Boolean algebras, splitting theorems, and Δ_2^0 sets

by

Michael D. Morley (Ithaca, N. Y.) and **Robert I. Soare** (Chicago, Ill.) *

Abstract. For a set $S \subseteq \omega$ define $\mathcal{E}_S$, the lattice (under inclusion) of the recursively enumerable (r.e.) sets restricted to S, to be $\{W \cap S\colon W \text{ r.e.}\}$. Our principal result is a general splitting theorem for noncomplemented elements of $\mathcal{E}_S$, S in Δ_2^0, which simultaneously extends the well-known splitting theorems of Friedberg, Sacks, and Owings, although its proof is not difficult. A corollary is that if S is infinite and in Δ_2^0 (or even in Σ_2^0) then S is hyperhyperimmune if and only if $\mathcal{E}_S$ is a Boolean algebra. This immediately yields Lachlan's remarkable characterization of *hh*-simple sets as those coinfinite r.e. sets whose lattice of supersets forms a Boolean algebra. It follows that a (Turing) degree $\boldsymbol{a} \leqslant \boldsymbol{0}'$ is *high* ($\boldsymbol{a}' = \boldsymbol{0}''$) if and only if $\mathcal{E}_S$ forms a Boolean algebra for some infinite set $S \in \boldsymbol{a}$.

Introduction. Standard notation and terminology may be found in Rogers [11]. Let $\{\varphi_e\colon e \in \omega\}$ be an acceptable numbering of partial recursive functions and let $W_e =$ domain φ_e. Given $S \subseteq \omega$, and r.e. set W let W_S denote the set $W \cap S$ in $\mathcal{E}_S$. A member A_S of $\mathcal{E}_S$ is *complemented in* $\mathcal{E}_S$ if there exists an r.e. set B such that $A_S \cup B_S = S$ and $A_S \cap B_S = \emptyset$, and A is *noncomplemented in* $\mathcal{E}_S$ otherwise. (Of course, $\mathcal{E}_S$ is a Boolean algebra just if every member is complemented.) A set $S \subseteq \omega$ is *Boolean* just if S is infinite and $\mathcal{E}_S$ is a Boolean algebra. A set A is in Δ_2^0 if A can be put in both two number quantifier forms or equivalently [11, p. 314] if A is recursive in $\emptyset'$ (denoted $A \leqslant_T \emptyset'$).

Our principal result (Theorem 2.1) asserts that for S in Δ_2^0 (or even in Σ_2^0) any noncomplemented member A_S of $\mathcal{E}_S$ can be uniformly decomposed as the disjoint union of noncomplemented members B_S and C_S where B and C are Turing incomparable. This yields the splitting theorems of Sacks [12] and Friedberg [3] for $S = \omega$, and the splitting theorem of Owings [8] for S co-r.e. A corollary is that a Δ_2^0 (or even Σ_2^0) set S is Boolean if and only if it is hyperhyperimmune (*hh*-immune), which answers a question of Jockush and Soare. For S co-r.e., this is Lachlan's charac-

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terization [5] of hh-simple sets. The original proofs of these results were all different, although Owings showed [8] how Lachlan's result follows from his. We shall extend the method in the splitting theorem of Sacks to give one simple proof for all these results.

We shall then use results of Cooper and Jockush to prove that for any degree $\boldsymbol{a} \leqslant \boldsymbol{0}'$, $\boldsymbol{a}$ is high if and only if S is Boolean for some infinite set $S \in \boldsymbol{a}$. In contrast, the second author has shown [14] that if a degree $\boldsymbol{a}$ is *low* ($\boldsymbol{a}' = \boldsymbol{0}'$) then $\mathcal{E}_S \cong \mathcal{E}$ for every infinite $S \in \boldsymbol{a}$ (where $\mathcal{E}$ denotes $\mathcal{E}_\omega$).

1. Review of splitting theorems. We begin with an expository summary of related results. Post's famous problem [9] was to find a nonrecursive r.e. set A which is *incomplete* ($A <_T \emptyset'$). Specifically Post searched for a simple property on the complement $\bar{A}$ of an r.e. set A which would guarantee $A <_T \emptyset'$, and he made the following definition. A set S is *hyperhyperimmune* (*hh-immune*) if S is infinite and there is no recursive function f such that for all x and y,

(1.1) $W_{f(x)}$ is finite;

(1.2) $x \neq y \Rightarrow W_{f(x)} \cap W_{f(y)} = \emptyset$; and

(1.3) $W_{f(x)} \cap S \neq \emptyset$.

(An infinite set S for which there is no recursive function satisfying (1.2) and (1.3) is called *strongly hyperhyperimmune*. It is well-known that these properties coincide for S co-r.e. [15] or even Δ_2^0 [1].) An r.e. set A is *hyperhypersimple* (*hh-simple*) if $\bar{A}$ is hh-immune. Unfortunately, neither hh-simplicity [17] nor any property invariant under automorphisms of $\mathcal{E}$ can guarantee incompleteness [15], and Post's question was eventually answered by an entirely different method [2].

THEOREM 2.1 (Friedberg–Muchnik). *There exist r.e. sets A and B which are Turing incomparable.*

Friedberg then proved [3] several important results about r.e. sets, including the following.

THEOREM 1.2 (Friedberg Splitting Theorem). *If A is any nonrecursive r.e. set then there exist r.e. sets B and C such that*

i) $A = B \cup C$ *and* $B \cap C = \emptyset$; *and*

ii) B *and* C *are each r.e. but nonrecursive.*

Using the fact that an r.e. set A is recursive if and only if $\bar{A}$ is r.e., Friedberg's theorem asserts that any noncomplemented set $A \in \mathcal{E}$ "splits" as the disjoint union of two sets noncomplemented in $\mathcal{E}$. Later Sacks [12] simultaneously extended both of Friedberg's theorems.

THEOREM 1.3 (Sacks Splitting Theorem). *If A is any nonrecursive r.e. set then there exist r.e. sets B and C such that*

i) $A = B \cup C$ *and* $B \cap C = \emptyset$; *and*

ii) B *and* C *are Turing-incomparable.*

Even though hh-simple sets did not solve Post's problem, they played an important role in the development of the theory of r.e. sets and numerous interesting characterizations of them arose [16, Theorem 6] and [10, Theorems 2 and 4]. The most dramatic and important of these was discovered by Lachlan [5, Theorem 3]. For any $A \in \mathcal{E}$, Lachlan defines $\mathcal{L}(A) = \{W\colon\ W \supseteq A \;\&\; W \text{ r.e.}\}$, which forms a lattice under inclusion. (Note that $\mathcal{L}(A) \cong \mathcal{E}_{\bar{A}}$ for A r.e.)

THEOREM 1.4 (Lachlan). *For any coinfinite r.e. set A, A is hh-simple if and only if $\mathcal{L}(A)$ (or equivalently $\mathcal{E}_{\bar{A}}$) forms a Boolean algebra.*

One half of this theorem can be done much more generally in the following lemma, although its proof is the same as in Lachlan [5, p. 13].

LEMMA 1.5. *If S is infinite (not necessarily in Δ_2^0) and $\mathcal{E}_S$ is a Boolean algebra then S is strongly hh-immune (and hence hh-immune).*

Proof. If S is not strongly hh-immune as witnessed by $\{W_{f(n)}\colon n \in \omega\}$ for some recursive function f, then A_S is noncomplemented in $\mathcal{E}_S$ where $A = \bigcup \{W_n \cap W_{f(n)}\colon\ n \in \omega\}$. ■

The other direction of Lachlan's theorem was strengthened by Owings [8], with the help of R. W. Robinson.

THEOREM 1.6 (Owings Splitting Theorem). *Let A and D be r.e. sets such that $A - D$ is not co-r.e. Then there exist r.e. sets B and C, whose indices may be obtained uniformly from that of A, such that*

i) $A = B \cup C$ *and* $B \cap C = \emptyset$; *and*

ii) *neither* $B - D$ *nor* $C - D$ *is co-r.e.*

(Theorem 1.2 is, of course, the special case $D = \emptyset$.) The other direction of Lachlan's theorem now easily follows because if D is a coinfinite r.e. set such that $\mathcal{L}(D)$ is not a Boolean algebra then there exists r.e. $A \supseteq D$ such that $A - D$ is not co-r.e. Using Theorem 1.6 repeatedly, A may be split to produce a weak array $\{B_n\}_{n \in \omega}$ which satisfies (1.2) and (1.3) for $S = \bar{D}$ and therefore witnesses that $\bar{D}$ is not hh-immune. Namely, split A into $B_0 \cup C_0$, then split C_0 into $B_1 \cup C_1$, and so forth. The sequence $\{B_n\}_{n \in \omega}$ is r.e. by the uniformity of Theorem 1.6.

2. Splitting r.e. sets restricted to Δ_2^0 sets. We now give a single framework and proof for all the theorems mentioned in Section 1. Owings' theorem asserts for S co-r.e. (namely $S = \bar{D}$) that every noncomplemented A_S of $\mathcal{E}_S$ splits into the disjoint union of two noncomplemented members B_S and C_S of $\mathcal{E}_S$. We now obtain the same conclusion for the weaker hypothesis "S in Δ_2^0." Curiously, the proof becomes easier by adding the additional conclusion of the Sacks Splitting Theorem.

THEOREM 2.1. *For any set S in Δ_2^0 and any r.e. set A, if A_S is a noncomplemented member of $\mathcal{E}_S$ then there exist r.e. sets B and C, whose indices* *may be found uniformly from that of A, such that:*

i) $A = B \cup C$ *and* $B \cap C = \emptyset$;

ii) B_S *and* C_S *are noncomplemented in* $\mathcal{E}_S$; *and*

iii) B *and* C *are Turing incomparable.*

This theorem simultaneously generalizes the splitting theorems of Sacks (let $S = \omega$) and Owings (let $S = \bar{D}$) and leads to the following generalization of Lachlan's theorem.

THEOREM 2.2. *For any infinite set S in Δ_2^0, S is hh-immune if and only if $\mathcal{E}_S$ is a Boolean algebra (i.e. S is Boolean).*

Proof. If $\mathcal{E}_S$ contains a noncomplemented element A_S where A is r.e., then as in Theorem 1.6 we can repeatedly apply Theorem 2.1 to decompose A into an array $\{B_n: n \in \omega\}$ which witnesses that S is not hh-immune. The converse follows from Lemma 1.5. ∎

To prove Theorem 2.1 it will suffice to decompose A into disjoint r.e. sets B and C which satisfy the following "requirements" [7] for all $e \in \omega$

$$R_e^B\colon\ W_e^B \cap S \neq \bar{A} \cap S\,, \qquad R_e^C\colon\ W_e^C \cap S \neq \bar{A} \cap S\,,$$

as we now verify for B. The case of C is similar.

LEMMA 1. *If B satisfies requirements $\{R_e^B: e \in \omega\}$, then C_S is noncomplemented in $\mathcal{E}_S$.*

Proof. If $C \cap S = \bar{W}_e \cap S$ for some $e \in \omega$, then $W_i^B = W_e \cap \bar{B}$ contradicts R_i^B.

LEMMA 2. *If B satisfies requirements $\{R_e^B: e \in \omega\}$, then C is not recursive in B.*

Proof. If C is recursive in B then $\bar{C} = W_e^B$ for some $e \in \omega$, and hence $W_i^B = W_e^B \cap \bar{B}$ contradicts R_i^B.

Following Sacks [12] we shall satisfy requirement R_e^B by attempting to preserve *agreement* of $W_e^B \cap S$ with $\bar{A} \cap S$ rather than disagreement as one might suppose.

Proof of Theorem 2.1. Given S in Δ_2^0 there exists [13, p. 29] a recursive sequence $\{S_s: s \in \omega\}$ of finite sets such that for all x, $\lim_s S_s(x)$ exists and equals $S(x)$. Let f be a 1:1 recursive function with range A and define $A_s = \{f(0), f(1), \ldots, f(s)\}$. Let B_s and C_s denote the integers enumerated in B and C respectively by the end of stage s in the construction below. Define the recursive functions,

$$l^B(s, e) = \max\{x\colon\ x \leq s \,\&\, (\forall y < x)[y \in S_s \Rightarrow [y \in W_{e,s}^{B_s} \Leftrightarrow y \notin A_s]]\},$$

$$m^B(s, e) = \max\{l^B(t, e)\colon\ t \leq s\}\,,$$

$$r^B(s, e) = \max\{z\colon\ (\exists x)[x \leq m^B(s, e) \,\&\, x \in W_{e,s}^{B_s} \,\&\, z \text{ is used in the latter computation}]\}\,,$$

and similarly $l^C(s, e)$, $m^C(s, e)$, $r^C(s, e)$ with C in place of B.

Stage $s = 0$. Enumerate $f(0)$ in B.

Stage $s+1$. Let $e(B) = \mu e[f(s+1) \leq r^B(s, e)]$ and $e(C) = \mu e[f(s+1) \leq r^C(s, e)]$. Enumerate $f(s+1)$ in B if either $e(B)$ is undefined or if both $e(B)$ and $e(C)$ are defined and $e(B) > e(C)$. Enumerate $f(s+1)$ in C otherwise. This completes the construction.

We say that a requirement R_e^B is *injured at stage* $s+1$ if some $y \leq r^B(s, e)$ is enumerated in B at stage $s+1$.

LEMMA 3. *For each $e \in \omega$, if R_e^B is injured at most finitely often, then $W_e^B \cap S \neq \bar{A} \cap S$.*

Proof. Fix e and assume that R_e^B is never injured after some stage s_0. Now if $W_e^B \cap S = \bar{A} \cap S$, then $\lim_s l^B(s, e) = \lim_s m^B(s, e) = \infty$. Define the r.e. set,

$$W_i = \{x\colon\ (\exists s \geq s_0)[x \in W_{e,s}^{B_s} \,\&\, x \leq m^B(s, e)]\}\,.$$

We claim $W_i = W_e^B$ contrary to our assumption that A_S is noncomplemented in $\mathcal{E}_S$. Clearly $W_i \supseteq W_e^B$ because $\lim_s m^B(s, e) = \infty$. However, $W_i \subseteq W_e^B$ by the choice of s_0, the definition of $r^B(s, e)$ and the fact that $\lambda s[m^B(s, e)]$ is nondecreasing.

LEMMA 4. Each requirement is injured at most finitely often.

Proof. Fix e and assume by induction that for all $i < e$, requirements R_i^B and R_i^C are injured at most finitely often. Then by Lemma 3 and its analogue for C in place of B, the conditions of R_i^B, R_i^C are satisfied for all $i < e$. Thus, $\{m^C(s, i)\colon i < e \,\&\, s \in \omega\}$ is bounded, say by m_0, because $\lambda s[m^C(s, i)]$ is nondecreasing as a function of s. But if $f(s) > m_0$ for all $s >$ some s_0, then R_e^B cannot be injured after stage s_0. The case of R_e^C is similar. ∎

Theorem 2.1 now follows from the lemmas.

COROLLARY 2.3. *If degree $\boldsymbol{a} \leq \boldsymbol{0}'$, then*

$$\boldsymbol{a}' = \boldsymbol{0}'' \Leftrightarrow (\exists \text{ infinite set } A \in \boldsymbol{a})\ [\mathcal{E}_A \text{ is a Boolean algebra}]\,.$$

Proof. If $\boldsymbol{a} \leq \boldsymbol{0}'$ and $\boldsymbol{a}' = \boldsymbol{0}''$ then by Jockusch [4, p. 491] $\boldsymbol{a}$ contains a cohesive (and hence hh-immune) set C. But $\mathcal{E}_C$ is isomorphic to the Boolean algebra of finite and cofinite sets.

Conversely, if $\boldsymbol{a}$ contains an infinite set A such that $\mathcal{E}_A$ is a Boolean algebra, then by Theorem 2.2, A is hh-immune. But then by Cooper [1] $\boldsymbol{a}' = \boldsymbol{0}''$.

3. Further remarks and questions. Note that some hypothesis such as "S in Δ_2^0" was necessary for Theorem 2.2, because there exists a set S (even $\leq_T \emptyset''$) such that S is hh-immune but not Boolean. (Use the method of [11, Exercise 12–51] to construct $S \leq_T \emptyset''$ which is hh-immune, but not strongly hh-immune, and thus cannot be Boolean by Lemma 1.5.) The same method easily yields more.

THEOREM 3.1. *If A is any nonrecursive r.e. set there exists a hh-immune set $S \leqslant_T \emptyset''$ such that A_S is noncomplemented in $\mathcal{E}_S$.*

Proof. Since A is nonrecursive the set $V_e = (A \cap W_e) \cup (\bar{A} \cap \bar{W}_e)$ is infinite for each e. A routine non-constructive diagonalization (recursive in $\emptyset''$) produces a set S which intersects each V_e but is disjoint from at least one set in every recursively enumerable sequence of disjoint nonempty finite sets. ∎

Carl Jockusch has observed that there is also a *strongly* hh-immune set S which is not Boolean, and indeed $S = D \cup D'$ is such a set, where D and D' are as in [11, Theorem XVII on p. 242]. This is because $D \cup D'$ is infinite, quasi-cohesive, and indecomposable (defined in [11, p. 240]) but not cohesive. However, the definitions trivially imply that if S is Boolean and indecomposable then S is cohesive. Now $D \cup D'$ being both quasi-cohesive and indecomposable possesses all the "almost-finiteness" properties in Rogers' diagram [11, p. 243] which are strictly weaker than cohesiveness.

One might hope to obtain the full relativization of the Sacks' splitting theorem (for the universe S in place of ω) by replacing conclusion iii) of Theorem 2.1 by "B_S and C_S are Turing incomparable." Carl Jockusch has observed that this is impossible even for S co-r.e. Let S be a complete co-r.e. set which is introreducible (recursive in each of its infinite subsets) but not hh-immune. (For example, let $\bar{S}$ be the deficiency set [11, p. 140] of a complete r.e. set.) For such a set S note that $A_S \equiv_T S$ whenever A_S is infinite.

It is natural to ask exactly *which* Boolean algebras are realized by $\mathcal{E}_S$ as S ranges through all sets in Δ_2^0. John Norstad has observed that for S in Δ_2^0, a simple Tarski-Kuratowski algorithm shows that $\mathcal{E}_S^*$, the lattice $\mathcal{E}_S$ modulo the ideal $\mathcal{F}$ of finite sets, is always an $\exists\forall\exists$-lattice as defined by Lachlan [5, p. 21]. Therefore, no new Boolean algebras are obtained by $\mathcal{E}_S^*$ if Lachlan's condition "S co-r.e." [5, Theorem 6] is replaced by "S in Δ_2^0." We close with the following open questions.

QUESTION 1. Does there exist a characterization, analogous to Corollary 2.3 for those degrees $\boldsymbol{a} \leqslant \boldsymbol{0}'$ which are high$_2$, i.e., which satisfy $\boldsymbol{a}'' = \boldsymbol{0}'''$? Lachlan [6, Theorem 3] has shown that for such degrees $\boldsymbol{a}$ which are r.e., there exists an infinite co-r.e. $S \in \boldsymbol{a}$ such that $\mathcal{E}_S^*$ contains no maximal elements (coatoms).

In [14] it is shown for $\boldsymbol{a} \leqslant \boldsymbol{0}'$,

$$\boldsymbol{a}' = \boldsymbol{0}' \Leftrightarrow (\forall \text{ infinite set } A \in \boldsymbol{a})\,[\mathcal{E}_A^* \cong^{\text{eff}} \mathcal{E}^*]\,,$$

where $\cong^{\text{eff}}$ denotes an effective automorphism in the obvious sense.

QUESTION 2. Does there exist a similar characterization for degrees $\boldsymbol{a} \leqslant \boldsymbol{0}'$ which are low$_2$, i.e., which satisfy $\boldsymbol{a}'' = \boldsymbol{0}''$? Lachlan [6, Theorem 4] has shown that for every infinite co-r.e. set S in such a degree, $\mathcal{E}_S^*$ contains maximal elements. In particular, is $\mathcal{E}_S \cong \mathcal{E}$ for all infinite sets S in such degrees?

Added May 27, 1974. Carl Jockusch has noted that Theorems 2.1 and 2.2 can easily be strengthened by replacing "S in Δ_2^0" by "S in Σ_2^0", and Theorem 3.1 can be sharpened by adding "S in Π_2^0" to the nonclusion. To prove Theorem 2.1 for S in Σ_2^0, note that for any r.e. A such that A_S is noncomplemented in $\mathcal{E}_S$ there is a Δ_2^0 set $S_0 \subseteq S$ such that A is noncomplemented in $\mathcal{E}_{S_0}$. (Use a $\emptyset'$ oracle to enumerate S_0 in increasing order so that S_0 intersects each set of the form $(\bar{W}_e \mathbin{\Delta} A) \cap S$. These sets are all infinite because A is noncomplemented in $\mathcal{E}_S$.) The requirements R_e^B, R_e^C hold for S_0 and thus *a fortiori* for S. Furthermore, S_0 is obtained uniformly from S, so that Theorem 2.1 extended to "$S \in \Sigma_2^0$" is still uniform. Theorem 2.2 extended to "$S \in \Sigma_2^0$" follows because a Σ_2^0 set is hh-immune if and only if it is strongly hh-immune.

References

[1] S. B. Cooper, *Jump equivalence of the Δ_2^0 hyperhyperimmune sets*, J. Symbolic Logic 37 (1972), pp. 598–600.

[2] R. M. Friedberg, *Two recursively enumerable sets of incomparable degrees of unsolvability*, Proceedings of the National Academy of Sciences 43 (1957), pp. 236–238.

[3] — *Three theorems on recursive enumeration: I, Decomposition, II, Maximal set, III, Enumeration without duplication*, J. Symbolic Logic 23 (1958), pp. 309–316.

[4] C. G. Jockusch, Jr., *The degrees of hyperhyperimmune sets*, J. Symbolic Logic 34 (1969), pp. 489–493.

[5] A. H. Lachlan, *On the lattice of recursively enumerable sets*, Trans. Amer. Math. Soc. 130 (1968), pp. 1–37.

[6] — *Degrees of recursively enumerable sets which have no maximal superset*, J. Symbolic Logic 33 (1968), pp. 431–443.

[7] — *On some games which are relevant to the theory of recursively enumerable sets*, Ann. of Math. 91 (1970), pp. 291–310.

[8] J. C. Owings, Jr., *Recursion, metarecursion and inclusion*, J. Symbolic Logic 32 (1967), pp. 173–178.

[9] E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. 50 (1944), pp. 284–316.

[10] R. W. Robinson, *Simplicity of recursively enumerable sets*, J. Symbolic Logic 32 (1967), pp. 162–172.

[11] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, New York 1967.

[12] G. E. Sacks, *Degrees of unsolvability* (revised edition), Ann. of Math. Studies 55, Princeton, New Jersey, 1966.
[13] J. R. Shoenfield, *Degrees of Unsolvability*, Amsterdam 1971.
[14] R. I. Soare, *Automorphisms of the lattice of recursively enumerable sets II. Low sets*, to appear.
[15] — *Post's program and complete sets*, to appear.
[16] C. E. M. Yates, *Recursively enumerable sets and retracing functions*, Z. Math. Logic Grundlagen Math. 8 (1962), pp. 331–345.
[17] — *Three theorems on the degrees of recursively enumerable sets*, Duke Math. J. 32 (1965), pp. 461–468.

CORNELL UNIVERSITY, Ithaca, New York
UNIVERSITY OF CHICAGO, Chicago, Illinois

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Remarks on invariant descriptive set theory *

by

John Burgess and **Douglas Miller** ** (Madison, Wisc.)

Abstract. Let X be a separable, completely metrizable space and E an analytic equivalence relation on X. $A \subset X$ is E-invariant if $y \in A$ whenever $x \in A$ and $E(x, y)$. We prove that the classes of E-invariant coanalytic sets and of E-invariant PCA sets each satisfy the Reduction Principle, and give E-invariant versions of other classical theorems. Our results generalize work of Vaught and others.

Let X be a Polish (separable, completely metrizable) space with $E \subseteq X \times X$ an equivalence relation on X. $B \subseteq X$ is *invariant* (with respect to E) provided $y \in B$ whenever $x \in B$ and $x E y$.

It is known (cf. [1]) that if E is a *countably separated* Σ^1_1 (analytic) equivalence, then X/E is Borel isomorphic to an analytic space (a metrizable continuous image of ω^ω) and, hence, that most theorems of descriptive set theory hold in invariant form.

Invariant version of several classical theorems have been proved under much weaker assumptions that countable separatedness. It has long been known (cf. our remarks after 1.2 below) that the invariant first separation principle, *Disjoint invariant Σ^1_1 sets can be separated by an invariant Borel set*, could be derived quite simply from the classical (non-invariant) theorem assuming only that E be Σ^1_1.

As 1.1 and 1.3 below we prove the invariant reduction principles:

If E is a Σ^1_1 equivalence then both the classes of invariant Π^1_1 (coanalytic) subsets of X and of invariant Σ^1_2 (PCA) subsets of X have the reduction property.

These results extend recent work of Y. N. Moschovakis ([18] and [19]) and R. L. Vaught ([23] and [24]). Vaught had proved the invariant reduction principles on the assumption that E be a "Polish action" equiva-

* Theorems 1.1, 1.7, 2.5 and all the new results in § 3 are due to Burgess. 1.3, 1.4, 4.2, the preliminary version of 1.6 and all of § 2 except 2.5 are due to Miller. All other results were proved jointly.

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