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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Local expansions, derivatives, and fixed points

by

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Abstract. Using covering space techniques, we obtain a fixed point theorem for open local expansions defined on compact, connected metric spaces. We then use this result to obtain some fixed point theorems for manifolds and some fixed point theorems using the derivative.

Introduction. Let (X, d) be a metric space. The statement that the function $f\colon X \to X$ is a *local expansion* means that f is continuous and that for each $x \in X$, there is an open set U containing x and a real number $M > 1$ so that if y and z belong to U, then $d(f(y), f(z)) \geqslant \mathrm{M}d(y, z)$. In this paper we prove the following:

THEOREM. *If (X, d) is a compact, connected metric space and $f\colon X \to X$ is an open* (1) *local expansion, then f has a fixed point.*

(The proof uses covering space techniques (!) and is constructive. But the reader will note that the technique used in the Banach Contraction Theorem, namely taking iterates, will almost never work for local expansions — once a point gets close to a fixed point its iterates start moving away. In fact, roughly the idea is to do the "opposite" of taking iterates, to very carefully take "roots".) We use this theorem to obtain some fixed point theorems for manifolds and some fixed point theorems using the derivative. The reader should consult a paper of M. Edelstein [1] for corresponding results on local contractions.

Proof of the theorem. Suppose that f is an open local expansion from the compact connected metric space (X, d) to itself. Then, in particular, f is a local homeomorphism of X onto X. So, by a lemma of S. Eilenberg [2], there exist two positive number η_0 and ε_0 such that each subset A of X of diameter less than ε_0 determines a decomposition of the set $f^{-1}(A)$ with the following properties:

1) $f^{-1}(A) = A_1 \cup A_2 \cup \ldots \cup A_m$,
2) f maps each A_j homeomorphically onto A,
3) each A_j has diameter less than η_0,
4) if $j \neq k$, then no point of A_j is closer than $2\eta_0$ to a point of A_k.

(1) By "open" here we mean that f takes open subsets of X to open subsets of X.

Hence, for our open local expansion, there is a positive number δ and an $M>1$ such that:

a) if $d(x,y)<\delta$, then $d(f(x),f(y))\geqslant Md(x,y)$,

b) if V is an open set of diameter less than δ and if $q\in f^{-1}(V)$, then there is a unique open set containing q of diameter less than $(1/M)\delta$ which maps homeomorphically onto V.

This last condition enables us to "lift" chains of small open sets, one link at a time, just as if f were a covering map in the classical sense (see Massey [3] for an excellent treatment of covering maps).

Let $V_1, V_2, \ldots, V_n$ be a fixed finite open cover of X so that the diameter of each V_j is less than δ. And now suppose that x_0 is any point of X and that $f(x_0)\neq x_0$. Then since X is connected, there is a chain of open sets chosen from among $V_1, V_2, \ldots, V_n$ from $f(x_0)$ to x_0. Lift this chain to a chain of open sets from x_0 to a point x_1 such that $f(x_1)=x_0$. Then lift this chain to a chain of open sets from x_1 to a point x_2 such that $f(x_2)=x_1$, etc.

$$f(x_0) \quad\longrightarrow\quad x_0 \quad x_1 \quad x_2 \quad \ldots$$

Notice that since the original chain had length at most $n\delta$, the length of the first lifted chain is at most $\frac{1}{M}n\delta$, and, in general, the length of the kth lifted chain is at most $\frac{1}{M^k}n\delta$.

In addition, the chains intersect, so if $j<k$,

$$d(x_j,x_k)\leqslant \frac{1}{M^{j+1}}n\delta+\frac{1}{M^{j+2}}n\delta+\ldots+\frac{1}{M^k}n\delta<\frac{n\delta}{M^{j+1}}\sum_{i=0}^{\infty}\frac{1}{M^i}=\frac{1}{M^j}\left(\frac{n\delta}{M-1}\right).$$

Thus $\{x_j\}$ is a Cauchy sequence, which therefore converges to a point $y\in X$. But since $f(x_j)=x_{j-1}$ and f is continuous,

$$f(y)=f(\lim_{j\to\infty}x_j)=\lim_{j\to\infty}f(x_j)=\lim_{j\to\infty}x_{j-1}=y,$$

and, as promised, f has a fixed point.

Remark. Look at how beautifully this works in practice! Let X be the unit circle in the plane, i.e. $\{z|\ |z|=1\}$, and let $f\colon X\to X$ be defined by $f(z)=z^2$. Starting with $x_0=i$, $f(x_0)=-1$, so if we consider the short arc from -1 to i, the proof above essentially lifts this to the short arc from i to $e^{\pi i/4}$. This, in turn, gets lifted to the short arc from $e^{\pi i/4}$ to $e^{\pi i/8}$, etc. This sequence converges to 1 which, in this case, happens to be the unique fixed point of the function. (In general, the fixed point will not be unique.) Notice, though, that this taking "roots" process must be done carefully, or you end up all over the place. It is more delicate than simply picking any point in the inverse image and repeating.

Some consequences. In this section, we use the theorem to obtain some fixed point theorems for manifolds and some fixed point theorems using the derivative. By the word "manifold", we always mean "compact, connected manifold without boundary".

COROLLARY 1. *If f is a local expansion from an n-dimensional manifold to itself, then f has a fixed point.*

Proof. Here, the Invariance of Domain Theorem, gives us openness "for free", so we just apply our theorem.

Remark. It is interesting to note here that if a manifold admits a local expansion, then it definitely does not have the fixed point property.

COROLLARY 2. *Suppose M is an n-manifold in $\mathbf{R}^k$, and f is a C^1 function from a neighborhood of M to $\mathbf{R}^k$ and taking M to M. Suppose also that for each $a\in M$, the differential at a is an expansion (i.e. there is an $m>1$ such that $|df(a)h|\geqslant m|h|$ for $h\in\mathbf{R}^n$). Then f has a fixed point.*

Proof. In the course of proving the Inverse Function Theorem, K. Smith proves the following fact in his Primer of Modern Analysis (see Smith [5], pp. 234–235):

Let $f\colon \mathbf{R}^m\to\mathbf{R}^n$ be C^1 at a. Then for each positive number ε, there is a positive number δ such that if $|x-a|<\delta$ and $|y-a|<\delta$, then

$$|f(x)-f(y)-df(a)(x-y)|\leqslant\varepsilon|x-y|.$$

Using our assumption that $|df(a)h|\geqslant m|h|$ and taking $\varepsilon=\frac{1}{2}(m-1)$, we see that there is a positive number δ so that if $|x-a|<\delta$ then

$$|f(x)-f(y)|\geqslant\tfrac{1}{2}(m+1)|x-y|.$$

Thus f is a local expansion from M to itself, and so, by Corollary 1, f has a fixed point.

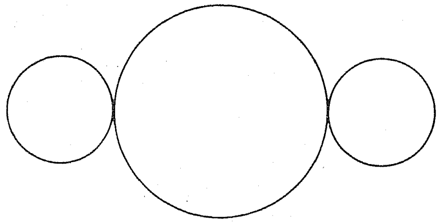
COROLLARY 3. *Let M be a smooth n-manifold, and suppose $f\colon M\to M$ is a C^1 map whose differential is an expansion at each point. Then f has a fixed point.*

We omit the proof of Corollary 3. The interested reader can supply a proof using the techniques of this paper. However, a proof using different techniques can be found in M. Shub's paper [4].

EXAMPLE. We conclude with an example of a local expansion of a compact, connected metric space to itself which has no fixed points. This shows that the somewhat unfortunate hypothesis that the map be open is essential.

Let X denote the following subset of the plane:

$$\{z|\ |z|=1\}\cup\{z|\ |z-\tfrac{3}{2}|=\tfrac{1}{2}\}\cup\{z|\ |z+\tfrac{3}{2}|=\tfrac{1}{2}\}.$$



Give X the arc-length metric. The map is constructed roughly as follows: stretch each of the small circles onto the big circle; stretch each of the upper and lower semi-circles of the big circle first around a smaller circle, then across the other semi-circle, and finally around the other smaller circle.

For those who prefer a formula, we let $f\colon X\to X$ be defined by:

$$f(z)=\begin{cases} 2(z-\frac{3}{2}) & \text{if} \quad \operatorname{Re}(z)\geqslant 1\,,\\ 2(z+\frac{3}{2}) & \text{if} \quad \operatorname{Re}(z)\leqslant -1\,,\\ \frac{1}{2}z^{-6}-\frac{3}{2} & \text{if} \quad \frac{1}{2}\leqslant \operatorname{Re}(z)\leqslant 1\,,\\ z^3 & \text{if} \quad -\frac{1}{2}\leqslant \operatorname{Re}(z)\leqslant \frac{1}{2}\,,\\ -\frac{1}{2}z^{-6}+\frac{3}{2} & \text{if} \quad -1\leqslant \operatorname{Re}(z)\leqslant -\frac{1}{2}\,. \end{cases}$$

The reader can check that this is indeed a local expansion with no fixed points.

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On the *topology and its application

by

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Abstract. The purpose of the present paper is to study the relation between the set of the first category and the null set by introducing the *topology to T_1-space. As a result of this application, we made clearer the similarity and difference of the set having the Baire property and the measurable set in the sense of Lebesgue.

§ 1. Introduction. Let l be a T_1 space defined by the closure operation $X\to\overline{X}$. We denote by P some property about the subsets of l, and by $\boldsymbol{P}$ the family of all subsets of l which satisfy P. We say that a subset X has the *property* P at a point $p\in l$, if there exists a neighbourhood $V(p)$ of p such that $V(p)X\in\boldsymbol{P}$. We denote by X^* the set of points at which X does not have the property P, namely $X^* = \{p/\forall V(p),\ V(p)X\notin\boldsymbol{P}\}$. Assume that the family $\boldsymbol{P}$ is an ideal, i.e.,

(1) (i) the conditions $X\in\boldsymbol{P}$ and $Y\subset X$ imply $Y\in\boldsymbol{P}$,
(ii) the conditions $X\in\boldsymbol{P}$ and $Y\in\boldsymbol{P}$ imply $X+Y\in\boldsymbol{P}$,

then the operation $X\to X^*$ has the following properties

(2) (a) X^* is closed, (b) if $X\subset Y$, then $X^*\subset Y^*$,
(c) $X^{**}\subset X^*\subset\overline{X}$, (d) if G is open, then $GX^* = G(GX)^*$,
(e) $(XY)^*\subset X^*\cdot Y^*$, (f) $X^*-Y^*\subset(X-Y)^*$.

Assume further that the property P satisfies the relation

(3) $$\{X\in\boldsymbol{P}\}\equiv\{XX^*=0\}\equiv\{X^*=0\}\,,$$

then

(4) (a) $(X-X^*)^*=0$, namely $X-X^*\in\boldsymbol{P}$,
(b) $X^{**}=X^*$,
(c) if $Y\in\boldsymbol{P}$, then $(X\pm Y)^*=X^*$

(see [2], [3]).

In the following, we shall now assume that P satisfies the conditions (1) and (3) and that every single element-subset of l belongs to $\boldsymbol{P}$. Two important examples of the family $\boldsymbol{P}$ of this kind are the family of the sets of the first category and the family of the sets of measure zero (in the sense of Lebesgue).