Association and fixed points

by

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Abstract. Let ab be a continuum irreducible between a and b and let Ab denote the set of irreducibility related to the point a, i.e., the set of all points x such that ab is irreducible between x and b. The continuum ab is said to be associated with ac at the point a, in symbols $ab \multimap ac$, if Ab is equal to Ac and the intersection $ab \cap ac$ is a continuum different from this common set of irreducibility. The notion of association, introduced and investigated in part I of this paper, serves to prove in part II that if X is a hereditarily unicoherent and hereditarily decomposable metric continuum, then for every upper semi-continuous continuum-valued function F which maps X into itself there exists a fixed point, i.e., a point x such that $x \in F(x)$.

The present paper consists of two parts.

In part I we investigate a binary relation due to Lelek [4], p. 134, between irreducible continua in a λ-dendroid. This relation, called by me association and denoted by $\multimap$, is a generalization of the relation holding between segments ab and ac in a Euclidean space when they are collinear and the point a does not lie between b and c.

The properties of the association described in part I may perhaps be of some interest in themselves, but they are used to prove in part II that if X is a λ-dendroid, then for every upper semi-continuous continuum-valued function F which maps X into itself there exists a fixed point x, i.e., such a point x that $x \in F(x)$.

This theorem (see Corollary, § 6) generalizes an analogous result of Ward (see [5]) and simultaneously answers a question of Charatonik (see [1]). In the particular case where F is single-valued it gives, moreover, a solution to a problem raised by Professor B. Knaster in the New Scottish Book (problem 526 dated November 22, 1960).

I. Association

§ 1. Preliminaries on irreducible continua. In this paper a *continuum* means a metric, connected and compact space.

A continuum X is said to be *irreducible between the points a and b* provided that X contains a and b and no other subcontinuum of X contains both these points; then a (and also b) is called a *point of irreducibility* of X. This definition is equivalent to saying that X contains a and is not a union of two proper subcontinua which both contain a (see [3], § 48, I, Theorem 4 and [2], p. 270).

A continuum is said to be *unicoherent* if the intersection of every two of its subcontinua whose union gives the whole continuum is a continuum. A continuum is said to be *decomposable* if it is a union of two continua not contained in one another. A property of a continuum X is said to be *hereditary* if every subcontinuum of X that is non-trivial (i.e., contains more than one point) has this property. Following Charatonik (see e.g. [1]), a hereditarily unicoherent and hereditarily decomposable continuum is called a λ*-dendroid. In what follows X will always denote an arbitrary* λ*-dendroid.*

According to the well-known Brouwer reduction theorem (see e.g. [3], § 42, IV) for every two points of any continuum there exists an irreducible subcontinuum between them (see e.g. [3], § 48, I, Theorem 1). It follows from the hereditary unicoherence of X that for every two points $a, b \in X$ such an irreducible subcontinuum is unique; we denote it by ab.

Therefore we have for every continuum $K \subset X$

(i) $$a \in K \text{ and } b \in K \quad \text{imply} \quad ab \subset K,$$

and consequently for every continuum $ac \subset X$

(ii) $$ab \subset ac \quad \text{implies} \quad ab \cup bc = ac.$$

Definition 1. The set of all points $x \in X$ such that $xb = ab$ is called the *set or irreducibility* related to the point a in ab; we denote this set by Ab.

From this definition it follows that $a \in Ab \subset ab$. Also $b \notin Ab$ and $Ab \neq ab$ whenever ab is non-trivial. Moreover, by the assumed hereditary decomposability of X, the set Ab is a continuum (see [2], § 1, p. 239). In view of (i), for every continuum $K \subset X$

(iii) $$a \notin K \text{ and } b \in K \quad \text{imply} \quad Ab \cap K = \emptyset.$$

In the sequel we do not use the results of the structural theory of irreducible continua by Kuratowski (see e.g. [2]). In particular we do not use the main concept of that theory, namely the concept of a layer. Instead we express relations between irreducible subcontinua of X by the corresponding relations between sets of irreducibility.

§ 2. Basic properties of association. Given a point $a \in X$, consider non-trivial subcontinua of X which contain a as a common point of irreducibility, i.e., non-trivial continua of the form ab for some b.

Definition 2. We say that ab and ac are *associated* at a if

$$Ab = Ac \quad \text{and} \quad Ab \neq ab \cap ac.$$

Then we write $ab \mathrel{-\!\!<} ac$, setting a as first in the denotation of these irreducible continua, instead of using a as an index in $\mathrel{-\!\!<}$.

Example 1. Let X be the union of a non-trivial segment ab and segments bc and bd in the Euclidean plane such that the intersection of every two of them equals b. Then $ab \mathrel{-\!\!<} ac$ and also $ac \mathrel{-\!\!<} ad$ (Fig. 1).

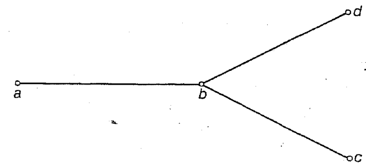

Fig. 1

Example 2. Let X be the union of two curves of the Euclidean plane, namely $y = \sin(\pi/x)$ where $0 < |x| \leqslant 1$, and of the segment with end-points $b = (0, -1)$ and $d = (0, 2)$. Put $a = (-1, 0)$ and $c = (1, 0)$. Then $ab \mathrel{-\!\!<} ac$ and $ac \mathrel{-\!\!<} ad$, whereas neither $bc \mathrel{-\!\!<} bd$ nor $bc \mathrel{-\!\!<} ba$ (Fig. 2).

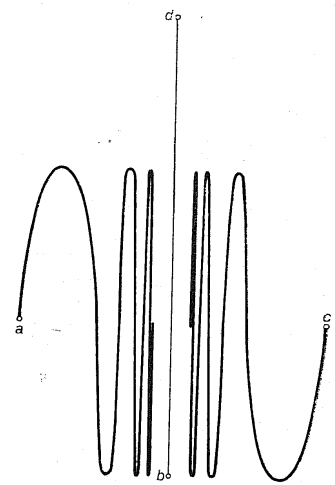

Fig. 2

We have the following propositions, related to the notion of association.

Proposition 1. *In order that* $ab \mathrel{-\!\!<} ac$, *it is necessary and sufficient that* $a \notin bc$.

Proof. Necessity. By Definition 2, there exists an $x \in ab \cap ac$ such that $x \notin Ab$ and $x \notin Ac$. Then $xb \subsetneq ab$ and $xc \subsetneq ac$ according to Definition 1 and (i). Hence $a \notin xb \cup xc$ by the irreducibility of ab and ac. Since $bc \subset xb \cup xc$ by (i), it follows that $a \notin bc$.

Sufficiency. We have $Ab \subset ab$ and $ab \subset ac \cup bc$ by (i). Moreover, the assumption $a \notin bc$ implies by (iii) that

$$Ab \cap bc = \emptyset . \tag{1}$$

Therefore if $x \in Ab$, then $x \in ac$, and so $xc \subset ac$ by (i). Then also $ab = xb$ by Definition 1. Since $xb \subset xc \cup bc$ according to (i), it follows from the assumption that $a \in xc$, so that $ac \subset xc$ by (i). Hence $xc = ac$, i.e., $x \in Ac$ by Definition 1. Thus $a \notin bc$ implies $Ab \subset Ac$. By symmetry, it also implies $Ac \subset Ab$; therefore $Ab = Ac$.

Now, by (i), $bc \subset ab \cup ac$. Hence $bc = ab \cap bc \cup ac \cap bc$, and therefore $ab \cap ac \cap bc \neq \emptyset$. It follows from (1) that $Ab \neq ab \cap ac$.

PROPOSITION 2. *The association* $-\!\!<$ *is symmetric, reflexive and transitive* (see Lelek [4], p. 134).

Proof. The symmetry follows directly from Definition 2.

Reflexivity. If $ab = ac$, then $bc \subset Ba$ according to Definition 1 and to (i). Since $a \notin Ba$ for any non-trivial ab, it follows that $a \notin bc$, and hence $ab -\!\!< ac$ by Proposition 1.

Transitivity. If $ab -\!\!< ac$ and $ac -\!\!< ad$, then, by Proposition 1, $a \notin bc \cup cd$. But $bd \subset bc \cup cd$ in view of (i). It follows that $a \notin bd$, and hence $ab -\!\!< ad$ by Proposition 1.

PROPOSITION 3. *For* $ab \subset ac$, *in order that* $ab -\!\!< ac$, *it is necessary and sufficient that* $b \notin Ac$.

Proof. Since $bc \subset ac$ by (ii), it follows from the irreducibility of ac that $a \notin bc$ means $bc \neq ac$, i.e., $b \notin Ac$ by Definition 1. Applying Proposition 1, we complete the proof.

DEFINITION 3. We say that ab is a *segment* (*with an initial point* a) of ac if $ab \subset ac$ and $ab -\!\!< ac$, i.e., by Proposition 3 and (i), $b \in ac - Ac$.

Directly from Definitions 3 and 2, taking Proposition 2 into account, we obtain two propositions.

PROPOSITION 4. *The relation* ab *being a segment of* ac *is reflexive and transitive.*

PROPOSITION 5. *In order that* $ac -\!\!< ad$, *it is necessary and sufficient that there exist a common segment* ab *of* ac *and* ad.

Not losing sight of the point a, consider association also at another point b.

PROPOSITION 6. *If* $ab \subset ad$ *and* $bc -\!\!< bd$, *then* $ab \subset ac$.

Proof. We have $b \in ad$ and $b \notin cd$ according to Proposition 1. Since $ad \subset ac \cup cd$ by (i), it follows that $b \in ac$. Therefore, by (i), $ab \subset ac$.

PROPOSITION 7. *If* ab *is a segment of* ad *and if* $bc -\!\!< bd$, *then* ab *is a segment of* ac.

Proof. In view of Proposition 6, it suffices to show that $b \notin Ac$.

We have $b \notin Ad$ according to Definition 3. Hence $a \notin Bd$ by Definition 1. By Definition 2, the assumed association implies $Bd = Bc$. Therefore $a \notin Bc$, and consequently, by Definition 1, $b \notin Ac$.

Proposition 7 immediately implies

PROPOSITION 8. *If* ab *is a segment of* ad *and if* bc *is a segment of* bd, *then* ab *is a segment of* ac.

PROPOSITION 9. *If* ab *is a segment of* ad *and if* bc *is a segment of* bd, *then* ac *is a segment of* ad.

Proof. We have to prove that $ac -\!\!< ad$ and $ac \subset ad$.

Applying Proposition 8, we see that ab is a common segment of ac and of ad. Hence $ac -\!\!< ad$ by Proposition 5. By Definition 3 and (ii), also $ab \cup bc = ac$ and $ab \cup bd = ad$. Since $bc \subset bd$, bc being a segment of bd by assumption, it follows that $ac \subset ad$.

PROPOSITION 10. *If* $ab = \overline{\bigcup ab_\tau}$, *where* τ *runs over an arbitrary set, then for each point* $p \in ab - Ba$ *there exists an index* τ *such that* pb_τ *is a segment of* pb *and that*

$$p \in ab_\tau - B_\tau a . \tag{2}$$

Proof. In view of (i) and Definition 1 we have $ap \subsetneq ab$. Hence by (ii),

$$ap \cup pb = ab , \tag{3}$$

and $b \notin ap$ by the irreducibility of ab; also pb is non-trivial, so that $b \notin Pb$. It follows that $ap \cup Pb$ is a proper subcontinuum of $ab = \overline{\bigcup ab_\tau}$. Then, taking (i) into account, we infer that there exists a point

$$b_\tau \in ab - (ap \cup Pb) , \tag{4}$$

and hence $b_\tau \in pb - Pb$ by (3). Therefore by Definition 3, pb_τ is a segment of pb, whence in particular $pb_\tau -\!\!< pb$. Since $ap \subset ab$ by (3), it follows by Proposition 6 that $ap \subset ab_\tau$. But $b_\tau \notin ap$ by (4); thus $ap \subsetneq ab_\tau$, and therefore (2) by Definition 1.

Remark 1. One can see that in the above proof of Proposition 10 it is essential to assume that Pb is a continuum, namely, that the continuum ab is hereditarily decomposable.

Remark 2. It is worth noticing that the following statement is true:

If a sequence of irreducible subcontinua ab_j *of any hereditarily unicoherent continuum is increasing, then* $\overline{\bigcup ab_j} = ab$ *for some point* b.

In fact, for every proper subcontinuum of the continuum $\overline{\bigcup ab_j}$ which contains a there exists, by (i), a point b_j not belonging to this subcontinuum. Since the sequence of ab_j is increasing by assumption, it follows that the continuum $\overline{\bigcup ab_j}$ is not a union of two proper subcontinua which both contain a. Thus a is a point of irreducibility of $\overline{\bigcup ab_j}$, i.e., $\overline{\bigcup ab_j} = ab$ for some point b.

§ 3. **Prolongable segments.** Given a non-trivial continuum $ab \subset X$, consider the set $ab - Ba$. Obviously $a \in ab - Ba$, and $ab - Ba$ is the union of all proper subcontinua of ab which contain a (in fact, this is simply the *composant* of a in ab; for this concept see [3], § 48, VI). It is known (see ibidem [3], Theorem 2) that

(1) $$\overline{ab - Ba} = ab\,.$$

Now observe that, in view of (i) and Definition 1,

(2) $$p \in ab - Ba \quad \text{means} \quad ap \underset{\neq}{\subset} ab\,;$$

thus the set $ab - Ba$ is also the union of all proper subcontinua of ab which contain a as a point of irreducibility.

If p runs over $ab - Ba$, then the sets Bp are constant, namely

(3) $$Bp = Ba$$

according to Proposition 3 and Definition 2, while the opposite sets Pb give a decomposition of $ab - Ba$ into disjoint continua, which follows from Definition 1. Defining Pb as earlier than $P'b$ if pp' is a segment of pb, we obtain an order between members of this decomposition which directs them from Ab to Ba. Indeed, transitivity follows directly from Proposition 9. The same argument as in the proof of Proposition 10 can be used to show directivity, i.e., that

PROPOSITION 10*. *If $p, p' \in ab - Ba$, then there exists a $p'' \in ab - Ba$ such that pp'' is a segment of pb and $p'p''$ is a segment of $p'b$.*

Indeed, properties (1) and (2) of the set $ab - Ba$ imply the assumptions of Proposition 10; namely $ab = \overline{ab - Ba}$ by (1) and $ab - Ba = \bigcup ap$, where p runs over $ab - Ba$ in view of (2) and (iii). Therefore, in the same way as the point b_τ is found in the proof of Proposition 10 (see (4)), we may find a point p'' which satisfies the conditions required.

Now for non-trivial continua $ab \subset ad \subset X$ we are interested not only in the sets Ab and Ad but also in the orders: in the order over $ab - Ba$ and in the restriction to $ab - Ba$ of the order over $ad - Da$. *These two orders may be different.*

EXAMPLE 3. Let X be the union of the curve $x = \sin(\pi/y)$, where $0 < y \leqslant 1$, and of the segment with end-points $a = (-2, 0)$ and $b = (1, 0)$ in the Euclidean plane. Put $d = (0, 1)$, $p = (-1, 0)$ and $p' = (0, 0)$. Then ab is a segment of ad, whereas pp', being a segment of pb, is not a segment of pd. Moreover, no segment of pb is a segment of pd, i.e., in view of Proposition 5, the association $pb -\!\!< pd$ does not hold (Fig. 3).

DEFINITION 4. We say that a non-trivial continuum ab is a *prolongable segment* (*with an initial point* a) of ad if for each $p \in ab - Ba$ the association $pb -\!\!< pd$ holds.

Then clearly $ab -\!\!< ad$. Since for each $p \in ab - Ba$ we have $ap \subset ab$ by (2), whence $pb -\!\!< pd$ implies $ap \subset ad$ by Proposition 6, it follows that also $ab \subset ad$

by (1). Thus ab is a segment of ad, which justifies the term *prolongable segment* proposed. We omit the easy proof of the above-mentioned identity of these orders.

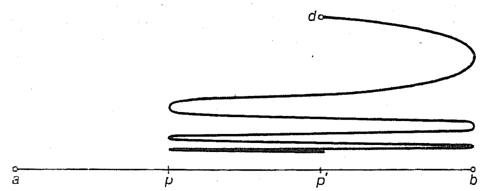

Fig. 3

PROPOSITION 11. *A non-trivial continuum ab is a prolongable segment of ad if and only if for each $p \in ab - Ba$ there exist segments pq and pr of pb such that $pq \subset pr$ and $qr -\!\!< qd$.*

Proof. Necessity. Let $p \in ab - Ba$. Putting in Proposition 10* $p' = p$ and $p'' = q$, we infer that there exists a segment pq of pb such that $q \in ab - Ba$. It follows according to Definition 4 that $qb -\!\!< qd$. Since pb is a segment of pb, the condition holds.

Sufficiency. For each $p \in ab - Ba$ there exist, by assumption, segments $pq \subset pr$ of pb such that $qr -\!\!< qd$. By Proposition 2 and Definition 3, pq is a segment of pr; therefore pq is a segment of pd by Proposition 7. By Proposition 5, the association $pb -\!\!< pd$ holds.

PROPOSITION 12. *A non-trivial continuum ab is a prolongable segment of ad if and only if for each $p \in ab - Ba$ there exist segments pp' of pb and $p'q$ and $p'r$ of $p'b$ such that $p'q \subset p'r$ and $qr -\!\!< qd$.*

The necessity follows from Propositions 10* and 11. To show the sufficiency it is enough, by Proposition 11, to prove the following

PROPOSITION 12*. *If pp' is a segment of pb and if $p'q \subset p'r$ are segments of $p'b$, then pq and pr are segments of pb such that $pq \subset pr$.*

Indeed, pq and pr are segments of pb by Proposition 9. By Proposition 8, also pp' is a segment of pq and of pr, whence $pp' \cup p'q = pq$ and $pp' \cup p'r = pr$ according to Definition 3 and (ii). Therefore the assumed inclusion $p'q \subset p'r$ implies that $pq \subset pr$.

Observe now that Proposition 1 implies, directly by Definition 4, the following criterion for prolongable segments.

PROPOSITION 13. *A non-trivial continuum ab is a prolongable segment of ad if and only if $ab \cap bd \subset Ba$.*

The above criterion enables us to prove the following

PROPOSITION 14. *Let subcontinua ab and bd of X be given. If $Bd \underset{\neq}{\subset} Ba$, then ab is a prolongable segment of ad.*

Proof. If, on the contrary, there exists a point $p\in(ab-Ba)\cap bd$, then $Bp = Ba$ by (3). Also $p\in bd-Bd$ by the assumed inclusion, whence similarly $Bp = Bd$. Thus $Bd = Ba$, which contradicts the assumption.

PROPOSITION 15. *Let $K\subset X$ be a continuum, $a\notin K$ and $d\in K$. Then $\overline{ad-K}\cap K \neq \emptyset$ and, for each point b of this set, $\overline{ad-K} = ab$ and the continuum ab is a prolongable segment of ad.*

Proof. We have $\emptyset\neq ad\cap K\subsetneq ad$ by assumption. Hence we have for the boundary of $ad\cap K$ in ad the inequality $\overline{ad-K}\cap ad\cap K = \overline{ad-K}\cap K\neq\emptyset$. The intersection $ad\cap K$ is a continuum by the hereditary unicoherence of X, and it contains d by assumption. Hence, for each point b of this boundary, the closure $\overline{ad-K} = \overline{ad-ad\cap K}$ is an irreducible continuum between a and b (see e.g. [3], § 48, II, Theorem 7), i.e., $\overline{ad-K} = ab$. Consequently $ab\cap K\subset Ba$ by Definition 1. Since $bd\subset K$ by (i), it follows that $ab\cap bd\subset Ba$. Thus, by Proposition 13, ab is a prolongable segment of ad.

PROPOSITION 16. *For a non-trivial continuum ad the set Ad is the intersection of a decreasing sequence of prolongable segments of ad.*

Proof. There exists (see e.g. [3], § 48, VI, Theorem 1) a sequence of continua $K_j\subset X$ where $j = 1, 2, \ldots$ such that

(4) $$a\notin K_j \quad\text{and}\quad d\in K_j\,,$$

(5) $$ad-Ad = \bigcup K_j\,,$$

(6) $$K_j\subset K_{j+1}\,.$$

By Proposition 15, considering (4) and (6), it suffices to prove that $Ad = \bigcap\overline{ad-K_j}$.

From (5) it follows that $Ad\subset\bigcap\overline{ad-K_j}$. Now let $b\in\bigcap\overline{ad-K_j}$. Suppose on the contrary that $b\notin Ad$. Then $b\in ad-Ad$, and thus, by (5) and (6), there exists a j_1 such that $b\in K_j$ for all $j>j_1$. Hence by Proposition 15,

(7) $$ab = \overline{ad-K_j}\quad\text{for all } j>j_1$$

and simultaneously ab is a segment of ad, whence $Ab = Ad$ by Definitions 2 and 3. Also ab is non-trivial, whence $Ab\subsetneq ab-Ba$ according to (iii). Consequently there exists a point $p\notin Ad$ such that

(8) $$p\in ab-Ba\,.$$

Then $p\in ad-Ad$ by (7) and (8), and therefore by (5) and (6) there exists an index $j_2>j_1$ such that $p\in K_j$ for all $j>j_2$. Since $p\in\overline{ad-K_j}$ by (7) and (8), it follows from Proposition 15 that $\overline{ad-K_j} = ap$. Then by (7), $ab = ap$. Therefore $p\in Ba$ by Definition 1, which contradicts (8).

II. Fixed points

§ 4. **Preliminaries on multi-valued functions.** Let F be a *multi-valued* function which maps X into itself, i.e., a function assigning to each point $x\in X$ a non-empty closed set $F(x)\subset X$. A *fixed point* of F is defined by $x\in F(x)$. We call a multi-valued function F *continuum-valued* provided that $F(x)$ is a continuum for each $x\in X$. If the equality $\lim x_j = x$ implies the inclusion $\mathrm{Ls}\,F(x_j)\subset F(x)$, then F is called *upper semi-continuous* (see e.g. [3], § 43, I and II). In the single-valued case upper semi-continuity simply becomes continuity.

In the reasoning which follows we use a considerable abbreviation: *instead of an arbitrary upper semi-continuous continuum-valued function F which maps X into itself we write simply F.*

Setting for any $K\subset X$

$$F(K) = \bigcup\{F(x)\colon x\in K\}\,,$$

we next prove that the assumptions about F imply the following two properties:

(I) *The set $F(K)$ is a continuum whenever K is a continuum, by virtue e.g. of* [5], p. 161.

(II) *If a sequence of continua $K_j\subset X$ is decreasing, then the inequality $K_j\cap F(K_j)\neq\emptyset$ for all j implies $\bigcap K_j\cap F(\bigcap K_j)\neq\emptyset$.*

In order to deduce (II), it suffices to verify that

$$\bigcap F(K_j)\subset F(\bigcap K_j)\,.$$

If $y\in\bigcap F(K_j)$, then for each j we have $y\in F(K_j)$, whence there exists an $x_j\in K_j$ such that $y\in F(x_j)$. By the compactness of X, we may assume that the sequence of x_j converges to some point x, so that $x\in\bigcap K_j$ and $\mathrm{Ls}\,F(x_j)\subset F(x)$ by the upper semi-continuity of F. Since $y\in\mathrm{Ls}\,F(x_j)$, then $y\in F(\bigcap K_j)$.

Note also that for every continuum $K\subset X$

(III) $$a\notin F(K)\ \text{and}\ d, d'\in F(K)\quad\text{imply}\quad ad\mathrel{-\!\!<} ad'$$

which follows from Proposition 1, considering (1) and (I).

§ 5. **Basic properties of families $\mathscr{P}_a$.** The sets belonging to these families will play an important part in the establishing the fixed point required.

Given an arbitrary point $a\in X$, we define $\mathscr{P}_a$ as the family of all non-trivial continua $ab\subset X$ satisfying

(IV) $$ab\mathrel{-\!\!<} ad\quad\text{for all } d\in F(a)$$

and such that for each $p\in ab-Ba$ there exist segments pq and pr of pb satisfying $pq\subset pr$ and

(V) $$qr\mathrel{-\!\!<} qt\quad\text{for all } t\in F(q)\,.$$

We verify first that the family $\mathcal{P}_a$ is correctly defined, i.e.,

$$\text{if } ab \in \mathcal{P}_a \text{ and } ab' = ab, \quad \text{then} \quad ab' \in \mathcal{P}_a.$$

By Proposition 2, ab' satisfies condition (IV).

In view of Definition 1,

$$(1) \qquad ab' = ab \quad \text{means} \quad B'a = Ba,$$

whence if $p \in ab' - B'a$, then $p \in ab - Ba$. Since $ab \in \mathcal{P}_a$, there exist segments $pq \subset pr$ of pb such that (V) holds. It remains to show that pq and pr are segments of pb'. For this purpose it suffices to verify, in view of Proposition 4, that $pb = pb'$.

By Proposition 3 and Definition 2,

$$(2) \qquad p \in ab - Ba \quad \text{implies} \quad Bp = Ba;$$

therefore $Bp = Ba$ and $B'p = B'a$. Thus $Bp = B'p$ whence by (1) $pb = pb'$.

We now prove four lemmas.

LEMMA 1. *Let $d \in F(a)$. If $Ad \cap F(Ad) = \emptyset$, then there exists a segment ab of ad such that $ab \in \mathcal{P}_a$.*

Proof. By Proposition 16, the set Ad is an intersection of a decreasing sequence of prolongable segments of ad. It follows from the assumed equality and from (II) that there exists a segment ab of ad with the property that

$$(3) \qquad ab \cap F(ab) = \emptyset.$$

Then $ab \prec ad$ by Definition 3. Since $d \in F(a)$ by assumption, also $ad \prec ad'$ for all $d' \in F(a)$ by (III). It follows by the transitivity of association that $ab \prec ad'$ for all $d' \in F(a)$. Thus condition (IV) holds.

For each $p \in ab - Ba$ there exist, by Proposition 11, segments $pq \subset pr$ of pb such that $qr \prec qd$. It remains to show, in view of the transitivity of association, that $qd \prec qt$ for all $t \in F(q)$. But $q \in pb$ by Definition 3, and $pb \subset ab$ by (i), whence $t \in F(ab)$ and $q \notin F(ab)$ by (3). Since $d \in F(ab)$ by assumption, applying (III) we complete the proof.

It can now be seen that the condition of Lemma 1 not only suffices for the existence of an element of $\mathcal{P}_a$, but also ensures the existence of a maximal element of $\mathcal{P}_a$. That is, the following is true

LEMMA 2. *If $d \in F(a)$ and $Ad \cap F(Ad) = \emptyset$, then there exists an ab maximal in $\mathcal{P}_a$.*

Proof. Given any increasing sequence of continua $ab_j \subset X$, observe that by the proposition in Remark 2, § 2, we have $\overline{\bigcup ab_j} = ab$ for some point b.

If, moreover, $ab_j \in \mathcal{P}_a$ for all j, then $ab \in \mathcal{P}_a$.

Indeed, taking $p = a$ in Proposition 10, we see that there exists an ab_j which is a segment of ab, whence ab_j is associated at a with ab by Definition 3. Since ab_j satisfies condition (IV), it follows by Proposition 2 that ab satisfies condition (IV). By the same Proposition 10, for each $p \in ab - Ba$ there exists an index j such that pb_j is a segment of pb and that $p \in ab_j - B_j a$. Since $ab_j \in \mathcal{P}_a$, there exist segments $pq \subset pr$ of pb_j such that (V) holds. By Proposition 4, both pq and pr are segments of pb.

Thus, the family $\mathcal{P}_a$ is closed with respect to the operation of closure of a union of increasing sequences. In view of Lemma 1, to finish the proof it is enough to apply the following theorem, which is dual to the Brouwer reduction theorem:

For any non-empty family $\mathcal{P}$ of closed subsets of X which is closed with respect to the operation of closure of a union of increasing sequences, there exists an element maximal in $\mathcal{P}$.

We prove the above statement.

There exists a countable base $B_1, B_2, \ldots$ of X, because of compactness. Taking $P_0 \in \mathcal{P}$, we define a sequence $P_1, P_2, \ldots$ as follows:

Let P_1 be any element of $\mathcal{P}$ which contains P_0 and meets B_1 if such exists; in the opposite case, set $P_0 = P_1$. Assuming, by an inductive step, $P_1 \subset P_2 \subset \ldots \ldots \subset P_{j-1}$ to be defined, let P_j be any element of $\mathcal{P}$ which contains P_{j-1} and meets B_j if such exists; in the opposite case, set $P_{j-1} = P_j$.

The sequence of $P_j \in \mathcal{P}$ defined in such a manner has the property that for every $Q \in \mathcal{P}$

$$(4) \qquad B_j \cap Q \neq \emptyset \text{ and } P_{j-1} \subset Q \quad \text{imply} \quad B_j \cap P_j \neq \emptyset.$$

Simultaneously it is increasing, so that $P = \overline{\bigcup P_j}$ belongs to $\mathcal{P}$ by assumption on $\mathcal{P}$. We show that P is maximal in $\mathcal{P}$.

If, on the contrary, there exists a $Q \in \mathcal{P}$ such that $P \subsetneq Q$, then there exists an element B_j of the base such that $P \cap B_j = \emptyset$ and that $B_j \cap Q \neq \emptyset$. Since $P_{j-1} \subset Q$, it follows from (4) that $B_j \cap P_j \neq \emptyset$, which contradicts the equality $P \cap B_j = \emptyset$.

LEMMA 3. *Let $ab \in \mathcal{P}_a$. If $Ba \cap F(Ba) = \emptyset$, then for each $d \in F(Ba)$ the continuum ab is a prolongable segment of ad.*

Proof. We verify that the condition of Proposition 12 holds. Given $p \in ab - Ba$, we have to show that there exist segment pp'' of pb and segments $p''q \subset p''r$ of $p''b$ such that $qr \prec qd$.

Adapting Proposition 16 to the set Ba, we get, by the assumed equality and by (II), a segment $p'b$ of ab with the initial point b such that

$$(5) \qquad p'b \cap F(p'b) = \emptyset.$$

Then $p' \in ab - Ba$ according to Definition 3, and thus by Proposition 10* there exists a $p'' \in ab - Ba$ such that pp'' is a segment of pb and $p'p''$ is a segment of $p'b$, whence

$$(6) \qquad p''b \subset p'b$$

according to Definition 3 and (ii). Since $ab \in \mathscr{P}_a$ by assumption, there exist segments $p''q \subset p''r$ of $p''b$ such that (V) holds. It remains to show that $qr -\!\!< qd$. For this purpose it suffices to verify, in view of (V) and of the transitivity of association, that $qt -\!\!< qd$ for any point $t \in F(q)$.

We have $q \in p''b$, $p''q$ being a segment of $p''b$; hence $t \in F(p'b)$ and $q \notin F(p'b)$ by (5) and (6). Since $p' \in ab - Ba$, whence $Ba = Bp'$ by (2), it follows by assumption that $d \in F(Bp')$; thus naturally $d \in F(p'b)$. Therefore by (III), $qd -\!\!< qt$.

LEMMA 4. *Let a continuum $ab \in \mathscr{P}_a$ be a segment of ac. If $bc \in \mathscr{P}_b$, then $ac \in \mathscr{P}_a$.*

In fact we prove the following more general

LEMMA 4*. *Let a continuum ab satisfy condition* (IV) *and be a segment of ac. If $bc \in \mathscr{P}_b$, then $ac \in \mathscr{P}_a$.*

Proof. By Definition 3 we have $ab -\!\!< ac$; therefore, applying Proposition 2, we see that ac satisfies condition (IV).

Also $ab \subset ac$ by Definition 3, and for a point

(7) $$p \in ac - Ca$$

consider two cases: $b \in Ca$ and $b \in ac - Ca$.

In the first case we get, applying (i),

(8) $$bc \subset Ca\,.$$

Since $bc \in \mathscr{P}_b$ by assumption, we infer that (putting $a = b$ and $p = b$ in the definition of $\mathscr{P}_a$) there exist segments bq and br of bc such that (V) holds. Then $q, r \in bc$ by Definition 3, and thus by (8) $q, r \in Ca$. But $Ca = Cp$ by (7) and (2), and therefore $q, r \in Cp$, i.e., $pq = pr = pc$ by Definition 1. By Proposition 4, pq and pr are segments of pc.

In the second case we have $b \in ac - Ca$, whence

(9) $$Cb = Ca$$

according to (2). By Proposition 10*, considering (7), there exists a segment pp' of pc such that $p' \notin Ca$ and bp' is a segment of bc, so that $p' \in bc - Cb$ by Definition 3 and (9). Since $bc \in \mathscr{P}_b$ by assumption, there exist segments $p'q \subset p'r$ of $p'c$ such that (V) holds. By Proposition 12*, pq and pr are segments of pc satisfying $pq \subset pr$.

§ 6. **The fixed-point theorem.** First the following three auxiliary theorems would be proved.

Let $\mathscr{K}$ be the family of all continua $K \subset X$ with the property that $K \cap F(K) \neq \emptyset$. By (II), $\mathscr{K}$ is inductive, i.e., closed with respect to the operation of the intersection of decreasing sequences. Since $X \in \mathscr{K}$, it follows by the Brouwer reduction theorem that there exists a minimal element of $\mathscr{K}$.

THEOREM I. *Let K be minimal in $\mathscr{K}$. If K is non-trivial, then there exist a point a and an ab maximal in $\mathscr{P}_a$ such that $Ab \subsetneqq K$.*

Proof. We have $K \cap F(K) \neq \emptyset$, whence there exists a point $a \in K$ such that $K \cap F(a) \neq \emptyset$. Then $a \notin F(a)$ by assumption on K, and therefore for a point

(1) $$d \in F(a)$$

such that $d \in K$, the continuum ad is non-trivial, i.e., $Ad \subsetneqq ad$, and simultaneously $ad \subset K$ by (i). Consequently

(2) $$Ad \subset K$$

and therefore by the minimality of K, we have $Ad \cap F(Ad) = \emptyset$. Thus by Lemma 2, considering (1), there exists an ab maximal in $\mathscr{P}_a$. Then $ab \in \mathscr{P}_a$ so that (IV) is fulfilled. Thus it follows from (1) that $ab -\!\!< ad$, and further $Ab = Ad$ by Definition 2. Therefore by (2), $Ab \subsetneqq K$.

The following lemma is stronger than Lemma 3 and serves only to prove the next theorem.

LEMMA 5. *If $ab \in \mathscr{P}_a$ and $Ba \cap F(Ba) = \emptyset$, then there exists an $ab' = ab$ such that $ab' \cap b'd' = B'a \cap B'd'$ for all $d' \in F(B'a)$.*

Proof. In view of Proposition 13 it has been shown in Lemma 3 that $ab \cap bd \subset Ba$, i.e., $(ab - Ba) \cap bd = \emptyset$, for all $d \in F(Ba)$. Hence $(ab - Ba) \cap F(Ba) = \emptyset$; therefore the assumed equality $Ba \cap F(Ba) = \emptyset$ implies

(3) $$ab \cap F(Ba) = \emptyset\,.$$

Simultaneously, for a given point

(4) $$d \in F(Ba)$$

we have

(5) $$ab \cap bd \subset Ba\,,$$

and we prove first that there exists an $ab' = ab$ such that the equality $ab' \cap b'd = B'a \cap B'd$ holds.

The intersection $Ba \cap bd$ is a continuum by the hereditary unicoherence, and $d \notin Ba \cap bd$ by (3) and (4). Clearly $b \in Ba \cap bd$. Thus, applying to this continuum the same argument as in the proof of Proposition 15 to K, we get a point $b' \in Ba \cap bd$ such that

(6) $$Ba \cap bd \cap b'd \subset B'd\,.$$

Since $b' \in Ba$, then by Definition 1 we have the equalities

(7) $$ab' = ab\,,$$

(8) $$B'a = Ba\,.$$

Since $b' \in bd$, whence $b'd \subset bd$ by (i), it follows from (5)–(8) that $ab' \cap b'd \subset B'a$ and $B'a \cap b'd \subset B'd$. Consequently $ab' \cap b'd \subset B'a \cap B'd$. The converse inclusion is trivial.

Now for an arbitrary point

(9) $$d' \in F(B'a)$$

it suffices to verify two equalities: $B'd = B'd'$ and $ab' \cap b'd = ab' \cap b'd'$.

We have $b' \notin F(Ba)$ by (3) and (7), and $d, d' \in F(Ba)$ by (4), (8) and (9). Therefore $b'd \mathrel{-\!\!<} b'd'$ by (III), and thus, by Definition 2, the equality $B'd = B'd'$ holds.

Since $d, d' \in F(Ba)$, we have $dd' \subset F(Ba)$ by (i) and (I), and hence $ab' \cap dd' = \emptyset$ by (3) and (7). But $b'd' \subset b'd \cup d'd$ by (i), and thus $ab' \cap b'd' \subset ab' \cap b'd$. By symmetry, the converse inclusion is also satisfied.

THEOREM II. *If ab is maximal in $\mathscr{P}_a$ and if $Ba \notin \mathscr{K}$, then there exists a fixed point.*

Proof. We have, by assumption,

(10) $$ab \in \mathscr{P}_a ,$$

(11) $$Ba \cap F(Ba) = \emptyset ;$$

thus, by Lemma 5, there exists an $ab' = ab$ such that $ab' \cap b'd' = B'a \cap B'd'$ for all $d' \in F(B'a)$. Then $ab' \in \mathscr{P}_a$ by (10), because $\mathscr{P}_a$ is correctly defined, and also ab' is maximal in $\mathscr{P}_a$. Then also $B'a = Ba$ by Definition 1. Thus, ab' satisfies all the assumptions which we make with respect to ab. Therefore it is not necessary to consider ab' in the sequel. We may assume, without change of notation, it is ab that satisfies

(12) $$ab \cap bd = Ba \cap Bd$$

for any $d \in F(Ba)$, and therefore also for

(13) $$d \in F(b) .$$

Then we have

(14) $$Bd \cap F(Bd) \neq \emptyset .$$

Suppose the contrary. By (13) and Lemma 1, then there exists a continuum $bc \in \mathscr{P}_b$ which is a segment of bd. Since ab is a segment of ad by (12) and by Proposition 13, it follows by Proposition 8 that ab is a segment of ac. Thus by Lemma 4, considering (10), $ac \in \mathscr{P}_a$. However, according to Definition 3, $ab \subset ac$ and also $c \in bd - Bd$, whence $c \notin ab$ by (12). Consequently $ab \subsetneqq ac$, which contradicts the maximality of ab in $\mathscr{P}_a$.

We prove now that there exists a continuum bs satisfying the following three conditions

(15) $$bs \subset Bd ,$$

(16) $$Bd \subset F(bs) ,$$

(17) $$Bd \cap F(bs - Sb) = \emptyset .$$

For this purpose, note that the set $G = \{x \in X\colon Bd \cap F(x) = \emptyset\}$ has properties: a) $G \cap \mathrm{Fr}(G) = \emptyset$, b) $Ba \subset G$, c) $Bd \cap F(G) = \emptyset$, d) $Bd \cap G \subsetneqq Bd$.

Indeed, property a) means that G is open, and therefore it follows from the upper semi-continuity of F. By (11), we have $b \notin F(Ba)$ and simultaneously $d \in F(bs)$ by (13), whence $Bd \cap F(Ba) = \emptyset$ according to (iii), and consequently, by the definition of G, we get b). Further, c) follows directly from this definition. Hence $Bd \cap F(Bd \cap G) = \emptyset$ and therefore $Bd \cap G \neq Bd$ by (14), i.e., d).

By b), there exists a component of $Bd \cap G$ which contains b. Let K denote the closure of this component. Then we have $b \in K$, and by virtue of d) and of a theorem of Janiszewski (see [3], § 47, III, Theorems 1 and 2), the continuum K meets the boundary of $Bd \cap G$ in Bd, and thus of course K meets the set $Bd \cap \mathrm{Fr}(G)$. Consider a continuum $L \subset K$ which is minimal with respect to the property: $b \in L$ and $L \cap Bd \cap \mathrm{Fr}(G) \neq \emptyset$ (such a minimal continuum L exists by the Brouwer reduction theorem). Clearly $L = bs$ for any point $s \in L \cap Bd \cap \mathrm{Fr}(G)$. Since $s \in Bd$, it follows by (i) that (15) is satisfied.

Since $s \in \mathrm{Fr}(G)$, we have $s \notin G$ by a), and consequently we have $Bd \cap F(s) \neq \emptyset$ by the definition of G, hence naturally $Bd \cap F(bs) \neq \emptyset$. Since $d \in F(bs)$ by (13), we have $b \in F(bs)$ by (I) and (iii). Therefore $bd \subset F(bs)$ by (i), whence (16) holds.

To prove (17), recall that $bs = L \subset K$ and $K \subset \overline{Bd \cap G} \subset Bd \cap (G \cup \mathrm{Fr}(G))$, whence $bs \subset Bd \cap G \cup Bd \cap \mathrm{Fr}(G)$. Moreover, by the minimality of $L = bs$, we have for every $bp \subsetneqq bs$ the equality $bp \cap Bd \cap \mathrm{Fr}(G) = \emptyset$, and consequently $bp \subset Bd \cap G$. Therefore $bs - Sb \subset Bd \cap G$, and thus by c) we get (17).

From (15)–(17) it follows that $Sb \subset F(Sb)$; hence naturally $Sb \cap F(Sb) \neq \emptyset$, i.e., $Sb \in \mathscr{K}$. By the Brouwer reduction theorem there exists a continuum K_1 minimal in $\mathscr{K}$ such that

(18) $$K_1 \subset Sb .$$

Suppose, to get a contradiction, that K_1 is non-trivial.

By Theorem I, there exists a continuum

(19) $$b'c \in \mathscr{P}_b$$

such that $B'c \subsetneqq K_1$, so that by (18)

(20) $$B'c \subsetneqq Sb .$$

Consider the continuum as. We have $ab \cap bs \subset Ba$ by (12) and (15); therefore, by Proposition 13, ab is a segment of as. By Definition 3,

(21) $$ab \mathrel{-\!\!<} as$$

and $ab \subset as$. For $ab = as$ we get $bs \subset Ba$ according to (i) and Definition 1, which involves a contradiction of (11), (15) and (16); it follows that

(22) $$ab \subsetneqq as .$$

Therefore $b \in as - Sa$ by Definition 1, and thus $Sb = Sa$ in view of Proposition 3 and Definition 2. It follows from (20) that $B'c \subsetneqq Sa$, whence $b' \in Sa$ and further

$$as = ab', \tag{23}$$

and also $Sa = B'a$ by Definition 1. Therefore

$$B'c \subsetneqq B'a . \tag{24}$$

In view of Proposition 2, from (10), (21) and (23) it follows that ab' satisfies condition (IV) from the definition of $\mathscr{P}_a$. By Proposition 14, considering (24), ab' is a segment of ac. Thus, by (19) and Lemma 4*, we get $ac \in \mathscr{P}_a$. However, $ab' \subset ac$ by Definition 3 and therefore by (22) and (23) we have $ab \subsetneqq ac$, which contradicts the maximality of ab in $\mathscr{P}_a$.

This contradiction shows that K_1 is trivial. Since $K_1 \cap F(K_1) \neq \emptyset$, the unique point of K_1 is a fixed point.

Remark. This fixed point belongs to Bd for d as in (13), because we have $K_1 \subset Bd$ by (15), (18) and obvious inclusion $Sb \subset bs$.

Denote by $\mathscr{M}$ the family of all continua $M \subset X$ with the property that for every $b'c \in \mathscr{P}_{b'}$

$$B'c \subsetneqq M \quad \text{implies} \quad b'c \subset M .$$

Clearly $\mathscr{M}$ is inductive, and so $\mathscr{K} \cap \mathscr{M}$ is inductive.

THEOREM III. *If ab is maximal in $\mathscr{P}_a$, then $Ba \in \mathscr{M}$.*

Proof. Let $b'c \in \mathscr{P}_{b'}$ and

$$B'c \subsetneqq Ba . \tag{25}$$

Then $b' \in Ba$, and thus in view of (i) it suffices to show that $c \in Ba$.

Since $b' \in Ba$, we have $ab = ab'$ and $Ba = B'a$ by Definition 1. Then $B'c \subsetneqq B'a$ by (25), and hence, by Proposition 14, ab' is a segment of ac. Then also $ab' \in \mathscr{P}_a$, $\mathscr{P}_a$ being correctly defined, and consequently we have $ac \in \mathscr{P}_a$ by Lemma 4. However, $ab' \subset ac$ by Definition 3, whence the equality $ab = ab'$ gives $ab \subset ac$. The maximality of ab in $\mathscr{P}_a$ implies $ab = ac$, and therefore, by Definition 1, $c \in Ba$.

Theorems I–III imply the existence of a fixed point, i.e.

COROLLARY. *If X is a λ-dendroid, then for every upper semi-continuous continuum-valued function F which maps X into itself there exists a fixed point.*

Proof. There exists by virtue of the Brouwer reduction theorem, a continuum K which is minimal in $\mathscr{K}$, i.e., minimal with respect to the property that $K \cap F(K) \neq \emptyset$. Then, for trivial K, the unique point of K is a fixed point.

If K is non-trivial, then by Theorem I there exists an ab maximal in $\mathscr{P}_a$, and in the case where $Ba \notin \mathscr{K}$ there exists a fixed point by Theorem II. In the opposite case we have $Ba \in \mathscr{K} \cap \mathscr{M}$ in view of Theorem III. Therefore, by the Brouwer reduction theorem, there exists a continuum M which is minimal in $\mathscr{K} \cap \mathscr{M}$, and simultaneously there exists a continuum $L \subset M$ which is minimal in $\mathscr{K}$. Then, for trivial L, the unique point of L is a fixed point.

If L is non-trivial, then according to Theorem I there exists a $b'c$ maximal in $\mathscr{P}_{b'}$ such that $B'c \subsetneqq L$, and so that $B'c \subsetneqq M$. It follows by virtue of the definition of the element M of $\mathscr{M}$ that $b'c \subset M$. Since $b'c$ is non-trivial as an element of $\mathscr{P}_{b'}$, we then have $Cb' \subsetneqq M$. Consequently $Cb' \notin \mathscr{K} \cap \mathscr{M}$ by the minimality of M, and therefore $Cb' \notin \mathscr{K}$ by Theorem III. Then, by Theorem II, there exists a fixed point.

All the assumptions in the above Corollary are essential (see [1], p. 336).

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Accepté par la Rédaction le 28. 5. 1974