in the case \( M = \text{Set} \) we have \( X \times \Delta[1] \approx X \otimes \Delta[1] \), where \( \otimes \) is defined in [7], and in this case our homotopy is identical with that defined in [7].

4. Godement homotopy in categories \((\Delta^*_n, M)\). The definition of homotopy in the category \((\Delta^*_n, \text{Set})\) given by Godement in [4] admits an obvious generalization for categories \((\Delta^*_n, M)\). For each object \( X \) in \((\Delta^*_n, M)\) we define an object \( X \times \Delta[1] \) in \((\Delta^*_n, M)\) as follows:

\[
(X \times \Delta[1])_n = \bigsqcup_{s \geq 0} X_{s \times n}
\]

where \( X_{s \times n} = X_{s} \), \( s \) runs over all maps \( \sigma : [n] \to [1] \) in \( \Delta_n \) and any map \( \sigma : [n] \to [n] \) in \( \Delta_n \) induces a map \( \hat{\sigma} : (X \times \Delta[1])_n \to (X \times \Delta[1])_n \) determined by the conditions \( \hat{\sigma} \circ w_{sn} = w_{sn} \circ \sigma \) for all \( w_{sn} : X_{s} \to (X \times \Delta[1])_n \) denotes an imbedding). Each map \( f : X \to X' \) induces a map \( X \times \Delta[1] \to X' \times \Delta[1] \) and maps \( \rho_i, i = 1, \ldots, \rho_i : X \times X' \times \Delta[1] \) are imbeddings corresponding to two constant maps \( [n] \to [1] \).

**Definition 5.** A \( G \)-homotopy of maps of an object \( X \) in \( Y \) in a category \((\Delta^*_n, M)\) is a map \( h : X \times \Delta[1] \to Y \) and it joins maps \( h, h', \gamma, \alpha \).

If \( M = \text{Set} \) then a standard computation shows that the functor \( \cdot \times \Delta[1] : (\Delta^*_n, \text{Set}) \to (\Delta^*_n, \text{Set}) \) is a Kan extension of a functor \( Q : \Delta^*_n \to (\Delta^*_n, \text{Set}) \) along the Yoneda map \( \Delta[1] \to \Delta^*_n \). Consider a map \( \Phi : [n] \to [m] \), where \( \Phi \) is the Yoneda map \( \Delta[1] \to \Delta^*_n \) and for each \( \Phi : [n] \to [m] \) there exist unique maps \( \gamma \) such that \( \gamma : [n] \to [m] \) and \( \gamma : [n] \to [n] \). It is easy to see that \( Q \) is equivalent to a functor \( \{n] \to \Delta^*_n \} \).

5. \( \tau \)-homotopy in categories \((\Delta^*_n, M)\). We have observed that in the case of categories \((\Delta^*_n, \text{Set})\), \((\Delta^*_n, M)\) and \((\Delta^*_n, M)\), where \( \text{Set} \) is assumed to be closed with respect to direct limits (this assumption is not essential).

A choice of one of the functors \( P_n, P, \) and \( Q \) and of morphisms \( i, i' \) of a Yoneda map \( P_n, P, Q \) determines a "model" for homotopy. The functors \( P_n, P, \) and \( Q \) correspond to traditional "prismatic models" \( \Delta^*_n \times I \) with maps \( \Delta^*_n \) into lower and upper faces of \( \Delta^*_n \times I \). The functor \( Q \) corresponds to "simplicial models" \( \Delta^*_n \times I \) with maps \( \Delta^*_n \) into faces \([0, 1], [n] \), \([n \times 2], [n \times 2], \ldots, [n + 2], [n + 2] \).

Now we describe a \( \tau \)-homotopy of another type in categories \((\Delta^*_n, M)\) which correspond to "prismatic models". A construction of a required functor \( \Delta^*_n \to (\Delta^*_n, \text{Set}) \) is in fact a construction of some special triangulation of prisms \( \Delta^*_n \times I \), appropriately related to maps induced by \( \Delta^*_n \). We describe one such triangulation, called \( \tau \)-triangulation. It is not as good as the standard triangulation, which corresponds to the functor \( P \), because maps \( |\delta^{}| \times 1 : \Delta^*_n \times I \to \Delta^*_n \times I \) map \( \tau \)-simplices onto \( \tau \)-simplices, but are not affine in general (compare formula (192)).

We can obtain another useful functor \( \Delta^*_n \to (\Delta^*_n, \text{Set}) \) by constructing another triangulation of prisms. Over each simplex \( \sigma \) of a barycentric subdivision of \( \Delta^*_n \) (with natural ordering of vertices induced by an inclusion of faces) we build the standard triangulation of a prism \( \sigma \times I \). The sum of all such triangulations of a prism \( \sigma \times I \) is appropriately related to maps in \( \Delta^*_n \). Imbeddings of \( \Delta^*_n \) into lower and upper faces of \( \Delta^*_n \times I \) are simpfical if we consider \( \Delta^*_n \) as a barycentric triangulation. Consequently, we have to replace objects \( X \) in \((\Delta^*_n, M)\) by their simplicial subdivisions, generalizing the Kan construction in \((\Delta^*_n, \text{Set})\). We shall not discuss this subject here.

For fixed \( n \) and \( m \) we consider sequences \( (i_0, i_1, \ldots, i_{m-1}) \) such that \( 0 \leq i_0 \leq \ldots \leq i_{m-1} \leq n \) and \( 0 \leq i_{m} \leq n \) for \( k = 0, 1, \ldots, m-1 \). For each such sequence we denote by \( i_0, i_1, \ldots, i_{m} \) such that \( \{i_0, \ldots, i_{m}\} = \{0, 1, \ldots, n\} \times \{i_0, \ldots, i_{m-1}\} \) and \( i_{m} \leq \ldots \leq i_{m} \). We can identify a sequence \( (i_0, i_1, \ldots, i_{m}) \) with a monomorphic map \( i_0, i_1, \ldots, i_{m} : [n-1] \to [m] \) in \( \Delta^*_n \). We denote by \( \tau_{m} \) such an affine map that

\[
\tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) = \begin{cases} \{0(i_0, i_1, \ldots, i_{m-1}) \} & \text{for } 0 \leq i_0 \leq m, \\
 \{i_{m-1}, \delta \} & \text{for } m < i_0 < n \end{cases}
\]

where \( b(i_0, i_1, \ldots, i_{m}) \) denotes a barycenter of a face of \( \Delta^*_n \) determined by the vertices \( i_0, i_1, \ldots, i_{m} \). If \( m = 0 \), then we have

\[
\tau_{m}(0, i_0, \ldots, i_{m-1}) = \{0(0, 1, \ldots, n) \}
\]

for \( 0 < j < n \). It is easy to see that all maps \( \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) \) determine a triangulation of a prism \( \Delta^*_n \times I \) and we call it the \( \tau \)-triangulation.

**Proposition 6.** Maps \( \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) \) satisfy the following relations:

\[
\begin{align*}
(13) & \quad \tau_{m+1}((\cdot) ; \delta) = \{i_{m+1}\}, & \delta = 0, 1, \\
(14) & \quad \text{if } m > 0, i, i' \text{ are monomorphic maps and the diagram} & \\
& \begin{bmatrix} n-2 & e \cr r \end{bmatrix} & \begin{bmatrix} n-1 \end{bmatrix} \\
& \begin{bmatrix} n-2 & e \cr r \end{bmatrix} & \begin{bmatrix} n-1 \end{bmatrix} \\
& \text{is commutative, then} & \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) & = \{i_{m+1}\} \times \tau_{m}((i_0, \ldots, i_{m-2}) ; \delta), \\
(15) & \quad \text{if } 0 < j < m, & \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) & = \{i_{m+1}\} = \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta), & |i_{m}| \cr (16) & \quad \text{if } m < j \leq n+1, m < n, & \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) & = \{i_{m+1}\} & \tau_{m}((i_0, \ldots, i_{m-1}) ; \delta) & = \{i_{m+1}\}.
\end{align*}
\]
§ 3. Preservation of homotopy in categories of simplicial objects

Let Top denote the category of topological spaces and continuous maps. By Proposition 1.3 of Chapter II of [3] it follows that there exist pairs of adjoint functors \((\Delta^*, \mathbf{Set}) \rightleftarrows \text{Top}, \quad (\Delta^*, \mathbf{Set}) \rightleftarrows \text{Top})

where \(\mathbf{Sing}, \mathbf{i}, \mathbf{Sing}, \mathbf{S}g\) are functors "simplicial set of singular simplexes", \(|\left[\pi,-\pi\right]|\) and \(|\left[\pi,-\pi\right]|\) are geometric realization functors and we have

\[ |\pi,\left[\pi,-\pi\right]| \approx |\pi,\left[\pi,-\pi\right]| \]

giving rise to the dual adjunction (3.1)

It is well-known that \(|\pi,\left[\pi,-\pi\right]| \approx |\pi,\left[\pi,-\pi\right]| \) and this implies that \(|\pi,\left[\pi,-\pi\right]| \approx |\pi,\left[\pi,-\pi\right]| \)

Thus the functor \(|\pi,\left[\pi,-\pi\right]| \) preserves homotopy. The definitions imply

Corollary 1. The forgetful functor \(z_1: (\Delta^*, \mathbf{M}) \rightarrow (\Delta^*, \mathbf{M}) \) preserves homotopy.

Corollary 2. The forgetful functors \(z_2: (\Delta^*, \mathbf{M}) \rightarrow (\Delta^*, \mathbf{M}), \quad z: (\Delta^*, \mathbf{M}) \rightarrow (\Delta^*, \mathbf{M}) \)

To prove that the functor \(z_1\) preserves homotopy we need the following

Proposition 3. There exists such an equivalence of functors that the diagrams

\[
\begin{array}{c}
z_1(X \times \Delta^1) \xrightarrow{z_1} z_1(X) \times \Delta^1 \\
\end{array}
\]

\[
\begin{array}{c}
z_1(X) \times \Delta^1 \xrightarrow{z_1} z_1(X) \\
\end{array}
\]

\[ \delta = 0, 1 \] are commutative for all objects \(X \in (\Delta^*, \mathbf{M}) \); \(i_x, i_y\) denote the appropriate inclusions.

Proof. Let \(w_{\pi,\Delta}^*: z_1(X) \rightarrow z_1(X) \times \Delta^1\), \(k = 0, 1, \ldots, q\), denote inclusions; then we define a natural transformation \(\psi\) and its inverse \(\psi\) by the formulas

\[ \phi_\pi(X)w_\pi w_{\pi,\Delta} = \psi_\pi w_{\pi,\Delta} \]

for all epimorphic maps \(\pi: [n] \rightarrow [m]\) in \(\Delta\), and if \(\eta = \eta^1 \cdots \eta^p\) with \(0 \leq j < \cdots < j < \Delta\) then

\[ \psi_\pi(X)w_\pi w_{\pi,\Delta} = \psi_\pi w_{\pi,\Delta} \]

A standard and lengthy computation shows that \(\psi\) and \(\psi\) are in fact natural transformations and that \(\psi\) is inverse to \(\phi\).

Theorem 4. The functor \(z_1: (\Delta^*, \mathbf{M}) \rightarrow (\Delta^*, \mathbf{M}) \) preserves homotopy.

Proof. Let a homotopy \(h_{\pi,\Delta}\) join maps \(f_0, f_1: X \rightarrow Y\) in \((\Delta^*, \mathbf{M})\). By Theorem 4 of part 2 to this homotopy corresponds a map \(h: X \times \Delta^1 \rightarrow Y\) such that \(h \circ j_0 = f_0\),

Les us remark that \(|\pi,\left[\pi,-\pi\right]|(x+(1-t)) = x+(1-t)|\pi,\left[\pi,-\pi\right]|(y).
Lemma 8. There exists a natural transformation \( \sigma \) of functors that satisfies the diagrams

\[
\begin{array}{ccc}
|X| \times |\Delta^1| & \xrightarrow{\sigma(X)} & |X \times \Delta^1| \\
|\Delta^0| & \xrightarrow{\Delta^0} & |X| \times |\Delta^1| \\
\end{array}
\]

\( \delta = 0, 1 \) are commutative, where \( \delta \) denote the appropriate embeddings.

Proof. At first we consider the case \( X = \Delta^n \). Then it is easy to see that there exists an isomorphism \( \Delta^1 \simeq \Delta^0 \times \Delta^0 \) which maps \( 1_{[2n+1]} \) onto \( g = (g_1, g_2) \), where \( g_1 : [2n+1] \mapsto [n], g_2 : [2n+1] \mapsto [1] \) satisfy \( g_1(i) + (n+1)g_2(i) = i \) for all \( i \in [2n+1] \). We denote \( b_i = (1/(n+1)) \) for all \( i \in [2n+1] \) and define

\[
\sigma(\Delta^n)_{i_0, \ldots, i_n} = \begin{cases} 1 & \text{if } \sum_i i_b_i = 0 \\ 0 & \text{if } \sum_i i_b_i = 1 \end{cases}
\]

where \( a_0, \ldots, a_n \) denote vertices of \( [2n+1] \) and \( t_0, \ldots, t_n \) belong to the unit interval and \( t_0 + \cdots + t_n = 1 \). For an arbitrary object \( X \) in \( (\Delta^n, S) \) we extend the definition of \( \sigma \) as follows. Let \( x_n \in X^n \); then \( \sigma(X) \) is the only map such that

\[
\sigma(X)(\Delta^n)_{i_0, \ldots, i_n} = (\sum_{i_n} x_n) = \sigma(X)_{i_0, \ldots, i_n}.
\]

Theorem 9. The functor \( |\cdot| : (\Delta^n, S) \to \text{Top} \) preserves G-homotopy.

Proof. Let \( h : X \times \Delta^n \to Y \) be a map which joins \( f_0, f_1 \); then \( |f_0| = |h| \) and \( |f_1| = |h| \) are \( \sigma(X) \)-homotopic for \( \delta = 0, 1 \); consequently \( |h| \) is \( \sigma(X) \)-homotopic.

Theorem 10. Let \( f_0, f_1 : X \to Y \) be maps in the category \( (\Delta^n, S) \) and suppose that the maps \( \ast(f_0), \ast(f_1) \) are \( \tau \)-homotopic. Then the maps \( \ast(f_0|X), \ast(f_1|X) \) are homotopic.

Proof. We know that \( |\ast(X)| \simeq |X| \); thus we can represent \( \ast(f_0|X) \) as a cokernel of a pair of maps \( \overrightarrow{\pi} \Delta^n \to \Delta^n \times \Delta^n \), where \( : [n] \to [n] \) runs over maps in \( \Delta^n \), \( \pi_0 \) runs over \( \Delta^n \times \Delta^n \), \( \Delta^n \times \Delta^n \) is \( \Delta^n \), and \( \lambda \) is \( \lambda = \lambda_{|\ast(f_0)|} \). Let \( : \Delta^n \times \Delta^n \to |\ast(X)| \) be a natural map.

Reactions (15')-(17') correspond to all pairs of adjacent \((n+1)\)-dimensional simplices of \( \tau \)-truncation of a prism \( \Delta^n \times \Delta^n \). Using those relations, we show that for each \( x_n \in X^n \) there exists a unique map \( H_{x_n} : \Delta^n \times \Delta^n \to |\ast(X)| \), such that

\[
H_{x_n}(s_{n+1}) = (s_{n+1})_{i_0, \ldots, i_n} = \begin{cases} 1 & \text{if } \sum s_i = 0 \\ 0 & \text{if } \sum s_i = 1 \end{cases}
\]

where \( a_0, \ldots, a_n \) denote vertices of \( [2n+1] \) and \( t_0, \ldots, t_n \) belong to the unit interval and \( t_0 + \cdots + t_n = 1 \). For an arbitrary object \( X \) in \( (\Delta^n, S) \) we extend the definition of \( \sigma \) as follows. Let \( x_n \in X^n \); then \( \sigma(X) \) is the only map such that

\[
\sigma(X)(\Delta^n)_{i_0, \ldots, i_n} = (\sum_{i_n} x_n) = \sigma(X)_{i_0, \ldots, i_n}.
\]

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\[
H_{x_n}(s_{n+1}) = (s_{n+1})_{i_0, \ldots, i_n} = \begin{cases} 1 & \text{if } \sum s_i = 0 \\ 0 & \text{if } \sum s_i = 1 \end{cases}
\]

where \( a_0, \ldots, a_n \) denote vertices of \( [2n+1] \) and \( t_0, \ldots, t_n \) belong to the unit interval and \( t_0 + \cdots + t_n = 1 \). For an arbitrary object \( X \) in \( (\Delta^n, S) \) we extend the definition of \( \sigma \) as follows. Let \( x_n \in X^n \); then \( \sigma(X) \) is the only map such that

\[
\sigma(X)(\Delta^n)_{i_0, \ldots, i_n} = (\sum_{i_n} x_n) = \sigma(X)_{i_0, \ldots, i_n}.
\]
and similarly for (15'), (17). Thus the maps \( \tilde{h}(x_i) \) induce \( H_{\alpha}. \) By (14') it follows that the maps \( H_{\alpha} \) induce such a map \( H: [X_i \times A_1] \rightarrow [Z(Y)] \) that \( H \circ (v \times 1) \circ (w_{(a_0, a_1)}) \times 1) = H_{\alpha}. \) In fact, we have (abbreviation: \( H' = h_{\alpha} \circ \tau \circ \omega \)).

\[
H_{\alpha} = ([x_i] \times 1) \circ \tau \circ \omega \circ ([0, \ldots, 1, \ldots, \delta]) = H_{\alpha} \circ \tau \circ \omega \circ ([0, \ldots, 1, \ldots, \delta]) \circ ([x_i] \times 1) = [h(x_i)] \circ [x_i] = \tilde{h}'(\tilde{h}'(x_i)) \circ [x_i] = H_{\alpha} \circ \tau \circ \omega \circ ([0, \ldots, 1, \ldots, \delta])
\]

thus for any \( \epsilon \) holds \( H_{\alpha} \circ ([x_i] \times 1) = H_{\alpha_{\epsilon, \alpha}}. \) If we put \( H' = \bigcup H_{\alpha}, \bigcup A_{\alpha, \epsilon} \times A_1 \rightarrow [Z(Y)] \), then

\[
H' \circ (v \times 1) \circ (w_{(a_0, a_1)}) \times 1) \circ ([x_i] \times 1) = H_{\alpha} \circ ([x_i] \times 1) = H_{\alpha} \circ (v \times 1) \circ (w_{(a_0, a_1)}) \times 1)
\]

thus \( H' \) induces \( H. \) It is easy to see that \( H \) joins \([x_i] \times 1) \) and \([x_i] \times 1).\)

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