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Concerning unicoherence of continua

by

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Abstract. In this paper we investigate the unicoherence of a continuum M, knowing that the elements of a certain decomposition $\mathcal{G}$ of M are unicoherent. We confine ourselves to considering only upper semi-continuous monotone decompositions. In this case, if the decomposition space $M/\mathcal{G}$ is a dendroid and for each subcontinuum K of M each element G of $\mathcal{G}$ the intersection $K \cap G$ is a continuum, then M is unicoherent.

If M is a hereditarily decomposable continuum which is irreducible about a finite set and $\mathcal{G}$ is an admissible decomposition of M, the suppositions may reduce to the single one that the elements of $\mathcal{G}$ are unicoherent.

We obtain analogous assertions concerning the hereditary unicoherence of continua if the elements of $\mathcal{G}$ are such.

In this paper a continuum means a compact connected metric space.

Let M be a continuum. A family $\mathcal{G}$ of closed disjoint subsets of M covering M is said to be a decomposition of M.

The decomposition $\mathcal{G}$ of M is said to be *monotone* if its elements are continua.

The decomposition $\mathcal{G}$ of M is said to be *upper semi-continuous* if for each open subset U of M containing some element G of $\mathcal{G}$ there exists an open subset V of M such that $G \subset V \subset U$ and V is the union of the elements of $\mathcal{G}$ intersecting it. For equivalent definitions of this concept see [3], pp. 183–185, or [8], p. 122.

Let I be a continuum irreducible from a to b. Suppose that one can define a non-trivial upper semi-continuous monotone decomposition $\mathcal{G}$ of I such that each element of $\mathcal{G}$ not containing a and b separates I. It is shown in [7] that in this case there exists a unique decomposition which is minimal with respect to the above properties. Its elements are called *layers* of I.

The upper semi-continuous monotone decomposition $\mathcal{G}$ of the continuum M is said to be *admissible* if, for each irreducible continuum $I \subset M$ and for each layer T of I, there exists an element G of $\mathcal{G}$ containing T (compare [2], p. 115).

A dendroid means an arcwise connected and hereditarily unicoherent continuum.

A λ-dendroid means a hereditarily decomposable and hereditarily unicoherent continuum.

Let M be a hereditarily decomposable continuum which is irreducible about a finite set W and let $w \in W$. It is shown in [6] that the set $E = \{x \in M: M \text{ is irreducible about } \{x\} \cup (W \setminus \{w\})\}$ is a continuum. We call E an *end-continuum* of M. This result of [6] generalizes an analogous result for irreducible continua obtained in [5]. In this paper we purpose to show that a continuum M is unicoherent or hereditarily unicoherent if the elements of a certain decomposition of M possess the same property. In particular, we consider the case where M is hereditarily decomposable and irreducible about the finite set.

THEOREM 1. *If an upper semi-continuous monotone decomposition $\mathcal{G}$ of the continuum M is such that*

1° *the elements of $\mathcal{G}$ are unicoherent,*

2° *the decomposition space $M/\mathcal{G}$ is a dendroid,*

3° *for each subcontinuum K of M and each element G of $\mathcal{G}$ the intersection $K \cap G$ is a continuum,*

then M is unicoherent.

Proof. Denote the decomposition space $M/\mathcal{G}$ by T and the quotient map of M onto T by f. Thus $f\colon M \to M/\mathcal{G} = T$ is monotone.

Let $M = K \cup L$, where K and L are proper subcontinua of M. The set $K \cup L$ is closed. We shall prove that it is strongly connected.

Let a and b be arbitrary points of $K \cap L$ and put $a' = f(a)$ and $b' = f(b)$. By 2° the space T is a dendroid, so there exists a unique arc (a', b') T having a' and b' as its end-points. Let $S = f^{-1}((a', b'))$. Since f is monotone, S is a continuum.

Let $K_1 = K \cap S$ and $L_1 = L \cap S$. We shall prove that K_1 and L_1 are continua.

Let K^* denote a subcontinuum of K which is irreducible from a to b. Put $f_1 = f|_{K^*}$. If $t \in f_1(K^*)$, then $f_1^{-1}(t) = K^* \cap f^{-1}(t)$. The set $f^{-1}(t)$ is an element of $\mathcal{G}$ and from 3° we see that $f_1^{-1}(t)$ is a continuum. Thus the map f_1 is monotone. Since K^* is irreducible from a to b, and since f_1 is monotone, $f(K^*) = f_1(K^*)$ is irreducible from $a' = f_1(a) = f(a)$ to $b' = f_1(b) = f(b)$ (see [4], § 48, Theorem 3, p. 192). Thus $f(K^*) \subset T$ is irreducible from a' to b', and since T is a dendroid, there exists a unique subcontinuum of T irreducible from a' to b', namely the arc (a', b'). Thus $f(K^*) = (a', b')$ and therefore $K^* \subset f^{-1}(f(K^*)) = f^{-1}((a', b')) = S$.

Suppose $K_1 = K \cap S = A \cap B$, where A and B are closed disjoint sets and $A \neq \emptyset \neq B$. Let $a \in A$. The continuum K^* is contained in both S and K and therefore K^* is contained in $K \cap S$. Since K^* is connected and K^* contains both a and b, we have $K \subset A$ and $b \in A$.

Let $c \in B$ and let G_c be the element of $\mathcal{G}$ which contains c, i.e., $c \in G_c \in \mathcal{G}$. Since $G_c \cap S$ contains c, we have $G_c \subset S$, whence $K \cap G_c \subset K \cap S$. By 3° the set $K \cap G_c$ is connected and since c belongs to both B and $K \cap G_c$, we have $K \cap G_c \subset B$. The points a and b on the one hand and c on the other hand belong to different components of K_1; therefore $a \neq c \neq b$. Moreover, since for each $G \in \mathcal{G}$ the set $K \cap G$ is connected by 3°, $K \cap G_a$ and $K \cap G_b$ on the one hand and $K \cap G_c$ on the other are contained in different components of K_1, and therefore $G_a \neq G_c \neq G_b$ (here G_a and G_b denote the elements of $\mathcal{G}$ containing a and b, respectively).

Observe that the set $S \setminus G_c$ is not connected by the construction, and that points a and b are in different components of $S \setminus G_c$. Since both a and b belong to K, we see that $K \cap (S \setminus G_c)$ is not connected and it contains a and b in different components of it. Since $K^* \subset A$, $K \cap G_c \subset B$ and $A \cap B = \emptyset$, we have $K^* \cap (K \cap G_c) = \emptyset$, and we conclude from $K^* \subset K \cap S$ that $K^* \subset K \cap S \setminus (K \cap G_c) = K \cap (S \setminus G_c)$. Therefore a and b, as points of the continuum K^*, belong to the same component of $K \cap (S \setminus G_c)$: a contradiction. This proves that K_1 (and similarly L_1) is connected. Since they are closed by definition, we have just proved that K_1 and L_1 are continua.

Let us consider sets of the type $G \cap K_1 \cap L_1$, where G is an element of $\mathcal{G}$. For each $G \in \mathcal{G}$ such that $G \subset S$ the following properties hold.

1° *The intersections $G \cap K_1$ and $G \cap L_1$ are both non-empty.*

In fact, suppose that $G \cap K_1 = \emptyset$ for some G. Thus

$$K_1 = K_1 \setminus G \cap K_1 = K \cap S \setminus G \cap K \cap S = K \cap (S \setminus G)\,. \tag{1}$$

Since $a, b \in K_1$, we see by the assumption that a and b are not in G; therefore they are in different components of the set $S \setminus G$, and thus of the intersection $K \cap (S \setminus G)$; a contradiction of (1).

2° *The intersection $G \cap K_1 \cap L_1$ is non-empty.*

In fact, by property 1° we have $G \cap K_1 \neq \emptyset$ and $G \cap L_1 \neq \emptyset$.

$$\begin{aligned}(G \cap K_1) \cup (G \cap L_1) &= G \cap (K_1 \cup L_1) = G \cap (K \cap S \cup L \cap S) \\ &= G \cap (S \cap (K \cup L)) = G \cap S \cap M = G\,.\end{aligned} \tag{2}$$

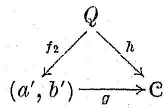
Thus G is represented as the union of two non-empty closed sets, namely $G \cap K_1$ and $G \cap L_1$. Since G is connected, we have

$$(G \cap K_1) \cap (G \cap L_1) \neq 0, \qquad \text{i.e.} \qquad G \cap K_1 \cap L_1 \neq \emptyset.$$

3° *The set $G \cap K_1 \cap L_1$ is a continuum.*

In fact, by 3° we infer that $G \cap K_1$ and $G \cap L_1$ are continua, since K_1 and L_1 are. Formula (2) shows that $(G \cap K_1) \cup (G \cap L_1) = G$. Since G is unicoherent by 1°, we conclude that $(G \cap K_1) \cap (G \cap L_1) = G \cap K_1 \cap L_1$ is a continuum.

We shall prove that $Q = K_1 \cap L_1$ is a continuum. To prove this it is sufficient to show that Q possesses only one component. Let us recall that the intersection of all open and closed (relatively to Q) subsets of Q which contain a point $p \in Q$ is called a *quasi-component* of Q containing p. If the set Q is compact, the concept of a component of Q and of a quasi-component of Q coincide (see [4], § 47, Theorem 2, p. 169). Further, if Q is separable, there exists a continuous map $h\colon Q \to \mathcal{C}$ (here $\mathcal{C}$ is the well-known Cantor discontinuum) such that point inverses under h are quasi-components of Q (see [4], § 46, Theorem 3, p. 148). Let us consider the map $f_2\colon Q \to (a', b')$ defined as the restriction of f on Q. Let t be an arbitrary point of (a', b') and let $G_t = f^{-1}(t) \in \mathcal{G}$. It follows by property 2° that $G_t \cap K_1 \cap L_1 \neq \emptyset$, i.e., $G_t \cap Q \neq \emptyset$. The latter implies that $f_2(G_t \cap Q) = f(G_t) = \{t\}$. Thus f_2 maps Q onto (a', b'). Observe further that according to properties 2° and 3° $f_2^{-1}(t) = f^{-1}(t) \cap Q = G_t \cap K_1 \cap L_1$ is a non-empty continuum. Thus $f_2^{-1}(t)$ is contained in only one component of Q and since the set $f_2^{-1}(t)$ is non-empty; therefore putting $g(t) = h(f_2^{-1}(t))$ we see that $g\colon (a', b') \to \mathcal{C}$ is a well-defined map of $(a' b')$ into $\mathcal{C}$. The map g is defined in such a way that the following diagram commutes:



The map f_2 is continuous as a restriction of a continuous map. Let F be an arbitrary closed subset of $\mathcal{C}$. Since h is continuous, $h^{-1}(F)$ is closed in Q, and since Q is compact, $f_2(h^{-1}(F))$ is closed. Thus the map g is continuous. Therefore, (a', b') being connected, its image $g((a', b'))$ is a connected subset of $\mathcal{C}$, and since $\mathcal{C}$ is totally disconnected, we conclude that $g((a', b'))$ is a set containing only one point. Further, $h(Q) = g(f_2(Q)) = g((a', b'))$; therefore $Q = K_1 \cap L_1$ has only one component. Thus the set $K_1 \cap L_1$ is connected. Since it is obviously closed, it is a continuum.

The points a and b belong to both K_1 and L_1, and so they belong to the intersection $K_1 \cap L_1$. Since $K_1 \subset K$ and $L_1 \subset L$, we have $K_1 \cap L_1 \subset K \cap L$. Thus the set $K \cap L$ is strongly connected, which completes the proof.

THEOREM 2. *If an upper semi-continuous monotone decomposition $\mathcal{G}$ of the continuum M has properties 2° and 3° of Theorem 1, and 1′ the elements of $\mathcal{G}$ are hereditarily unicoherent, then M is hereditarily unicoherent.*

Proof. Let P be an arbitrary subcontinuum of M. We consider a decomposition $\mathfrak{D}$ of P defined as follows:

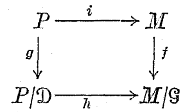
$$\mathfrak{D} = \{D \subset P : D = G \cap P \text{ for some } G \in \mathcal{G}\} .$$

The decomposition $\mathfrak{D}$ is monotone by 3°. It is upper semi-continuous, since $\mathcal{G}$ is upper semi-continuous and P is a continuum (see [1], Lemma 1, p. 117).

We shall prove that the decomposition $\mathfrak{D}$ has properties 1°, 2° and 3° of Theorem 1.

Since each element of $\mathfrak{D}$ is a subcontinuum of an element of $\mathcal{G}$, property 1° follows by 1′.

To prove 2°, let us consider the decomposition space $P/\mathfrak{D}$ and let $f\colon M \to M/\mathcal{G}$ and $g\colon P \to P/\mathfrak{D}$ be the quotient maps. We shall define a map $h\colon P/\mathfrak{D} \to M/\mathcal{G}$. Let $t \in P/\mathfrak{D}$ be an arbitrary point and let $G_t \in \mathcal{G}$ be such that $g(G_t \cap P) = t$. The set G_t is uniquely determined. We define $h(t) = f(G_t)$. The map h is defined in such a way that the following diagram commutes:



where i denotes the inclusion mapping.

The map h is injective. In fact, if $t_1 \neq t_2$, $t_j \in P/\mathfrak{D}$, then $G_{t_1} \neq G_{t_2}$ and therefore $h(t_1) \neq h(t_2)$.

Let F be an arbitrary closed set of $M/\mathcal{G}$. Since f is continuous, $f^{-1}(F)$ is a closed set in M; thus $P \cap f^{-1}(F)$ is closed in P and since P is a continuum and g is continuous, the set $g(P \cap f^{-1}(F))$ is closed in $P/\mathfrak{D}$. Thus the map h is continuous, and since it is injective, h is an imbedding of the continuum $P/\mathfrak{D}$ into the dendroid $M/\mathcal{G}$. Therefore $P/\mathfrak{D}$ is a dendroid.

To prove 3°, let Q be an arbitrary subcontinuum of P and let D be an element of $\mathfrak{D}$. The set $Q \cap D = Q \cap G \cap P = Q \cap G$ is a continuum by 3°, which is assumed for the decomposition $\mathcal{G}$ of M.

Thus we have just shown that the hypotheses of Theorem 1 are satisfied for the decomposition $\mathfrak{D}$ of P. Hence P is unicoherent and consequently M is hereditarily unicoherent.

THEOREM 3. *Let $\mathcal{G}$ be an admissible decomposition of the hereditarily decomposable continuum M. Then M is a λ-dendroid if and only if properties 1′, 2° and 3° assumed for $\mathcal{G}$ are satisfied.*

Proof. The sufficiency follows from Theorem 2. The necessity of properties 1′ and 3° follows at once by the hereditary unicoherence of the λ-dendroids. The necessity of 2° is established in [1], Theorem 5, p. 26.

LEMMA 1. *Let M be a continuum which is irreducible about a finite set containing n points and let $\mathcal{G}$ be an admissible decomposition of M. Then $M/\mathcal{G}$ is a dendroid with at most n ends.*

Proof. Let M be irreducible about the set $\{a_1, a_2, \ldots, a_n\}$, let $f\colon M \to M/\mathcal{G}$ be the quotient map and let $a_k' = f(a_k)$, $1 \leqslant k \leqslant n$. It is easy to verify that since f is monotone, the continuum $M/\mathcal{G}$ is irreducible about the set $\{a_1', a_2', \ldots, a_n'\}$ (the proof is exactly the same as in the case $n=2$ given in [4], § 48, Theorem 3, p. 192).

In [2] (see [2], Theorem 1, p. 116) it is shown that the decomposition space of an admissible decomposition is arcwise connected (even hereditarily arcwise connected). We denote by l_k, $1 \leqslant k \leqslant n-1$, a fixed arc of $M/\mathcal{G}$ such that both a_k' and a_n' belong to l_k. We define inductively continua L_k, $1 \leqslant k \leqslant n-1$. Let $L_1 = l_1$ and let $a_k \in l_{k+1} \cap L_k$ be such that if

$$(\beta_k, a_{k+1}') \subsetneqq (a_k, a_{k+1}') \subset l_k ,$$

then $\beta_k \notin L_k$. We define $L_{k+1} = L_k \cup (a_k, a_{k+1}')$, where $(a_k, a_{k+1}') \subset l_{k+1}$. For each k, $1 \leqslant k \leqslant n-1$, the set L_K is a dendroid with at most $k+1$ ends by construction.

Since L_{n-1} contains each point a_k', $1 \leqslant k \leqslant n$, and since $M/\mathcal{G}$ is irreducible about the set $\{a_1', a_2', \ldots, a_n'\}$, we see that $L_{n-1} = M/\mathcal{G}$, which finishes the proof.

Lemma 2. *Let M be a hereditarily decomposable continuum which is irreducible about a finite set containing n points and let $\mathcal{G}$ be an upper semi-continuous monotone decomposition of M such that the decomposition space $M/\mathcal{G}$ is a dendroid. If G is an element of $\mathcal{G}$, then*

(a) $M \setminus G = U_1 \cup U_2 \cup \ldots \cup U_k$, *where $k \leqslant n$ and where U_i, $1 \leqslant i \leqslant k$, are open connected sets.*

(b) *For each i, $1 \leqslant i \leqslant k$, the set U_i is irreducible about a finite set and it has $\overline{U}_i \cap G$ as its end-continuum.*

Proof. (a) Among the components of $M \setminus G$ we choose $U_1, U_2, \ldots, U_k$, where $k \leqslant n$, such that every end-continuum of M which is not contained in G intersects some U_i, $1 \leqslant i \leqslant k$.

For each i, $1 \leqslant i \leqslant k$, the union $U_i \cup G$ is a continuum. Indeed, either $U_i \cup G = M$ (in this case $k=1$) or G separates M into at least two different components. In the second case the above statement is established in [4], § 47, as Theorem 3 on p. 168.

It follows that $\bigcup_{i=1}^{k} U_i \cup G$ is a continuum and since it intersects all the end-continua of M, we have $M = \bigcup_{i=1}^{k} U_i \cup G$. Therefore $M \setminus G = \bigcup_{i=1}^{k} U_i$, where k is at most n. Obviously we have

$$U_i = M \setminus \bigcup \{U_j \cup G\colon j = 1, 2, \ldots, i-1, i+1, \ldots, k\}.$$

and since $\bigcup \{U_i \cup G\colon j = 1, 2, \ldots, i-1, i+1, \ldots, k\}$ is a continuum; therefore U_i is an open set.

(b) Let i denote a fixed natural number, $1 \leqslant i \leqslant k$, let U_i intersect s end-continua of M: $E_1, E_2, \ldots, E_s$, let $a_j \in E_j \cap U_i$, $1 \leqslant j \leqslant s$, and let $x \in \overline{U}_i \cap G$.

We denote by M' a continuum which is irreducible about the set $\{x, a_1, \ldots, a_s\}$ and which is contained in $\overline{U}_i$. Observe that $M' \cup (M \setminus U_i) = M$, since $M' \cup (M \setminus U_i)$ is a continuum intersecting all the ends of M. Thus $U_i \subset M'$, whence $\overline{U}_i \subset M'$, and consequently we have $M' = \overline{U}_i$. Therefore $\overline{U}_i$ is irreducible about $s+1$ points and the set $\overline{U}_i \cap G$ is contained in one of the end-continua of U_i.

We shall prove that if x_1 does not belong to $\overline{U}_i \cap G$, there exists a proper subcontinuum A of $\overline{U}_i$ such that A contains x_1 and all the points a_j, $1 \leqslant j \leqslant s$. Since $\mathcal{G}$ is monotone, the quotient map $f\colon M \to M/\mathcal{G}$ is monotone. We put $a_j' = f(a_j)$, $1 \leqslant j \leqslant s$. Let $x_1 \in U_i$ and let $x_1' = f(x_1)$. Since U_i is connected, $f(U_i)$ is connected, and since $a_j' \in f(U_j)$ and $M/\mathcal{G}$ is a dendroid, we conclude that $(a_j', x_1') \subset f(U_i)$ and consequently $A' = \bigcup_{j=1}^{s} (a_j', x_1) \subset f(U_i)$. The set A' is a continuum as a finite union of arcs which have the point x_1' in common. Since the map f is monotone, $A = f^{-1}(A')$ is a continuum contained in U_i and containing the set $\{x_1, a_1, \ldots, a_s\}$. Therefore x_1 does not belong to the end-continuum of U_i containing $\overline{U}_i \cap G$, which shows that the set $\overline{U}_i \cap G$ is an end-continuum of M. The proof of Lemma 2 is finished.

Lemma 3. *Let M be a hereditarily decomposable continuum which is irreducible about a finite set and let $\mathcal{G}$ be an upper semi-continuous monotone decomposition of M such that the decomposition space $M/\mathcal{G}$ is a dendroid. If K is a subcontinuum of M, then for each element G of $\mathcal{G}$ the intersection $K \cap G$ is a continuum.*

Proof. Suppose that for some subcontinuum K of M and for some $G \in \mathcal{G}$ the intersection $K \cap G$ is not a continuum, i.e., $K \cap G = X \cup Y$, where X and Y are closed disjoint non-empty sets. By Lemma 2 we have $M \setminus G = U_1 \cup U_2 \cup \ldots \cup U_k$, $K \leqslant n$, where U_i, $1 \leqslant i \leqslant k$, are open sets (here n is the cardinality of the minimal set W such that M is irreducible about W). It is clear that K intersects some of the sets U_i; otherwise $K \subset G$ and $K \cap G = K$ would be a continuum.

We shall prove that if $K \cap U_i \neq \emptyset$, then $\overline{U}_i \cap G$ is contained in K. Let $x_0 \in K \cap U_i$ and let $x_1 \in K \cap G$. We choose continua $I_1 \subset K$ irreducible from x_0 to x_1 and $I_2 \subset U_i$ irreducible about x_0 and the end-continua of M intersecting U_i. The existence of the latter is established by Lemma 2.

The set $I_1 \cup I_2$ is a continuum which intersects $M \setminus U_i$. Therefore $I_1 \cup I_2 \cup (M \setminus U_i)$ is a continuum intersecting all the end-continua of M, which ensures that $I_1 \cup I_2 \cup (M \setminus U_i) = U$. Thus $I_1 \cup I_2 \supset \overline{U}_i$. Therefore $\overline{U}_i \cap G \subset I_1 \cup I_2$ and since $I_2 \cap G = \emptyset$, we conclude that $\overline{U}_i \cap G \subset I_1$, i.e., $\overline{U}_i \cap G \subset K$ since $I_1 \subset K$.

We consider all the sets U_i such that $U_i \cap K \neq \emptyset$. By the inclusion $\overline{U}_i \cap G \subset X \cup Y$ and since $\overline{U}_i \cap G$ is a continuum by Lemma 2, we obtain either $\overline{U}_i \cap G \subset X$ or $\overline{U}_i \cap G \subset Y$.

Let $\{U_{p_1}, \ldots, U_{p_{s_0}}\}$ be all the sets among U_i, $1 \leqslant i \leqslant k$, such that $K \cap U_{p_j} \neq \emptyset$, $1 \leqslant j \leqslant s_0$, and $\overline{U}_{p_j} \cap G \subset X$.

Let $\{U_{q_1}, \ldots, U_{q_{s_0}}\}$ be all the sets among U_i, $1 \leqslant i \leqslant k$, such that $K \cap U_{q_j} \neq \emptyset$, $1 \leqslant j \leqslant s_1$, and $\overline{U}_{q_j} \cap G \subset Y$.

$$K = K \cap M = K \cap \Big(\bigcup_{i=1}^{k} U_i \cup G\Big) = K \cap G \cup \bigcup_{j=1}^{s_0} (K \cap U_{p_j}) \cup \bigcup_{j=1}^{s_1} (K \cap U_{q_j})$$

$$= K \cap G \cup \bigcup_{j=1}^{s_0} (K \cap \overline{U}_{p_j} \cap G) \cap \bigcup_{j=1}^{s_1} (K \cap \overline{U}_{q_j} \cap G) \cup$$

$$\cup \bigcup_{j=1}^{s_0} (K \cap U_{p_j}) \cup \bigcup_{j=1}^{s_1} K \cap U_{q_j}) ,$$

since the sets $\overline{U}_{p_j} \cap G$, $1 \leqslant j \leqslant s_0$, and $\overline{U}_{q_j} \cap G$, $1 \leqslant j \leqslant s_1$, are contained in $K \cap G$. Further, since $\overline{U}_i \subset U_i \cup G$, $1 \leqslant i \leqslant k$, we obtain:

$$K = K \cap G \cup \bigcup_{j=1}^{s_0} (K \cap \overline{U}_{p_j}) \cup \bigcup_{j=1}^{s_1} (K \cap \overline{U}_{q_j})$$

$$= X \cup \bigcup_{j=1}^{s_0} K \cap \overline{U}_{p_j} \cup Y \cup \bigcup_{j=1}^{s_1} K \cap \overline{U}_{q_j} = A \cup B ,$$

where

$$A = X \cup \bigcup_{j=1}^{s_0} K \cap \overline{U}_{p_j} \quad \text{and} \quad B = Y \cup \bigcup_{j=1}^{s_1} K \cap \overline{U}_{q_j} .$$

We see that A and B are closed. Further,

$$A = X \cap \bigcup_{j=1}^{s_0} K \cap \overline{U}_{p_j} = X \cup \bigcup_{j=1}^{s_0} K \cap (\overline{U}_{p_j} \cap G \cup U_{p_j}) = X \cup \bigcup_{j=1}^{s_0} K \cap U_{p_j} ,$$

$$B = Y \cup \bigcup_{j=1}^{s_1} K \cap \overline{U}_{q_j} = Y \cup \bigcup_{j=1}^{s_0} K \cap (\overline{U}_{q_j} \cap G \cup U_{q_j}) = Y \cup \bigcup_{j=1}^{s_0} K \cap U_{q_j} .$$

Since $X \subset G$ and $Y \subset G$, we have $U_i \cap K = \emptyset$ and $U_i \cap Y = \emptyset$. Observe that $X \cap Y = \emptyset$, and $U_{i_1} \cap U_{i_2} = \emptyset$, where $i_1 \neq i_2$. In accordance to these remarks we see that A and B are disjoint non-empty sets, which contradicts the assumption that K is a continuum.

Lemmas 1 and 3 show that hereditarily decomposable continua which are irreducible about a finite set possess properties 2° and 3°; hence we get the following theorems:

THEOREM 4. *Let M be a hereditarily decomposable continuum which is irreducible about a finite set and let $\mathcal{G}$ be an admissible decomposition of M. If the elements of $\mathcal{G}$ are unicoherent, so is M.*

THEOREM 5. *Let M be a hereditarily decomposable continuum which is irreducible about a finite set and let $\mathcal{G}$ be an admissible decomposition of M. Then M is a λ-dendroid if and only if the elements of $\mathcal{G}$ are hereditarily unicoherent.*

Theorems 4 and 5 are direct consequences of Lemmas 1 and 3 and Theorems 1 and 3.

We now discuss if all properties 1′, 2° and 3° in Theorem 3 are essential for proving that M is a λ-dendroid. In other words, we ask whether it is possible to omit one of these properties and still prove that M is a λ-dendroid.

EXAMPLE 1. If M_1 is a circle and if $\mathcal{G}$ is the point-decomposition, then both 1′ and 3° hold, but 2° does not.

EXAMPLE 2. If M_2 is the union of a circle and a ray approximating it and if $\mathcal{G}$ is an admissible decomposition of M_2, then both 2° and 3° hold, but 1′ does not.

Asking if property 3° is essential, we come to an open question. Does a hereditarily decomposable continuum M having properties 1′ and 2° have also property 3°?

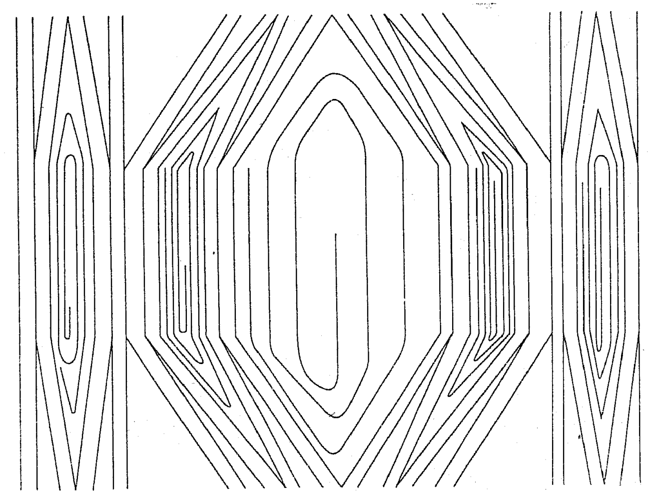


Fig. 1

EXAMPLE 3 (this example is due to J. J. Charatonik). If M_3 is the continuum shown in Fig. 1 and $\mathcal{G}$ is its minimal admissible decomposition, then 2° is satisfied, but 3° is not. In fact, M_3 is a continuum composed of two copies of the "accordion-like continuum" (see e.g. [7], p. 12), of uncountably many vertical straight line segments which join the corresponding layers of the copies, and of countably many spiral lines winding in to the simple closed curves contained in the continuum. To show the above proposition we just observe the subcontinuum of M_3 which is the union of both the "accordionlike continua" and an arbitrary segment connecting them.

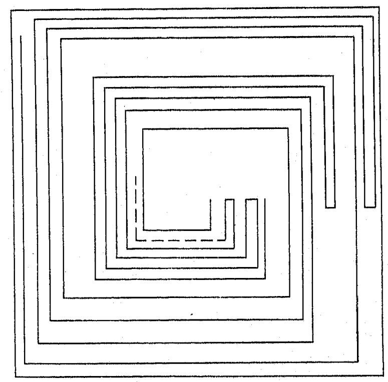


Fig. 2

EXAMPLE 4. We take the continuum M_3 and we approximate each simple closed curve which is an element of the minimal admissible decomposition of M together with the spiral winding in to it by a ray, as is shown in Fig. 2. We define the continuum M_4 as the union of M_3 and all the approximating rays. If $\mathcal{G}$ is the minimal admissible decomposition of M_4, then 1° and 2° hold, but 3° does not.

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