Mappings covered by products and pinched products

by

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Abstract. We say that a mapping $f\colon X \Rightarrow Y$ is covered by a product or (A, C) covered iff there is a space A and a mapping $c \in C$ (a class of mappings) such that $c\colon A \times Y \Rightarrow X$ and $c(A, y) = f^{-1}(y)$. The fibers of f are (A, C) covered iff there is a (non-empty) space K_y of mappings $m\colon A \Rightarrow f^{-1}(y)$ with $m \in C$. We say that f is fiber (A, C) covered and completely regular iff (1) the fibers of f are covered by (A, C), (2) if K denotes the collection of all such spaces K_y, then K^* is a complete metric space; and (3) for $y \in Y$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $z \in N_\delta(y)$ and $g \in K_y$, there is $h \in K_z$ and a mapping $s_{zy}\colon f^{-1}(z) \Rightarrow f^{-1}(y)$ which is a homeomorphism on a dense set $O_z \subset f^{-1}(z)$ open relative to $f^{-1}(z)$ such that (a) s_{zy} moves no point as much as ε, (b) $s_{zy}h = g$, and (c) for any $m \in K_y$, $s_{zy}^{-1}m$ (where s_{zy}^{-1} is defined on $s_{zy}(O_z)$ and $s_{zy}^{-1}m$ is defined on $m^{-1}(s_{zy}(O_z))$) extends to a mapping $w \in K_z$. We say that $f\colon X \Rightarrow Y$ is fiber (A, C) covered and LC^n completely regular iff in addition to the above, each K is LC^n. A typical theorem follows:

THEOREM. *All spaces are metric. Suppose that $f\colon X \Rightarrow Y$ is fiber (A, C) covered and* LC^n *completely regular and that* $\dim Y = n+1$. *Then f is locally (A, C) covered by $A \times Y$. If Y is contractible, separable, and locally compact, then f is (A, C) covered by $A \times Y$. Theorems of this kind are used to study mappings $f\colon X \Rightarrow [0, 1]$ such that* (a) *$f^{-1}(\frac{1}{2})$ is an n-sphere S^n with a k-sphere shrunk to a point,* (b) *$f^{-1}(x) = S^n$, $0 \leqslant x < \frac{1}{2}$, and* (c) *$f^{-1}(x)$ is a spherical modification of S^n of type $k+1$. Under certain conditions, $X \cong$ an $(n+1)$-manifold and f is (A, C) covered by $A \times I$ where $A = S^{n-(k+1)} \times I^{k+1}$.*

1. Introduction. There are a number of quite interesting (and, obviously, important) theorems in differential topology. Some of these characterize spheres, but their proofs use "smoothness" of both the manifold and the mapping. For example, the theorem of Reeb [16] (1952) and Milnor [11] (1956) later generalized by Milnor [12] (1959) and Rosen [17] (1960) is such a characterization.

THEOREM 1 (Reeb–Milnor–Rosen). *Suppose that M is smooth (C^∞) compact manifold and that f is a smooth real valued function on M with exactly two critical points (degenerate or not). Then M is homeomorphic to a sphere.*

This theorem has a topological version which we have in [9]. It is as follows.

THEOREM 2 (McAuley). *Suppose that M is a continuum (compact connected metric space) and that $f\colon M \Rightarrow I = [0, 1]$ is a (continuous) mapping. Furthermore, $f^{-1}(0) = a$ (point), $f^{-1}(1) = b$ (point), $f|(M - \{a, b\})$ is completely regular, and*

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$f^{-1}(x)$ *is homeomorphic to an n-sphere* S^n *for each* $x \in (0, 1)$. *Then M is homeomorphic to* S^{n+1}.

The condition that $f^{-1}(x)$ be an n-sphere S^n is quite natural in view of the following theorem from differential topology.

THEOREM 3. *If* $f\colon M \to N$ *is a smooth mapping between smooth manifolds of dimensions m and n, respectively, where* $m \geqslant n$ *and if* $y \in N$ *is a regular value, then the set* $f^{-1}(y) \subseteq M$ *is a smooth manifold of dimension* $m-n$.

One wonders just what are the topological properties of differential mappings? Also, under what reasonable (topological) conditions is a mapping differential? In the case of non-constant analytic mappings from the complex plane to the complex plane, the properties of *openness* and *lightness* actually characterize them. Whyburn [24] and Stoilow [19] have shown that if $f\colon M^2 \Rightarrow N^2$ is a light open mapping between 2-manifolds, then f is topologically equivalent to an analytic mapping. Several researchers, Church, in particular, have made considerable progress in obtaining topological properties of differentiable mappings. For references and results, see [2].

At the topology conference held at the University of Oklahoma, March, 1972, I gave a talk containing outlines of proofs of theorems for which Theorems 2 and 4 (below) are special cases. The manuscript for that talk has appeared in the Proceedings, *Topology Conference*, University of Oklahoma, 1972.

THEOREM 4 (McAuley). *Suppose that M is a continuum and that* $f\colon M \Rightarrow [0, 1]$ *is a mapping such that* (1) $f^{-1}(0) = a$ *(point)*, (2) $f^{-1}(1) = b$ *(point)*, (3) $f^{-1}(\frac{1}{4}) = f^{-1}(\frac{3}{4}) = a$ *figure eight (two circles with exactly one common point)*, (4) *for* $0<x<\frac{1}{4}$ *or* $\frac{3}{4}<x<1$, $f^{-1}(x) \cong$ *a circle*, (5) *for* $\frac{1}{4}<x<\frac{3}{4}$, $f^{-1}(x) \cong$ *a pair of disjoint circles, and* (6) *for* $0<x<1$, *there is a "triangulation" of* $f^{-1}(x)$ *which contains exactly four* 1*-simplexes (simple arcs) and f is completely regular with respect to the collection of all* 1*-simplexes (defined below). Then* $M \cong$ *Torus or Klein bottle.*

It is useful to define the concept of a mapping covered by a product. In fact, this concept and others given here seem to provide an approach to the study of certain kinds of mappings which will be given elsewhere.

2. Mappings covered by products. We say that a mapping $f\colon X \Rightarrow Y$ is *covered by a product* iff there exists a space A and a mapping $c\colon A \times Y \Rightarrow X$ such that $c(A, y) = f^{-1}(y)$ for each $y \in Y$. We also say that f is *covered by* (A, c).

If the closed (or quasi-compact) mapping $f\colon X \Rightarrow Y$ is covered by (A, c), then let G denote the collection of all $c^{-1}(x)$ for each $x \in X$. Thus, G is an upper semi-continuous (usc) decomposition of $A \times Y$ and the decomposition space $A \times Y/G$ is a *pinched product* (§ 7) homeomorphic to X.

QUESTION. What mappings are covered by products or locally covered by products? Clearly, mappings from products and certain fiber spaces are in this category.

3. Mappings whose fibers are covered by a fiber A with respect to a class C of mappings. Suppose that $f\colon X \Rightarrow Y$ is a mapping. We say that $f^{-1}(y)$, $y \in Y$, is a *fiber* even though (f, X, Y) may not be a fiber space. It may be true that any two fibers are homeomorphic or homotopically equivalent. We say that the *fibers of f are covered by* (A, C) iff A is a space (fiber) and C is a class of mappings such that for each $y \in Y$, there is a non-empty space K_y of mappings of A onto $f^{-1}(y)$ each belonging to the class C. (The class C may be, for example, the class of all homeomorphisms, all continuous mappings, all mappings which are homeomorphisms on a dense open subset of A, all finite-to-one mappings, etc.)

We also say that the fibers of f are *locally covered by* (A, C) iff for each $y \in Y$ there is an open set U_y in Y, $y \in U_y$, such that $f|f^{-1}(U_y)$ is covered by (A, C).

4. Mappings which are fiber (A, C) covered and LC^n completely regular. Suppose that $f\colon X \Rightarrow Y$. We say that f is fiber (A, C) *covered and completely regular* iff

(1) the fibers of f are covered by (A, C) (or f is fiber (A, C) covered), i.e., for each $y \in Y$, there is a non-empty space K_y of mappings (in the class C) of (A, y) onto $f^{-1}(y)$,

(2) if K denotes the collection of all such spaces K_y (specified by f being fiber (A, C) covered), then K^* (the union of the elements of K) is a complete metric space (for details, see [7]), and

(3) for each $y \in Y$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $z \in N_\delta(y)$ and $g \in K_y$, there is $h \in K_z$ and a mapping $s_{zy}\colon f^{-1}(z) \Rightarrow f^{-1}(y)$ which is a homeomorphism on a dense set $O_z(f^{-1}(z))$ and open relative to $f^{-1}(z)$ such that (a) s_{zy} moves no point as much as ε, (b) $s_{zy}h = g$, and (c) for any $m \in K_y$, $s_{zy}^{-1}m$ (where s_{zy}^{-1} is defined on $s_{zy}(O_z)$ and $s_{zy}^{-1}m$ is defined on $m^{-1}(s_{zy}(O_z))$) extends (uniquely) to a mapping which belongs to K_z, and (d) for each $e \in K_z$, $s_{zy}e \in K_y$.

We say that $f\colon X \Rightarrow Y$ is *fiber* (A, C) *covered and* LC^n *completely regular* iff (1) f is fiber (A, C) covered and completely regular (as above) and (2) each K_y is LC^n (locally connected in dimension n in the homotopy sense).

5. Certain fiber (A, C) covered and LC^n completely regular mappings are (A, c) covered. The theorem below gives conditions under which certain fiber (A, C) covered maps are *locally and globally* covered by a product.

THEOREM 5. *Suppose that* $f\colon X \Rightarrow Y$ *is fiber* (A, C) *covered and* LC^n *completely regular and that Y has dimension* $n+1$. *Then f is locally* (A, C) *covered by the product* $A \times Y$. *If Y is contractible, locally compact, and separable, then f is* (A, C) *covered by the product* $A \times Y$.

Proof. Since f is fiber (A, C) covered and LC^n completely regular, there exist space K_y of mappings satisfying the definitions above. The union K^* of the collection K of the various spaces K_y, $y \in Y$, is a complete metric space (hypothesis).

Now, K is lower semi-continuous (lsc) in the sense that if $y_i \in Y$ and $\{y_i\} \to y \in Y$, then each element $g \in K_y$ is the limit of a sequence $\{g_i\}$ where $g_i \in K_{y_i}$.

For $\varepsilon>0$, there exists $\delta>0$ such that if $y_i \in N(y)$, then there exists $h \in K_{y_i}$ and a mapping $g_{y_i y}\colon f^{-1}(y_i) \Rightarrow f^{-1}(y)$ (with certain properties including that no point is moved more than ε) such that $g_{y_i y} h = g$. Thus, $\varrho(h, g)<\varepsilon$ where ϱ is a complete metric for K^*. Using the above and the fact that $\{y_i\}\to y$, it should be apparent that K is lsc.

Next, we show that K is equi-LC^n. Suppose that $\varepsilon>0$ and $g \in K_y$. We must exhibit $\delta>0$ such that for each t, $0\leqslant t\leqslant n$ and any mapping $r\colon S^t \to N_\delta(g) \cap K_z$, then r can be extended to $R\colon I^{t+1}\to N_\delta(g) \cap K_z$ where I^{t+1} is a $(t+1)$-cell whose boundary is S^t.

Since K_y is LC^n, there is a $\delta_1>0$ such that each mapping r of S^t, $0\leqslant t\leqslant n$, into $K_y \cap N_{\delta_1}(g)$ can be extended to a mapping R of I^{t+1} into $K_y \cap N_{\delta_1/2}(g)$. From the fact that f is fiber (A, C) covered and LC^n completely regular, there is $\alpha>0$ such that if $\varrho(z, y)<\alpha$ (ϱ also denoting a metric for Y), there is $h \in K_z$ and a mapping $s_{zy}\colon f^{-1}(z) \Rightarrow f^{-1}(y)$ which is a homeomorphism on a dense set $O_z \subset f^{-1}(z)$ open relative to $f^{-1}(z)$ such that (a) s_{zy} moves no point as much as $\frac{1}{2}\delta_1$, (b) $s_{zy} h = g$, and (c) for any $m \in K_y$, $s_{zy}^{-1} m$ (where s_{zy}^{-1} is defined on $s_{zy}(O_z)$ and $s_{zy}^{-1} m$ is defined on $m^{-1}(s_{zy}(O_z))$) can be extended (uniquely) to a mapping belonging to K_z.

Choose δ, $0<\delta<\min(\frac{1}{2}\delta_1, \frac{1}{2})$, such that if $K_z \cap N_\delta(z) \neq \emptyset$, then $\varrho(z, y)<\alpha$. Now, let $\varphi\colon S^t \to K_z \cap N_\delta(g)$. We can define a mapping H_{zy} of K_z into K_y as follows: For e in K_z, $H_{zy}(e) = s_{zy} e$. Clearly, $H_{zy}|(K_z \cap N_\delta(g))$ maps $K_z \cap N_\delta(g)$ into $K_y \cap N_{\delta_1}(g)$. Furthermore, $r = [H_{zy}|\varphi(S^t)]\varphi$ maps S^t into $K_y \cap N_{\delta_1}(g)$ and can be extended to a mapping R of I^{t+1} into $K_y \cap N_{\varepsilon/2}(g)$.

There is a mapping H_{yz} of K_y into K_z defined as follows: For $m \in K_y$, let $H_{yz}(m)$ be defined as the extension of $s_{zy}^{-1} m$ since $s_{zy}^{-1} m$ is defined on $m^{-1}(s_{zy}(O_z))$ and $s_{zy}^{-1} m$ extends to an element of K_z. Thus, $H_{yz}(m) \in K_z$. Now, $\Phi = [H_{yz}|K_y \cap N_{\varepsilon/2}(g)]R$ maps I^{t+1} into $K_z \cap N_\varepsilon(g)$ and agrees with φ on S^t the boundary of I^{t+1}. Consequently, K is equi-LC^n.

Now, let $F\colon K^* \Rightarrow Y$ be the function defined by $F(g) = x$ iff $g \in K_x$. The collection of point inverses under F is the collection K which is lsc and equi-LC^n. Also, K^* is a complete metric space. Given $x \in Y$, let $m(x)$ be an element of K_x. By Michael's Selection Theorem [10], there is an open set $U \subset Y$, $x \in U$, such that there is an extension of m to U (denote it by M) with the property that $M(u) \in K(u)$ for $u \in U$. It is easy to see that there is a continuous mapping $c\colon A\times U \Rightarrow f^{-1}(U)$ such that $c(A, u) = f^{-1}(u)$ and $c|(A, u) = M(u) \in K_u$. Hence, f is *locally* (A, C) *covered by the product* $A\times Y$. If Y is contractible, then it follows by using a standard Theorem ([22], p. 53) that f is (A, C) covered by the product $A\times Y$ provided Y is locally compact and separable.

6. Spherical modifications via upper semi-continuous decompositions. We shall describe an usc *spherical modification of type* k. Although the term spherical modification of type k is used in the literature (cf. A. H. Wallace, *Differential Topology*, First Steps, Benjamin, 1968), our method involves upper *semi-continuous decompositions instead of surgery*.

In some of the following theorems (proofs, in particular), we shall use a special n-manifold $A = S^{n-(k+1)}\times I^{k+1}$, a product of an $(n-(k+1))$-sphere with a $(k+1)$-cell. Let $E = S^{n-(k+2)}$ be an equator of $S^{n-(k+1)}$. Notice that

$$S^{n-(k+1)}\times I^{k+1} = \big((S^{n-(k+1)}-E)\times I^{k+1}\big) \cup (E\times I^{k+1}) .$$

Now, $(S^{n-(k+1)}-E)\times I^{k+1}$ is the union of two disjoint n-cells with a part of their boundaries missing, namely, that identified in $E\times I^{k+1}$. Now, *extend* this identification to the whole boundary of each n-cell in a canonical manner. The result is an n-sphere S^n. Note that this is obtained by a mapping $g\colon A\Rightarrow S^n$. Note that this is obtained by a mapping $g\colon A\Rightarrow S^n$ which is at most 2–1.

For fixed $a \in E = S^{n-(k+2)} \subset S^{n-(k+1)}$, consider (a, I^{k+1}) and $g(a, \partial I^{k+1})$ where ∂I^{k+1} is denoted by S^k, a k-sphere. Consider the decomposition space S^n/G_a where the only non-degenerate element of the usc decomposition G_a of S^n is $g(a, S^k)$, that is, G_a identifies a k-sphere of S^n to a point. We shall say that S^n/G_a is a k-*pinched* n-*sphere*.

Return to $A = \big((S^{n-(k+1)}-E)\times I^{k+1}\big) \cup (E\times I^{k+1})$. As before, A consists of two n-cells, Z_1 and Z_2, disjoint except for that part of their boundaries identified in $E\times I^{k+1}$. Consider the point $a \in E$ and a small ε-neighborhood $N_\varepsilon(a)$ of a in $S^{n-(k+1)}$). Now, $N_\varepsilon(a)\times I^{k+1}$ is n-cell with part of its boundary missing, namely, $\partial N_\varepsilon(a)\times I^{k+1}$. Next, consider $g|\big(A-N_\varepsilon(a)\times I^{k+1}\big)$. That is, extend the identification of the boundaries of Z_1 and Z_2 as described earlier except for that part of the boundaries of Z_1 and Z_2 in $N_\varepsilon(a)\times I^{k+1}$. Let $G_{a,\varepsilon}$ denote the usc decomposition of A such that the only non-degenerate elements of $G_{a,\varepsilon}$ are (i) those sets $g^{-1}g(x)$ which are non-degenerate for $x \in A-\overline{N_\varepsilon(a)}\times I^{k+1}$, (ii) the sets $(x, \partial I^{k+1})$ for $x \in N_\varepsilon(a)$, and (iii) for $x \in \partial N_\varepsilon(a)$, those sets $g^{-1}g(x)\times \partial I^{k+1}$. Thus, $A/G_{a,\varepsilon}$ is what we call an usc *spherical modification* (of S^n) of type k. It is, of course, an n-manifold without boundary. If $k = 0$ and $n = 2$, then $A/G_{a,\varepsilon}$ is a torus or 2-sphere with a 2-handle attached.

Usually, a spherical modification (of S^n) of type k is obtained by taking a (regular) neighborhood N of a (canonical) k-sphere S^k in S^n. Thus, $N \cong S^k\times I^{n-k}$ and $\partial N \cong S^k\times S^{n-k-1}$. Note that $S^k\times S^{n-k-1}$ is the boundary of $I^{k+1}\times S^{n-k-1}$. Consequently, S^n-N and $I^{k+1}\times S^{n-k-1}$ have homeomorphic boundaries and therefore can be "joined" by identifying their boundaries. The result is an n-manifold M without boundary. Hence, we have changed S^n to M by a spherical modification of type k. The same result can be achieved by using the upper semi-continuous decomposition described above. We shall give a more detailed description below and describe what we call a *continuous* spherical modification of type k.

7. Pinched products. Cartesian products and other fiberings are especially nice and useful. Spaces which are almost products are more abundant than the product spaces.

Suppose that each of X and Y is a topological space. Let $X\times Y$ denote the usual cartesian product with its topology. Also, suppose that G is an upper semi-continuous (usc) decomposition of $X\times Y$ such that each *non-degenerate* element g of G lies in (X, y) for some $y\in Y$. That is, G decomposes $X\times Y$ on the "levels" over Y. We call the decomposition space a *pinched product*. Thus, a space S is a *pinched product* iff there is a quasi-compact mapping [24] φ of $X\times Y$ onto S such that for each $s\in S$, $(X, y)\supset\varphi^{-1}(s)$ for some $y\in Y$. Thus, if $G=\{\varphi^{-1}(s)|\ s\in S\}$, then G is an usc decomposition of $X\times Y$ and the decomposition space $X\times Y/G$ is homeomorphic to S. We say that a pinched product is *simple* iff for each $y\in Y$, (X, y) contains *at most one non-degenerate element of* G. In many interesting cases, the collection H of non-degenerate elements is finite.

In this paper, we make use of (1) $X\times I$ where X is a compact n-manifold, $I=[0,1]$, and (2) simple pinched products $X\times I/G$. Furthermore, in our applications, $X\times I/G$ is a compact $(n+1)$-manifold. Non-trivial answers to the following questions would be of interest:

QUESTION. Under what conditions is the simple pinched product of compact manifolds a manifold?

8. A special case. *Pinched products of certain manifolds with the interval* $I=[0,1]$ *and continuous spherical modifications of type* k. In applications, we encountered the following kind of pinched product of spheres and cells such that the pinched product is also a manifold.

Consider $A=S^{n-(k+1)}\times I^{k+1}$ — the product of an $n-(k+1)$ sphere with a $k+1$ cell. Now, consider the product $A\times I$ and the following usc decomposition G of $A\times I$.

Consider an equator (fixed for all that follows) $E=S^{n-(k+2)}$ of $S^{n-(k+1)}$. Notice (as before) that A consists of two n-cells Z_1 and Z_2 which are disjoint except for that part of their boundaries identified in $E\times I^{k+1}$. For $0\leqslant x<\frac{1}{2}$, consider (A, x) and extend this identification of $(\partial Z_1\cap(E\times I^{k+1}), x)$ with $(\partial Z_2\cap(E\times I^{k+1}), x)$ in a *canonical manner* to all of $(\partial Z_1\cup\partial Z_2, x)$. We use the "same" identification e for each x. The various pairs of points identified are non-degenerate elements of the usc decomposition G.

For $x\geqslant\frac{1}{2}$, consider a fixed $a\in E$. Now, identify $(\partial Z_1, \frac{1}{2})$ with $(\partial Z_2, \frac{1}{2})$ as above. Next, identify $((a, \partial I^{k+1}), \frac{1}{2})$ to a point p. All pairs of points identified in $(\partial Z_1\cup\partial Z_2-(a, \partial I^{k+1}), \frac{1}{2})$ are non-degenerate elements of G as well as the set $((a, \partial I^{k+1}), \frac{1}{2})$.

For $\frac{1}{2}<x\leqslant 1$, let a be as above. Let $\varepsilon_x=\frac{1}{2}(x-\frac{1}{2})$ and $N_{\varepsilon_x}(a)$ denote an ε_x-neighborhood of a. Now, $N_{\varepsilon_x}(a)\times I^{k+1}$ is an n-cell with part of its boundary missing, namely, $\partial N_{\varepsilon_x}(a)\times I^{k+1}$. On $(\partial Z_1\cup\partial Z_2)-N_{\varepsilon_x}(a)\times I^{k-1}$, use the identification e of ∂Z_1 with ∂Z_2 described above. Finally, (i) identify $((t, \partial I^{k+1}), x)$ to a point for each $t\in(N_{\varepsilon_x}(a)\cup\partial N_{\varepsilon_x}(a))$. Pairs of points of $\partial N_{\varepsilon_x}(a)\times\partial I^{k+1}$ have been identified by e. Thus, $\overline{N_{\varepsilon_x}(a)}$ under the identification e of points on $\partial N_{\varepsilon_x}$ becomes an $(n-(k+1))$-sphere. Next, identify $(t, \partial I^{k+1})$ to a point for each $t\in\overline{N_{\varepsilon_x}(a)}$. Call this identification e'. Now, *under* e and e', $\overline{N_{\varepsilon_x}(a)}\times I^{k+1}$ becomes N, say, which is homeomorphic to $I^{n-(k+1)}\times S^{k+1}$ with certain points of $\partial I^{n-(k+1)}\times S^{k+1}$ identified corresponding to the points of $\partial N_{\varepsilon_x}(a)\times I^{k+1}$ identified by e. Thus, ∂N is homeomorphic to S^{n-1} with a tame or flat k-sphere shrunk to a point. Let the non-degenerate elements of G at the x level be denoted by H_x. All of the point sets above which are identified with points constitute the non-degenerate elements H of an usc decomposition G of $A\times I$. The remaining elements of G are those singletons not identified with any other point. It should be observed (as it will be used soon) that the non-degenerate elements of G which intersect $\overline{N_{\varepsilon_x}(a)}$ constitute an $(n-(k+1))$-sphere in the decomposition space $A\times I/G$.

THEOREM 6. *The pinched product* $A\times I/G$ *is homeomorphic to a differentiable* $(n+1)$-*manifold* N. *In fact,* $A\times I/G$ *is what we call a continuous spherical modification of type* k.

Proof. Let P denote the usual projection mapping of $A\times I$ onto $A\times I/G$, the decomposition space (or quotient space). Let G_x, $0\leqslant x\leqslant 1$, denote the "restriction" of G to (A, x), i.e., G_x is the usc decomposition of $A\times I$ at level x. Clearly, $(A, 0)/G_0=M_0$ and $M_1=(A, 1)/G_1$ are compact connected n-manifolds. Furthermore, M_1 is a spherical modification of M_0 of type k. Now, let $\overline{M}_0$ and $\overline{M}_1$ be differentiable manifolds homeomorphic to M_0 and M_1, respectively. There is a differentiable manifold M whose boundary is the disjoint union of $\overline{M}_0$ and $\overline{M}_1$ and a differentiable function f on M equal to 0 on $\overline{M}_0$, equal to 1 on $\overline{M}_1$, and otherwise having values between 0 and 1 and having *exactly* one non-degenerate critical point p (with critical value $\frac{1}{2}$, say) with type number $k+1$ [21].

Now, for $0\leqslant x<\frac{1}{2}$, $f^{-1}(x)\cong S^n\cong M_0\cong\overline{M}_0$, $f^{-1}(\frac{1}{2})\cong S^n$ with a k-sphere shrunk to a point, and for $\frac{1}{2}<x\leqslant 1$, $f^{-1}(x)\cong M_1\cong\overline{M}_1$. Furthermore, there is a "smooth" closed and connected set C such that (1) for $0\leqslant x<\frac{1}{2}$, $C_x=C\cap f^{-1}(x)\cong S^k$, (2) $f^{-1}(\frac{1}{2})\cap C=p$, the critical point of f, (3) for $\frac{1}{2}<x\leqslant 1$, $C_x=C\cap f^{-1}(x)\cong S^{n-(k+1)}$, and (4) C is "canonical" in the sense of Wallace [21; p. 88]. Consider the trajectories to the level sets of f. The trajectories starting at points of $C_0\cong S^k$ all end at p. As we move through the levels of f from $\overline{M}_0$ to $\overline{M}_1$, the $C_x\cong S^k$ shrink to p along the orthogonal trajectories. As we continue above the critical level, $f^{-1}(\frac{1}{2})$, $C_x\cong S^{n-(k+1)}$ grows along the orthogonal trajectories from p to $C_1\subset\overline{M}_1$. Thus, in this sense, C is "canonical."

Observe that there is a continuum Z in $A\times I/G$ which is homeomorphic to C such that (letting $Z_x=(A, x)\cap Z$), (1) $Z_x\cong C_x$, (2) for $0\leqslant x\leqslant\frac{1}{2}$, $Z_x=((a, \partial I^{k+1}), x)$ where a is the same point for the various x (note that for $x=\frac{1}{2}$, Z_x is a non-degenerate element of G), and (3) for $\frac{1}{2}<x<1$, $\leqslant Z_x$ consists of exactly those non-degenerate elements of G_x which intersect $\overline{N_{\varepsilon_x}(a)}\times I^{k+1}$ (Z_x is homeomorphic to $S^{n-(k+1)}$ as a subspace of $A\times I/G$). Let h denote a homeomorphism of Z onto C such that $h(Z_x)=C_x$.

A mapping $g\colon(A, x)$ onto a space Y "generates" the usc decomposition G_x iff $\{g^{-1}(y)|\ y\in Y\}$ is the collection G_x corresponding to (A, x). The kernel of g is $\{b\in(A, x)|\ g^{-1}g(b)=b\}$.

In order to prove Theorem 6, it is convenient to consider certain mappings of A onto the various $f^{-1}(x)$, $x \in [0, 1]$. We could use Theorem 5 to prove Theorem 6 by showing the existence of certain spaces of mappings which satisfy the hypotheses of Theorem 5. In fact, Theorem 5 is set up for such an application. However, we will give a more direct proof which hopefully will provide insight to the somewhat complicated nature of this and other similar theorems.

For $x \neq \frac{1}{2}$, let K_x denote the space of all mappings $g: (A, x) \Rightarrow f^{-1}(x)$ such that (1) g generates G_x and (2) $g(P^{-1}(Z_x)) = C_x$. (Recall that $P: A \times I \Rightarrow A \times I/G$ is the usual projection or quotient mapping.) We further require that $h|Z_x = g|P^{-1}(Z_x)$. The metric for K_x is the usual sup. metric.

For $x = \frac{1}{2}$, let $K^0_{1/2}$ denote the space of *all* mappings g of $(A, \frac{1}{2})$ onto $f^{-1}(\frac{1}{2})$ such that (1) g generates $G_{1/2}$, (2) $g(p^{-1}(Z_{1/2})) = p$, and (3) g can be factored by first mapping A onto $f^{-1}(x)$ by $s \in K_x$, $0 \leqslant x < \frac{1}{2}$ and then mapping $f^{-1}(x)$ onto $f^{-1}(\frac{1}{2})$ by a mapping m such that m is a homeomorphism on $f^{-1}(x) - m^{-1}(p)$ where $m^{-1}(p) = s(a, \partial I^{k+1})$ — a k-sphere. In fact, for each *fixed* m and *any* $g \in K^0_{1/2}$, $g = ms$ for *some* $s \in K_x$.

Let $K^1_{1/2}$ denote the space of *all* mappings $g: (A, \frac{1}{2}) \Rightarrow f^{-1}(\frac{1}{2})$ such that $g = mz$ where $z \in K_x$, $\frac{1}{2} < x < 1$, and m is a mapping of $f^{-1}(x)$ onto $f^{-1}(\frac{1}{2})$ such that (1) $m(C_x) = p$, (2) $m|(f^{-1}(x) - C_x)$ is a homeomorphism, and (3) g is the limit of a sequence $\{z_i\}$, $z_i \in K_{x_i}$, where $\{x_i\} \to \frac{1}{2}$ with $\frac{1}{2} < x_i \leqslant 1$. In order that g be the limit of a sequence $\{z_i\}$, it is usually the case that z_i and g have the same domain. For this purpose, we could (a) identify (A, x) with A for all x or (b) we could, however, define $\{z_i\} \to g$ to mean that for $(a, \frac{1}{2}) \in (A, \frac{1}{2})$, $\{z_i(a, z_i)\} \to g(a, \frac{1}{2})$ where $\{x_i\} \to \frac{1}{2}$. We shall use whichever is convenient without reference either (a) or (b). This should be apparent in the arguments below.

We shall consider two collections: (1) The collection L_0 of all K_x, $0 \leqslant x \leqslant \frac{1}{2}$ with $K_{1/2} = K^0_{1/2}$ and (2) the collection L_1 of all K_x, $\frac{1}{2} \leqslant x \leqslant 1$ with $K_{1/2} = K^1_{1/2}$. Now, L^*_i will denote the union of the elements of L_i. Next, we define a metric for L^*_i (like the one *suggested* for the collection of spaces in Theorem 5). If $m \in L^*_i$, let $\hat{m}$ denote the graph of m in $(A \times I) \times M$. Thus, for each pair $m, n \in L^*_i$ where $m \in K_a$ and $n \in K_b$, let $D(m, n) = H(\hat{m}, \hat{n})$ where H denotes the Hausdorff metric on the space of all closed subsets of $(A \times I) \times M$. Now, (L^*_i, D) is a topologically complete metric space (which should be apparent, later). For a proof, see an argument in ([8], Theorem 1) for an analogous result. We let ϱ denote a complete metric for L^*_i.

LEMMA 6.1. *Each K_x is* LC^0 (*in the homotopy sense*). *Indeed, K_x is locally contractible.*

Proof. For $x \neq \frac{1}{2}$, $g \in K_x$ is a certain mapping of $A \Rightarrow f^{-1}(x)$ which generates G_x, and $g(P^{-1}(Z_x)) = C_x$, indeed, $h|Z_x = g|P^{-1}(Z_x)$. Note that g is a homeomorphism on Int$A$ (meaning $A - \partial A$ since A is an n-manifold with boundary). Now, g generates G_x and the decomposition space A/G_x is homeomorphic to $f^{-1}(x)$. Thus, g corresponds naturally to a homeomorphism $\bar{g}$ of $f^{-1}(x)$ onto itself which leaves C

fixed (pointwise). In fact, K_x is homeomorphic to the space H_x of all homeomorphisms of $f^{-1}(x)$ onto itself which leave C_x fixed. It follows from results of Edwards and Kirby [4] that H_x (and consequently K_x) is locally contractible.

If $x = \frac{1}{2}$, then we have two cases: (1) $K_{1/2} = K^0_{1/2}$ and (2) $K_{1/2} = K^1_{1/2}$. Although $f^{-1}(\frac{1}{2})$ is not a manifold, it is homeomorphic to an n-sphere with a certain k-sphere identified to a point. In case (1), it follows from using results of [25] that $K^0_{1/2}$ is locally contractible. Similarly, it follows that $K^1_{1/2}$ is locally contractible.

LEMMA 6.2. *The collections L_i, $i = 0, 1$, are equi*-LC^n.

Proof. Each L^*_i is a complete metric space with metric ϱ. Note that $f|[0, \frac{1}{2})$ is completely regular (in the sense of Dyer and Hamstrom [3]). It follows by an argument analogous to that given in [3] that the collection of all K_x, $0 \leqslant x < \frac{1}{2}$, is equi-LC^n (for all n). To show that L_0 is LC^n, we need only consider $\varepsilon > 0$ and $g \in K_{1/2} = K^0_{1/2}$.

Since $K_{1/2}$ is LC^n, there is a $\delta_1 > 0$ such that each mapping $r: S^k \to K_{1/2} \cap N_{\delta_1}(g)$, for $0 \leqslant k \leqslant n$, can be extended to a mapping $R: I^{k+1} \to K_{1/2} \cap N_{\varepsilon/2}(g)$. By observing that $f|(M - C)$ is completely regular, it is not difficult to show that there is $\alpha > 0$ such that if $\frac{1}{2} - b < \alpha$, $b \in [0, \frac{1}{2}]$, there is a mapping $m: f^{-1}(b) \Rightarrow f^{-1}(\frac{1}{2})$ such that $m(C_b) = C_{1/2}$, $m|(f^{-1}(b) - C_b)$ is a homeomorphism, and m moves no point as much as $\frac{1}{2}\delta_1$.

Choose δ, $0 < \delta < \min(\frac{1}{2}\delta_1, \frac{1}{2})$ such that if $K_b \cap N_\delta(g) \neq \emptyset$, then $\frac{1}{2} - b < \alpha$. Now, let $\varphi: S^k \to K_b \cap N_\delta(g)$. We wish to show that φ can be extended to $\Phi: I^{k+1} \to K_b \cap N_\varepsilon(g)$. Let $c = \frac{1}{2}$. We can define 1-1 mapping $H_{bc}: K_b \to K_c$ as follows: By choice of α, $\frac{1}{2} - b < \alpha$ and consequently there is a mapping $m: f^{-1}(b) \Rightarrow f^{-1}(\frac{1}{2})$ as described above. Now, m is fixed for the remainder of the argument. For $e \in K_b$, let $H_{bc}(e) = me \in K_c$. Clearly, $H_{bc}|(K_b \cap N_\delta(g))$ maps $K_b \cap N_\delta(g)$ into $K_c \cap N_{\delta_1}(g)$ since m moves no point as much as $\frac{1}{2}\delta_1$ and $\delta < \frac{1}{2}\delta_1$. In fact, H_{bc} maps K_b *onto* K_c since $K_c = K^0_{1/2}$ and from part (3) of the definition of K_c. Recall that for *fixed* m and *any* $u \in K_c$, there is $e \in K_b$ such that $u = me$. Furthermore, $r = [H_{bc}|\varphi(S^k)]\varphi$ maps S^k into $K_c \cap N_{\delta_1}(g)$ and can be extended to a mapping $R: I^{k+1} \to K_c \cap N_{\varepsilon/2}(g)$ such that for each $p \in I^{k+1}$, $R(p) \in H_{bc}(K_b) \subset K_c$ since $H_{bc}(K_b) = K_c$ is LC^n. Now, define $H_{cb}: H_{bc}(K_b) \to K_b$ as $H_{cb}(me) = e$. Clearly, H_{cb} is the inverse of H_{bc} and H_{bc} is a homeomorphism. Now, $\Phi = [H_{cb}|H_{bc}(K_b) \cap N_{\varepsilon/2}(g)]R$ maps I^{k+1} into $K_b \cap \cap N_\varepsilon(g)$ and agrees with φ on S^k the boundary of I^{k+1}. Thus, L_0 is equi-LC^n. Similarly, it follows that L_1 is equi-LC^n.

LEMMA 6.3. *The collections L_i are lower semi-continuous* (lsc) *in the sense that if $\{x_i\} \to x$ in $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$, then K_x is in the closure of $\bigcup K_{x_i}$.*

A proof follows easily from the facts that (1) $f|f^{-1}[0, \frac{1}{2})$ and $f|(\frac{1}{2}, 1)$ are completely regular and (2) the definitions of $K^0_{1/2}$ and $K^1_{1/2}$.

Next, let $F: L^*_0 \Rightarrow [0, \frac{1}{2}]$ be the *function* defined by $F(k) = x$ iff $k \in K_x$. Thus, the collection of point inverses under F is the collection L_0 which is lsc and equi-LC^n. Also, L^*_0 is a complete metric space. Given $x \in [0, \frac{1}{2}]$, let $\varphi(x) \in K_x$. By Michael's Selection Theorem [10], there is an open set U of $[0, \frac{1}{2}]$ with $x \in U$ on a continuous extension of φ to U (denote it by Φ) with the property that $\Phi(u) \in K$ for each $u \in U$. Clearly, $[0, \frac{1}{2}]$ is covered by a finite number of closed intervals $[a_i, b_i]$ where $a_0 = 0 < b_0 = a_1 < b_1 = a_2 < b_2 \ldots < b_t = \frac{1}{2}$ with homeomorphisms $h_i: P(A \times [a_i, b_i]) \Rightarrow f^{-1}[a_i, b_i]$. Next, we sew the pieces together in the obvious way. Identify $h_i(a, a_i)$ with $h_{i+1}(a, a_i)$ for $i = 0, 1, \ldots, t-1$. We obtain a homeomorphism $H_0: P(A \times [0, \frac{1}{2}]) \Rightarrow f^{-1}[0, \frac{1}{2}]$. In a similar way, we obtain a homeomorphism $H_1: P(A \times [\frac{1}{2}, 1]) \Rightarrow f^{-1}[\frac{1}{2}, 1]$. We sew these together to obtain a homeomorphism $H: A \times I/G \Rightarrow f^{-1}[0, 1] = M$. Theorem 6 is proved.

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