[5] S. Godlewski, *Some remarks concerning the mappings of the inverse limit into an absolute neighborhood retract and its applications to cohomotopy groups*, Fund. Math. 63 (1968), pp. 89–95.
[6] — *Homotopy dependence of fundamental sequences, relative fundamental equivalence of sets and generalization of cohomotopy groups*, Fund. Math. 69 (1970), pp. 63–91.
[7] — *Cohomotopy groups and shape in the sence of Mardešić*, Bull. Acad. Polon. Sci. 21 (1973), pp. 711–718.
[8] — and W. Holsztyński, *Some remarks concerning Borsuk's theory of shape*, Bull. Acad. Polon. Sci. 17 (1969), pp. 373–376.
[9] J. W. Jaworowski, *Some remarks on Borsuk generalized cohomotopy groups*, Fund. Math. 50 (1962), pp. 257–264.
[10] A. T. Lundell and S. Weingram, *The topology of CW complexes*, 1969.
[11] S. Mardešić, *Shapes for topological spaces*, General Topology and its Applications 3 (1973), pp. 265–282.
[12] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1974), pp. 251–259.
[13] — *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.
[14] — *Čech cohomology and covering dimension for topological spaces*, Fund. Math. 87 (1975), pp. 31–52.
[15] S. Nowak, *Some properties of fundamental dimension*, Fund. Math. 85 (1974), pp. 211–227.
[16] E. Spanier, *Borsuk's cohomotopy groups*, Ann. of Math. 50 (1949), pp. 203–245.

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Homeotopy groups of compact 2-manifolds

by

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Abstract. Let X be a 2-manifold and let $H(X)$ denote the homeotopy group of X. Several results have been obtained concerning $H(X)$ in the case X is of the form $M-F_n$ where M is a closed 2-manifold and F_n is a set of n distinct points in M. In this paper it is shown that these results give rise immediately to corresponding results for compact 2-manifolds. In particular, it is shown that if Y is the compact 2-manifold obtained by removing the interiors of n disjoint closed discs from some closed 2-manifold M, then $H(Y)$ is isomorphic to $H(M-F_n)$.

1. Introduction. Let X be a 2-manifold (connected, triangulated) and let $H(X)$ denote the homeotopy group (or mapping class group) of X, i.e. $H(X)$ is the group of all isotopy classes in the space of all homeomorphisms of X onto X. W. Magnus [4] and, more recently, J. Birman [1] have obtained several results concerning $H(X)$ in the case X is of the form $M-F_n$ where M is a closed 2-manifold and F_n is a set of n distinct points in M. In this paper we show that these results give rise immediately to corresponding results for compact 2-manifolds. In particular, we show that if Y is the compact 2-manifold obtained by removing the interiors of n disjoint discs from some closed 2-manifold M, then $H(Y)$ is isomorphic to $H(M-F_n)$.

2. Notation. Let X be a 2-manifold and F a finite subset of $\mathrm{Int}(X)$. The homeotopy group $H(X)$ can be defined as the quotient group $G(X)/G_0(X)$ where $G(X)$ is the group of all homeomorphisms of X onto X and $G_0(X)$ is the normal subgroup of $G(X)$ consisting of those homeomorphisms g in $G(X)$ which are isotopic to the identity (denoted $g \simeq 1_X$). Similarly, we can define $H(X, F)$ to be the quotient group $G(X, F)/G_0(X, F)$ where $G(X, F)$ is the subgroup of $G(X)$ consisting of those g in $G(X)$ which map F onto F and $G_0(X, F)$ is the normal subgroup of $G(X, F)$ consisting of those homeomorphisms h in $G(X, F)$ which are isotopic to the identity by an isotopy which keeps F pointwise fixed (denoted $h \simeq 1_X(\mathrm{rel} F)$).

Let M be a closed 2-manifold. Let D_i for $1 \leqslant i \leqslant n$ denote a family of disjoint closed discs in M with P_i a point in $\mathrm{Int}(D_i)$ for each i between 1 and n. Let $F_n = \{P_1, P_2, \ldots, P_n\}$. We will use the notation M_n to denote the compact 2-manifold $M - \bigcup_{i=1}^{n} \mathrm{Int}(D_i)$.

3. Homeotopy groups. Using the notation of Section 2, the main result of the paper can be stated as: $H(M_n) \cong H(M - F_n)$. The proof of this result uses the following lemmas.

LEMMA 1. *If X is a compact 2-manifold and F is a finite subset of X, then $H(X - F) \cong H(X, F)$.*

Proof. This follows from Theorem 4.21 of [5].

LEMMA 2. *If T is any permutation of n elements, then there exists a homeomorphism h: $M_n \to M_n$ such that $h(\partial D_i) = \partial D_{T(i)}$ for $1 \leq i \leq n$.*

Proof. Straightforward.

Remark. In the above lemma, h can be chosen to extend to a homeomorphism h' on M which is isotopic to 1_M. In addition, we can arrange that $h'(P_i) = P_{T(i)}$ where $P_i \in \mathrm{Int}(D_i)$ for $1 \leq i \leq n$.

LEMMA 3. *If X is a compact 2-manifold and f: $X \to X$ is a homeomorphism which fixes a finite set F in* $\mathrm{Int}(X)$, *then there exists a P. L. homeomorphism h: $X \to X$ such that $f \simeq h(\mathrm{rel} F)$.*

Proof. Theorem 4A of [3] establishes this result for F equal to a single point $P_1 \in \mathrm{Int}(X)$. In the case that X is a compact manifold, the proof given in [3] reduces to the following:

Given a disc D_1 in $\mathrm{Int}(X)$ with $P_1 \in \partial D_1$, it is shown first that there exists an ambient isotopy H_t of X satisfying: 1) H_t is the identity outside a compact neighborhood of ∂D_1, 2) $H_t(p_1) = p_1$ for all t and 3) $g = H_1 f$ is P. L. on ∂D_1. Next it is shown that if a homeomorphism is P. L. on a boundary component of a compact 2-manifold Y, then this homeomorphism can be isotoped to a P. L. homeomorphism of Y by an isotopy which keeps the component fixed. Thus g/D_1 and $g/X - \mathrm{Int}(D_1)$ can be isotoped, by isotopies which keep ∂D_1 fixed, to P. L. homeomorphisms h_1 on D_1 and h_2 on $X - \mathrm{Int}(D_1)$. Since $h_1/\partial D_1 = h_2/\partial D_1$, these isotopies can be fitted together to yield an isotopy of X which keeps p_1 fixed and takes g to a P. L. homeomorphism h.

The above proof can be adapted to work for any finite set $\{p_1, \ldots, p_n\}$. Renumbering the points if necessary, we can decompose X so that $X = \bigcup_{i=1}^{n} D_i \cup X - \mathrm{Int}(D_n)$ where each D_i is a disc with $D_i \subset \mathrm{Int}(D_{i+1})$, $p_i \in \partial D_i$, and $D_n \subset \mathrm{Int}(X)$. Using the result in the first part of the above proof we can find a sequence of isotopies $H_t^1, \ldots, H_t^n$ which are such that $(\prod_{i=1}^{k} H_1^i) f$ is P. L. on $\bigcup_{i=1}^{k} \partial D_i$ where each H_t^i keeps $\{p_1, \ldots, p_n\}$ fixed. Note that each H_t^i can be chosen so as to leave $D_{i-1} \subset \mathrm{Int}(D_i)$ fixed; thus, the action of H_t^i does not destroy the effect of H_t^{i-1} on ∂D_{i-1}. Finally, we use the technique of the second part of the above proof to get f P. L. on each of the subsets $D_i - \mathrm{Int}(D_{i-1})$ and $X - \mathrm{Int}(D_n)$.

Remark. Lemma 3 is true for arbitrary 2-manifolds. The only difference in the proof is that in the general case we may have to decompose X into a countable union of compact submanifolds.

LEMMA 4. *If D is a disc with $p \in \mathrm{Int}(D)$ and f: $D \to D$ is a homeomorphism with $f/\partial D \cup p = 1_{\partial D \cup p}$, then $f \simeq 1_D(\mathrm{rel}\ \partial D \cup p)$.*

Proof. This is the "Alexander trick". See Theorem 5.2 of [3].

LEMMA 5. *Let X be a 2-manifold such that every component of ∂X is compact and let h be a homeomorphism of X onto itself (which preserves orientations if X is a plane, closed cylinder or open cylinder). If h is homotopic to the identity, then h is isotopic to the identity.*

Proof. This is Theorem 6.4 of [3].

THEOREM 6. *In the notation of Section 2,*

$$H(M_n) \cong H(M, F_n).$$

COROLLARY 7. $H(M_n) \cong H(M - F_n)$.

Proof. This follows immediately from Theorem 6 in view of the fact that by Lemma 1, $H(M, F_n) \simeq H(M - F_n)$.

Proof of Theorem 6. We fix, for each i, a homeomorphism e_i from D_i to the unit disc in R^2 which takes p_i to the origin. With these coordinates, any homeomorphism of ∂D_i into ∂D_j can be extended by "coning" to a homeomorphism of (D_i, p_i) into (D_j, p_j). More generally, any homeomorphism $h \in G(M_n)$ can be extended by coning to a homeomorphism $h_C \in G(M, F_n)$. For example, if T is a permutation of n elements and $h(\partial D_i) = \partial D_{T(i)}$, then $h_C(x) = h(x)$ for $x \in M_n$ and $h_C\big(e_i^{-1}(te_i(x) + (1-t)e_i(p_i))\big) = e_j^{-1}\big(te_j(h(x)) + (1-t)e_j(p_j)\big)$ for $x \in \partial D_i$ and $j = T(i)$.

For $f \in G(M_n)$ and $g \in G(M, F_n)$, let $\bar{f}$ denote the equivalence class of f in $H(M_n)$ and $\tilde{g}$ denote the equivalence class of g in $H(M, F_n)$. Define ψ: $H(M_n) \to H(M, F_n)$ by $\psi(\bar{f}) = \tilde{f}_C$.

1) ψ is well defined.

If f and f' are homeomorphisms of M_n with $\bar{f} = \bar{f}'$, then $f'f^{-1}$ is isotopic to the identity by some isotopy H_t. If we let $H_t' = (H_t)$, then $f_C' \circ f_C^{-1} \simeq 1_X$ (rel F_n) by the isotopy H_t'. Hence $\tilde{f}_C = \tilde{f}_C'$.

2) ψ is a homomorphism.

This follows since $(ff')_C = f_C f_C'$.

3) ψ is an epimorphism.

Proof of 3): let $\tilde{g} \in H(M, F_n)$.

3a. We can assume $g(p_i) = p_i$ for all i.

Assume g performs the permutation T on F_n. By Lemma 1 there exists a homeomorphism h on M_n which performs the permutation T^{-1} on $\{\partial D_i: 1 \leqslant i \leqslant n\}$. Hence, $h_C g(p_i) = p_i$ for each i. But if we can find $\bar{f} \in H(M_n)$ with $\psi(\bar{f}) = \widetilde{h_c g}$, then $\psi(\overline{hf}) = \tilde{g}$.

3b. We can assume g is P. L.

By Lemma 3 there exists a P. L. homeomorphism g' of M with $g' \simeq g$ (rel F_n). Hence $\tilde{g}' = \tilde{g}$.

3c. We can assume $g(D_i) = D_i$ for each i.

Both $\bigcup_{i=1}^{n} D_i$ and $g(\bigcup_{i=1}^{n} D_i)$ are regular neighborhoods of F_n in M and hence by the regular neighborhood theorem (see Theorem 8 of [6]), there is an isotopy of M (rel F_n) which moves $g(\bigcup_{i=1}^{n} D_i)$ back to $\bigcup_{i=1}^{n} D_i$. That is, g is isotopic (rel F_n) to a homeomorphism which sends each D_i to itself, so that we can assume the equivalence class of g is represented by such a homeomorphism.

Now, if $g(D_i) = D_i$ for each i, then we can define g': $M_n \to M_n$ by $g' = g|M_n$. $g'_C g^{-1}$ is a homeomorphism of M which is the identity on M_n. In particular, $g'_C g^{-1}|D_i$ is a homeomorphism of D_i keeping ∂D_i fixed. By Lemma 4 there exists an isotopy H_t^i of D_i, fixed on ∂D_i and p_i, which takes $g'_C g^{-1}|D_i$ to the identity on D_i.

Define $H_t(x) = x$ for $x \in M_n$ and $H_t(x) = H_t^i(x)$ for $x \in D_i$, then $g'_C g^{-1} \simeq 1_M(\text{rel} F_n)$ by the isotopy H_t. That is, $g'_C \simeq g$ (rel F_n) and hence $\psi(\bar{g}') = \tilde{g}$.

4) ψ is a monomorphism.

Given $\bar{f} \in H(M_n)$, we must show $f_C \simeq 1_M(\text{rel} F_n)$ implies f is isotopic to the identity on M_n, let r: $M - F_n \to M_n$ be the map given by retracting each $D_i - p_i$ onto its boundary and extending by the identity, i.e. $r(x) = x$ if $x \in M_n$ and $r(x) = e_i^{-1}(e_i(x)/|e_i(x)|)$ if $x \in D_i - p_i$. Suppose $f_C \simeq 1_M(\text{rel}F_n)$ by the isotopy H_t. Let $H'_t = H_t|M_n$ and let $G'_t = rH'_t$. Note that H'_t maps M_n into $M - F_n$ and G'_t maps M_n into M_n. G'_t is a homotopy taking $G'_0 = rH'_0 = rH_0|M_n = rf_C|M_n = rf = f$ to $G'_1 = rG'_1 = rG|M_n$, which is the identity on M_n.

Since G'_t is a homotopy of M_n taking f to the identity, we can conclude f is isotopic to the identity once it is shown that f satisfies the hypothesis of Lemma 5, i.e. we must eliminate the possibility that M_n is a plane, open cylinder or closed cylinder and f is orientation reversing. Obviously M_n is neither a plane nor an open cylinder, but it might be a closed cylinder. If that were the case and f were orientation reversing, then f_C would be an orientation reversing homeomorphism of $M \simeq S^2$ and hence not isotopic to 1_M. Thus Lemma 5 can be applied and the proof of Theorem 6 is completed.

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References

[1] J. Birman, *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. 22 (March, 1969), pp. 213–238.

[2] M. Dehn, *Die Gruppe der Abbildungsklassen*, Acta Math. 69 (1938), pp. 135–206.

[3] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), pp. 83–107.

[4] W. Magnus, *Über Automorphismen von Fundamentalgruppen berandeter Flächen*, Math. Ann. (1934), pp. 617–646.

[5] G. S. McCarty, *Homeotopy groups*, Trans. Amer. Math. Soc. 106 (1963), pp. 293–304.

[6] E. C. Zeeman, *Seminar on combinatorial topology*, Inst. Hautes Études Sci. Publ. Math., Paris 1963.

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