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Filter characterizations of z-, C^*-, and C-embeddings

by

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To the memory of Lee W. Anderson

Abstract. The paper [6] by the author and A. W. Hager is supplemented here by a number of filter-theoretic characterizations of z-embedding, and of those conditions which must be added to z-embedding to produce C^*- or C-embedding. These lead to filter characterizations of C^*- and C-embedding which include results of J. W. Green [15].

1. Introduction. The subset S of the topological space X is *z-embedded* in X if each zero-set of S is the restriction to S of a zero-set of X. (A zero-set is the set of zeros of a real-valued continuous function.) The notion of z-embedding occurs (sometimes only implicitly) in some special contexts in the early papers [12], [16], [17], and [18]. In 1963 the author initiated the general theory of z-embedding, and at the same time introduced the term "z-embedding" itself. (See [2]; portions of [2] are incorporated in [3], [4], [5], and [6].) Subsequently, the theory has been developed by A. W. Hager and by the author (sometimes jointly), as well as by others; see [6] for a number of basic results and for a comprehensive bibliography of relevant papers.

This paper may be regarded as a sequel to both [6] and [15]: [6] is devoted to a study of z-embedding and its relation to C^*- and C-embedding, but convergence (i.e., filter-theoretic) considerations are ignored. [15], on the other hand, is devoted to filter characterizations of C^*- and C-embeddings, but with no mention of z-embedding. In the present paper we supplement both [6] and [15] by providing filter characterizations of z-embedding (see 3.1) and of those conditions which must be added to z-embedding to produce C^*- or C-embedding (see 4.1 and 4.2). The C^*- and C-embedding characterizations of [15] (as well as improvements thereon) are then deduced as consequences (see 5.1 and 5.3).

Except for 3.2(d), 3.5, 3.6, and 3.9, the results of this paper require no separation axioms.

2. Preliminaries. We assume familiarity with [11], whose notation and terminology will be used throughout.

Throughout this paper, X will denote a topological space. If $f \in C(X)$, then $Z(f)$ denotes the zero-set of f. The set of all zero-sets of X is denoted by $\mathfrak{Z}(X)$.

By a filter (resp. filter base) $\mathcal{F}$ we shall always mean a *proper* filter (resp. filter base) (i.e., $\emptyset \notin \mathcal{F}$). Let $\mathcal{B}$ and $\mathcal{B}'$ be filter bases on X. As usual, we say that $\mathcal{B}$ is *coarser* than $\mathcal{B}'$, and $\mathcal{B}'$ is *finer* than $\mathcal{B}$ (written $\mathcal{B} \leqslant \mathcal{B}'$ or $\mathcal{B}' \geqslant \mathcal{B}$), in case every member of $\mathcal{B}$ contains some member of $\mathcal{B}'$. If $S \subset X$, then (by abuse of language) we say that S *meets* $\mathcal{B}$ in case S meets every member of $\mathcal{B}$; and that $\mathcal{B}$ *meets* $\mathcal{B}'$ in case every member of $\mathcal{B}$ meets every member of $\mathcal{B}'$. The *trace of* $\mathcal{B}$ *on* S is the set $\mathcal{B}|S = \{S \cap B: B \in \mathcal{B}\}$.

A filter base $\mathcal{B}$ on X is *completely regular* in case for each $B \in \mathcal{B}$ there is $B' \in \mathcal{B}$ such that B' and $X-B$ are completely separated (cf. [8], Chap. IV, § 1, Ex. 8). A completely regular filter is *maximal* if there exists no strictly finer completely regular filter. By Zorn's lemma, every completely regular filter on X is coarser than some maximal completely regular filter on X. We collect some additional needed information in the following proposition (which will sometimes be used without explicit mention):

2.1. PROPOSITION. (a) *If $\mathcal{F}$ is a completely regular filter on X, then these are equivalent*: (i) *$\mathcal{F}$ is maximal.* (ii) *If $Z, Z' \in \mathfrak{Z}(X)$, and if Z and Z' meet $\mathcal{F}$, then $Z \cap Z' \neq \emptyset$.*

(b) *If $\mathcal{F}$ is a maximal completely regular filter on X, then there is a unique z-ultrafilter $\mathcal{U}$ on X finer than $\mathcal{F}$; and if $Z \in \mathfrak{Z}(X)$, then $Z \in \mathcal{U}$ iff Z meets $\mathcal{F}$.*

(c) *If $\mathcal{U}$ is a z-ultrafilter on X, then there is a unique maximal completely regular filter on X coarser than $\mathcal{U}$.*

(d) ([15], Lemma 4) *Let $S \subset X$. If $\mathcal{F}$ is a maximal completely regular filter on S, then there is a unique maximal completely regular filter on X coarser than $\mathcal{F}$.*

Proof. For (a), see the proof of [15], Lemma 3. To prove (b), note that $\mathcal{F} \cap \mathfrak{Z}(X)$ is a base for a z-filter, and is therefore contained in some z-ultrafilter (which is necessarily finer than $\mathcal{F}$). To prove (c), note that

$$\mathcal{B} = \{f^{-1}[0, r]: f \in C(X),\ f \geqslant 0,\ r > 0,\ Z \subset Z(f) \text{ for some } Z \in \mathcal{U}\}$$

is a completely regular filter base on X. Then $\mathcal{B} \subset \mathcal{G}$ for some maximal completely regular filter $\mathcal{G}$, and $\mathcal{G}$ is coarser than $\mathcal{U}$. We omit the details.

Let $S \subset X$ and $f \in C(S)$. For each real number r, we define (as in [20]) the Lebesgue sets $L_r(f)$ and $L^r(f)$ of f as follows:

$$L_r(f) = \{x \in S: f(x) \leqslant r\}, \qquad L^r(f) = \{x \in S: f(x) \geqslant r\}.$$

We shall say that f is *z-embedded* in X in case, for each r, there exist $Z, Z' \in \mathfrak{Z}(X)$ such that $L_r(f) = S \cap Z$ and $L^r(f) = S \cap Z'$. (The theory of z-embedded functions will be treated in detail elsewhere.) A filter $\mathcal{F}$ on S will be called *z-embedded* in X in case for every $F \in \mathcal{F}$ there exist $F' \in \mathcal{F}$ and a z-embedded function $f \in C(S)$ which completely separates F' and $S-F$ (i.e., $f = 1$ on F', $f = 0$ on $S-F$, and $0 \leqslant f \leqslant 1$). Every z-embedded filter on S is, of course, completely regular. The following is a partial converse:

2.2. PROPOSITION. *Let $S \subset X$. If $\mathcal{F}$ is a maximal completely regular filter on S, and if $\mathcal{F} \leqslant \mathcal{G}|S$ for some completely regular filter $\mathcal{G}$ on X which meets S, then $\mathcal{F}$ is z-embedded in X.*

Proof. By 2.1(d), there is a maximal completely regular filter $\mathcal{F}^*$ on X such that $\mathcal{F}^* \leqslant \mathcal{F}$. Then $\mathcal{G}$ meets $\mathcal{F}^*$, so $\mathcal{G} \subset \mathcal{F}^*$ by the maximality of $\mathcal{F}^*$. Let $F \in \mathcal{F}$. Then $F \supset G \cap S$ for some $G \in \mathcal{G}$, and there is $G' \in \mathcal{G}$ such that G' and $X-G$ are completely separated in X. Moreover, since $G' \in \mathcal{F}^*$ we have $G' \supset F'$ for some $F' \in \mathcal{F}$. Choose $f \in C(X)$ such that f completely separates G' and $X-G$. Then clearly $f|S$ is z-embedded in X and completely separates F' and $S-F$.

We recall one bit of notation from [11]: If $f: X \to Y$ is continuous, and if $\mathcal{F}$ is a z-filter on X, then $f^\#(\mathcal{F})$ denotes the z-filter on Y consisting of all $Z \in \mathfrak{Z}(Y)$ such that $f^{-1}(Z) \in \mathcal{F}$ (see [11], 4.12).

If $\mathcal{F}$ is a z-filter on a subset S of X, and if $\varphi: S \to X$ is the canonical injection of S into X, then we call $\varphi^\#(\mathcal{F})$ the *z-filter on X generated by $\mathcal{F}$*. If $\mathcal{F}$ is a z-ultrafilter on S, then $\varphi^\#(\mathcal{F})$ is a prime z-filter on X ([11], 4.12) and hence is contained in a unique z-ultrafilter $\mathcal{U}$ on X ([11], 2.13); we call $\mathcal{U}$ the *z-ultrafilter on X determined by $\mathcal{F}$*.

We shall make frequent use of the following:

2.3. PROPOSITION. (a) ([3], 2.2) *A z-filter $\mathcal{F}$ on X is a real z-ultrafilter on X if and only if $\mathcal{F}$ is prime and closed under countable intersection.*

(b) ([3], 2.3) *If $\mathcal{F}$ is a real z-ultrafilter on X and if $f: X \to Y$ is continuous, then $f^\#(\mathcal{F})$ is a real z-ultrafilter on Y.*

3. Characterizations of z-embedding. In 3.1 we give several characterizations of z-embedding. (For convenience, we include one that is not filter-theoretic.) Except for (d), each of these characterizations will be applied later in this paper. The equivalence of (a), (c), and (e) makes clear the convergence-theoretic reason for the importance of z-embedding (particularly in completely regular spaces, where z-filters play a central role).

3.1. THEOREM. *If $S \subset X$, then these are equivalent:*

(a) *S is z-embedded in X.*

(b) *If $\mathcal{G}$ is any z-ultrafilter on S, and if $\varphi: S \to X$ is the canonical injection of S into X, then $\mathcal{G} \subset \varphi^\#(\mathcal{G})|S$.*

(c) *If $\mathcal{F}$ is any z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a z-ultrafilter on S.*

(d) *If $\mathcal{F}$ is any z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a prime z-filter on S.*

(e) *If $\mathcal{F}$ is any real z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a real z-ultrafilter on S.*

(f) *If A_1 and A_2 are subsets of S that are completely separated in S, then there exist zero-sets $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $A_1 \subset Z_1$, $A_2 \subset Z_2$, and $Z_1 \cap Z_2 \cap S = \emptyset$.*

(g) *Every (maximal) completely regular filter on S is z-embedded in X.*

Proof. (a) $\Rightarrow$ (b): If $A \in \mathcal{G}$, then, by z-embedding, $A = S \cap Z$ for some $Z \in \mathfrak{Z}(X)$. Then clearly $A \in \varphi^{\#}(\mathcal{G})|S$.

(b) $\Rightarrow$ (c): If $\mathcal{F}$ is a z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a base for a z-filter on S. By Zorn's lemma, $\mathcal{F}|S \subset \mathcal{G}$ for some z-ultrafilter $\mathcal{G}$ on S. Clearly $\mathcal{F} \subset \varphi^{\#}(\mathcal{G})$, so $\mathcal{F} = \varphi^{\#}(\mathcal{G})$. Then $\mathcal{F}|S = \mathcal{G}$ by (b).

(c) $\Rightarrow$ (d): This is immediate since a z-ultrafilter is necessarily prime ([11], 2.13).

(d) $\Rightarrow$ (e): If $\mathcal{F}$ is a real z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is closed under countable intersection (because $\mathcal{F}$ is), and is a prime z-filter on X by (d). By 2.3(a), $\mathcal{F}|S$ is a real z-ultrafilter.

(e) $\Rightarrow$ (*f*): This is trivial if either A_1 or A_2 is empty, so assume neither is empty. Since A_1 and A_2 are completely separated in S, there exist $Z'_i \in \mathfrak{Z}(S)$ with $A_i \subset Z'_i$ $(i = 1, 2)$ and $Z'_1 \cap Z'_2 = \emptyset$. Pick $x \in A_1$ and let $\mathcal{F} = \{Z \in \mathfrak{Z}(X)\colon x \in Z\}$. $\mathcal{F}$ is a real z-ultrafilter on X, so $\mathcal{F}|S$ is a real z-ultrafilter on S by (e); and since Z'_1 meets $\mathcal{F}|S$, we have $Z'_1 \in \mathcal{F}|S$. Thus $Z'_1 = Z_1 \cap S$ for some $Z_1 \in \mathfrak{Z}(X)$. Similarly, $Z'_2 = Z_2 \cap S$ for some $Z_2 \in \mathfrak{Z}(X)$, and clearly $Z_1 \cap Z_2 \cap S = \emptyset$.

(*f*) $\Rightarrow$ (g): It suffices to show that every $f \in C(S)$ is z-embedded in X. If r is real and $n > 0$, the Lebesgue sets $L_r(f)$ and $L^{r+(1/n)}(f)$ are completely separated in S, so by (f) there is $Z_n \in \mathfrak{Z}(X)$ with $L_r(f) \subset Z_n$ and $Z_n \cap L^{r+(1/n)}(f) = \emptyset$. Then $\bigcap_n Z_n \in \mathfrak{Z}(X)$ and $L_r(f) = \bigcap_n Z_n \cap S$. The argument is similar for $L^r(f)$.

(g) $\Rightarrow$ (a): Let $f \in C(S)$. For each $n > 0$, set

$$A_n = \{x \in S\colon |f(x)| \geqslant 1/n\},$$

and set $A_0 = Z(f)$. For $n = 0, 1, 2, \ldots$, let $\mathcal{B}_n = \{g^{-1}[0, r]\colon r > 0,\ g$ a nonnegative z-embedded function in $C(S)$, and $A_n \subset Z(g)\}$. Each $\mathcal{B}_n$ is a (completely regular) filter base on S. (This follows from the identity

$$g^{-1}[0, r] \cap h^{-1}[0, s] = (g \vee h)^{-1}[0, r \wedge s] \quad (g, h \geqslant 0),$$

and the formulas

$$L_r(g \vee h) = L_r(g) \cap L_r(h), \quad L^r(g \vee h) = L^r(g) \cup L^r(h)$$

(which imply that $g \vee h$ is z-embedded if both g and h are z-embedded).)

Suppose that $\mathcal{B}_0$ meets some $\mathcal{B}_n$, $n > 0$. Then there is a maximal completely regular filter $\mathcal{F}$ on S that contains both $\mathcal{B}_0$ and $\mathcal{B}_n$. We claim that $\mathcal{F}$ meets A_0. If not, there is $F \in \mathcal{F}$ with $F \cap A_0 = \emptyset$; and then by (g) there exist $F' \in \mathcal{F}$ and a nonnegative z-embedded $g \in C(S)$ such that $g = 1$ on F' and $g = 0$ on $S - F$. But then $g^{-1}[0, \frac{1}{2}] \in \mathcal{B}_0 \subset \mathcal{F}$, so $g^{-1}[0, \frac{1}{2}] \cap F' \neq \emptyset$, a contradiction. Thus $\mathcal{F}$ meets A_0, and, similarly, $\mathcal{F}$ also meets A_n. But A_0 and A_n are disjoint zero-sets in S, which is contrary to 2.1(a). We conclude that $\mathcal{B}_0$ misses every $\mathcal{B}_n$, $n > 0$, and hence for each $n > 0$ there is a z-embedded $g_n \in C(S)$ with $A_0 \subset Z(g_n)$ and $Z(g_n) \cap A_n = \emptyset$. Since g_n is z-embedded, there is $Z_n \in \mathfrak{Z}(X)$ with $Z(g_n) = S \cap Z_n$. Then $\bigcap_n Z_n \in \mathfrak{Z}(X)$ and $Z(f) = \bigcap_n Z_n \cap S$, so the proof is complete.

3.2 Remarks. (a) The equivalence of (a)-(*f*) of 3.1 is proved in [2]; (a) $\Leftrightarrow$ (f) is recorded again in [1], Theorem 6(1).

(b) The implication (a) $\Rightarrow$ (c) is proved by Green ([14], Theorem 3) under the hypothesis that S is C^*-embedded. 3.1(c) can fail spectacularly if S is not z-embedded: the proof of [15], Lemma 9 shows that if S is the x-axis of the tangent circle space Γ (see [11], 3K), and if $\mathcal{F}$ is a z-ultrafilter on Γ which meets S, then $\mathcal{F}|S$ is never even a z-filter on S.

(c) If in 3.1(f) the requirement "$Z_1 \cap Z_2 \cap S = \emptyset$" is replaced by "$Z_1 \cap Z_2 = \emptyset$", then the resulting statement is precisely the condition for C^*-embedding given by the Gillman-Jerison version of Urysohn's Extension Theorem ([11], 1.17).

(d) Consider this modification of 3.1(c): (c′) *If $\mathcal{F}$ is any z-filter on X which meets S, then $\mathcal{F}|S$ is a z-filter on S.* It is easy to see that (a) $\Rightarrow$ (c′). We are indebted to the referee for the following observations:

(i) If X is Tychonoff, then (c′) $\Rightarrow$ (a). More generally, if there is a function $f \in C(X)$ which is nonconstant on S, then (c′) implies that S is z-embedded in X. To see this, pick $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$, and set $\mathcal{F}_i = \{Z \in \mathfrak{Z}(X)\colon x_i \in Z\}$. Then there exist $Z_i \in \mathcal{F}_i$ such that $Z_1 \cap Z_2 = \emptyset$, and, by (c′), $\mathcal{F}_i|S$ is a z-filter on S. Let $A \in \mathfrak{Z}(S)$. Since $(A \cup Z_i) \cap S \supset Z_i \cap S$, we have $(A \cup Z_i) \cap S \in \mathcal{F}_i|S$, and hence $(A \cup Z_i) \cap S = Z'_i \cap S$ for some $Z'_i \in \mathfrak{Z}(X)$. Then $A = Z'_1 \cap Z'_2 \cap S$, so S is z-embedded in X.

(ii) Let X be a regular T_1-space on which every real-valued continuous function is constant. If $a, b \in X$ with $a \neq b$, then $S = \{a, b\}$ satisfies (c′), but S is not z-embedded in X.

We next consider briefly the following weakening of 3.1(c) (see the discussions in [13], pp. 54ff, [15], pp. 103ff, and [21], p. 177):

(∗) If $\mathcal{F}$ is any z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a base for a z-ultrafilter on S.

Let us call a subset S of X *weakly z-embedded* in X in case whenever $A, B \in \mathfrak{Z}(S)$ with $A \cup B = S$, either $A = S \cap Z$ for some $Z \in \mathfrak{Z}(X)$ or

$B = S \cap Z'$ for some $Z' \in \mathfrak{Z}(X)$. Obviously every z-embedded set is weakly z-embedded.

3.3. Proposition. *If S is a weakly z-embedded subset of a pseudocompact space X, then S satisfies* $(*)$.

Proof. Let $\mathcal{F}$ be a z-ultrafilter on X which meets S. Then $\mathcal{F}|S$ is a base for a z-filter $\mathcal{G}$ on S. We show first that $\mathcal{G}$ is prime: Let $A, B \in \mathfrak{Z}(S)$ with $A \cup B = S$; it suffices to show that $A \in \mathcal{G}$ or $B \in \mathcal{G}$ ([11], 2E.2). By weak z-embedding, we may assume that $A = S \cap Z$ for some $Z \in \mathfrak{Z}(X)$. Now if $Z' \cap A = \emptyset$ for some $Z' \in \mathcal{F}$, then $Z' \cap S \subset B$, whence $B \in \mathcal{G}$. Thus we may assume that $\mathcal{F}$ meets A. But then $\mathcal{F}$ meets Z, so $A \in \mathcal{F}|S \subset \mathcal{G}$.

Observe next that $\mathcal{F}$ is real: If not, there exist $f_n \in C(X)$ with $0 \leqslant f_n \leqslant 1$, $Z(f_n) \in \mathcal{F}$, and $\bigcap_n Z(f_n) = \emptyset$; but then $f = \sum_n 2^{-n} f_n$ is in $C(X)$, $Z(f) = \emptyset$, and $1/f$ is unbounded. It follows that $\mathcal{G}$ is closed under countable intersection, so $\mathcal{G}$ is a (real) z-ultrafilter by 2.3(a). Thus $(*)$ holds.

The converse of 3.3 is false; see 3.8 below.

3.4. Remarks. Mrówka ([21], 2.7) has given an example of a zero-set S in a (completely regular, zero-dimensional) pseudocompact space X which satisfies a condition somewhat stronger than weak z-embedding, but which is not z-embedded. By 3.3, this S satisfies $(*)$ (this is noted by Mrówka), so $(*)$ does not imply z-embedding. The relationship between z-embedding, weak z-embedding, and $(*)$ for non-pseudocompact X is left open by Mrówka. On this question we can contribute two facts: for pseudocompact S, $(*)$ is equivalent to C-embedding (see 5.6 below); and the following:

3.5. Theorem. *Let S be a realcompact subset of a completely regular space X. If S is weakly z-embedded in X, then S is z-embedded in X (and hence $(*)$ holds).*

Proof. Let $A \in \mathfrak{Z}(S)$ and let

$$\mathcal{B} = \{(S - A) \cap Z \colon Z \in \mathfrak{Z}(X) \text{ and } A \subset Z\}.$$

It will suffice to show that $\emptyset \in \mathcal{B}$. Suppose, on the contrary, that $\emptyset \notin \mathcal{B}$. Then $\mathcal{B}$ is a base for a z-filter $\mathcal{F}$ on the space $S - A$, and clearly $\mathcal{F}$ is closed under countable intersection. Moreover, by complete regularity of X, $\bigcap \mathcal{F} = \emptyset$.

We claim that the z-filter $\mathcal{F}$ is prime: Let $F_1, F_2 \in \mathfrak{Z}(S - A)$ with $F_1 \cup F_2 = S - A$. Now $S - A$ is cozero in S and hence z-embedded in S (see [6], 1.1). Thus there is $E_i \in \mathfrak{Z}(S)$ with $F_i = (S - A) \cap E_i$. Then $A \cup F_i = A \cup E_i$, so $A \cup F_i \in \mathfrak{Z}(S)$. By weak z-embedding of S, it follows that either $A \cup F_1 = S \cap Z_1$ for some $Z_1 \in \mathfrak{Z}(X)$ or $A \cup F_2 = S \cap Z_2$ for some $Z_2 \in \mathfrak{Z}(X)$. But then either $F_1 \in \mathcal{B}$ or $F_2 \in \mathcal{B}$, so $\mathcal{F}$ is prime.

Now by 2.3(a), $\mathcal{F}$ is a real z-ultrafilter on $S - A$. Since $S - A$ is cozero in the realcompact space S, $S - A$ is itself realcompact ([11], 8.14). But then $\mathcal{F} \to x$ for some $x \in S - A$, which is a contradiction. This completes the proof.

A closed subset of a realcompact space is realcompact ([11], 8.10), so we have:

3.6. Corollary. *A weakly z-embedded closed subset of a realcompact space is z-embedded.*

We now show that the converse of 3.3 is false. (This improves the observation in [15], p. 103 to the effect that $(*)$ does not imply C^*-embedding.) Lemma 3.7, which follows, is due to A. W. Hager. (For 3.7 and related results, see [7].)

3.7. Lemma (Hager). *If A is a countable (Tychonoff) space with no countable base, and if B is a discrete space with cardinality $2^{\aleph_0}$, then $A \times B$ is not z-embedded in $\beta A \times \beta B$.*

3.8. Example. There is a compact Hausdorff space X and a (realcompact) subset S of X such that S satisfies $(*)$, but such that S is not weakly z-embedded in X.

Proof. Choose $A = \{a_1, a_2, \ldots\}$ and B as in 3.7. (For an example of such an A, see e.g. [10], Example 3.1.4.) Let $X = \beta A \times \beta B$ and $S = A \times B$. Since A and B are realcompact, S is realcompact ([11], 8.11). By 3.7 and 3.5, it will therefore suffice to show that S satisfies $(*)$.

Let $\mathcal{F}$ be a z-ultrafilter on X which meets S, and let $\mathcal{G}$ be the z-filter on S generated by $\mathcal{F}|S$. Let $g \in C(S)$, $0 \leqslant g \leqslant 1$, and assume that $Z(g)$ meets $\mathcal{G}$. For each n, let $A_n = \{a_1, \ldots, a_n\}$, and let $S_n = A_n \times B$. Extend $g|S_n$ to $h_n \in C(X)$ (first extend over $A_n \times \beta B$, and then over X). For each integer $i \geqslant 1$, there is $k_{ni} \in C(X)$ with $k_{ni} = 1$ on $\{a_{n+i}\} \times \beta B$, $k_{ni} = 0$ on $A_n \times \beta B$, and $0 \leqslant k_{ni} \leqslant 1$. Let $F_{ni} = h_n \vee k_{ni}$ and set $F_n = \sum_i 2^{-i} F_{ni}$. Then $F_n \in C(X)$, and one easily verifies that $Z(F_n) \cap S = Z(g) \cap S_n$.

Now suppose that $Z(F_n) \notin \mathcal{F}$ for all n. Then for each n there is $Z'_n \in \mathcal{F}$ with $Z(F_n) \cap Z'_n = \emptyset$. Let $Z' = \bigcap_n Z'_n$ and note that $Z' \in \mathcal{F}$ (since X is compact). Then $Z(g) \cap Z' \neq \emptyset$, so $Z(g) \cap S_n \cap Z' \neq \emptyset$ for some n. But then $Z(F_n) \cap Z' \neq \emptyset$, a contradiction. Thus $Z(F_m) \in \mathcal{F}$ for some m. Since $Z(F_m) \cap S \subset Z(g)$, we have $Z(g) \in \mathcal{G}$, and it follows that $\mathcal{G}$ is a z-ultrafilter on S. Thus $(*)$ holds, and the proof is complete.

We conclude this section with a simple application of (a) $\Rightarrow$ (c) of 3.1 to realcompactness. (For some other applications of a somewhat similar nature, see the proofs of [5], 2.2 and [3], 2.6, 6.3, and 8.1.)

Recall that a map $f\colon X \to Y$ is *z-closed* in case $f(Z)$ is closed in Y whenever $Z \in \mathfrak{Z}(X)$.

3.9. Theorem. *Assume that X and Y are Tychonoff. If Y is realcompact, and if there exists a continuous z-closed map $f\colon X \to Y$ such that $f^{-1}(y)$ is realcompact and z-embedded in X for every $y \in Y$, then X is realcompact.*

Proof. Let $\mathcal{F}$ be a real z-ultrafilter on X. Then $f^{\#}(\mathcal{F})$ is a real z-ultrafilter on Y (2.3(b)), so $f^{\#}(\mathcal{F}) \to y$ for some $y \in Y$. If $Z \in \mathcal{F}$ and $y \notin f(Z)$, then (since f is z-closed) there exists $Z' \in \mathfrak{Z}(Y)$ such that $f(Z) \subset Z'$ and $y \notin Z'$. But then $f^{-1}(Z') \in \mathcal{F}$, so $Z' \in f^{\#}(\mathcal{F})$; hence $y \in Z'$, a contradiction. Thus $S = f^{-1}(y)$ meets $\mathcal{F}$, so $\mathcal{F}|S$ is a real z-ultrafilter on S by 3.1. Since S is realcompact, $\mathcal{F}|S \to x$ for some $x \in S$. But then $\mathcal{F} \to x$, so X is realcompact.

3.9 generalizes [19], 5.3 and [9], 4.9. It should be remarked that 3.9 is only a fragment of the general theory of realproper maps (see [4]).

4. z-embedding versus C^*-and C-embedding. As in [6], § 3, we consider the following three conditions on an embedding $S \subset X$:

(α) Disjoint zero-sets of S are completely separated in X.

(β) If $Z_1, Z_2 \in \mathfrak{Z}(X)$ and $Z_1 \cap Z_2 \cap S = \emptyset$, then $Z_1 \cap S$ and $Z_2 \cap S$ are completely separated in X.

(γ) S is completely separated from each disjoint zero-set in X.

These are important because of the following: (α) is equivalent to C^*-embedding (the Gillman–Jerison version of Urysohn's Extension Theorem [11], 1.17, [6], 3.6A; cf. 3.2(c) above); z-embedding plus (β) is equivalent to C^*-embedding (4.1A below); C^*-embedding plus (γ) is equivalent to C-embedding ([11], 1.18); and (more generally) z-embedding plus (γ) is equivalent to C-embedding (4.1B).

Conditions (β) and (γ) are characterized in [6], 3.4 in terms of partial extendibility of functions. Here we supplement the results of [6] by giving filter characterizations of both (β) and (γ) (4.2 below). In § 5 these will be combined with 3.1 to yield filter characterizations of C^*- and C-embeddings.

The following is given a different proof in [6], 3.6. We base our argument here on the easy equivalence (a) ⇔ (f) of 3.1.

4.1. THEOREM. *Let $S \subset X$.*

A. *These are equivalent*:

(1) *S is z-embedded and* (β) *holds.*

(2) *S is C^*-embedded.*

B. *These are equivalent*:

(1) *S is z-embedded and* (γ) *holds.*

(2) *S is C-embedded.*

Proof. A. This is immediate from (a) ⇔ (f) of 3.1 and the equivalence of (α) with C^*-embedding.

B. If (1) holds, then by [11], 1.18 (above) it suffices to show that S is C^*-embedded. Let A_1 and A_2 be completely separated in S and choose $Z_1, Z_2 \in \mathfrak{Z}(X)$ as in 3.1(f). By (γ), S and $Z_1 \cap Z_2$ are completely separated, so there is $Z \in \mathfrak{Z}(X)$ with $S \subset Z$ and $Z \cap Z_1 \cap Z_2 = \emptyset$. Then $Z \cap Z_1$ and $Z \cap Z_2$ are disjoint zero-sets in X containing A_1 and A_2, so S is C^*-embedded by Urysohn's Extension Theorem. The converse is clear from [11], 1.18.

The deceptively simple 4.1B is actually a result of some importance in the theory of C-embedding; its depth is roughly that of Urysohn's Extension Theorem (the latter being used in an essential way in its proof). For some applications of 4.1B, see [6], 4.2 (which includes [14], Theorem 7), [6], 4.4, 4.5, and 4.6, [3], 3.2, 3.4, and 5.1, and 5.2 and 5.3 below. In the same vein, we note that [15], Theorem 6 asserts that a zero-set S in X is C-embedded in X if and only if S satisfies 3.1(c) above. Since a zero-set automatically satisfies (γ), this is an immediate consequence of 3.1 and 4.1B. (Two other hypotheses on S automatically ensure (γ); see 4.5(b) below.)

4.2. THEOREM. *Let $S \subset X$.*

A. *These are equivalent:*

(1) (β) *holds.*

(2) *If $\mathcal{F}$ is any maximal completely regular filter on X, and if $\mathcal{F}|S$ meets $Z_1, Z_2 \in \mathfrak{Z}(X)$, then $Z_1 \cap Z_2 \cap S \neq \emptyset$.*

B. *These are equivalent*:

(1) (γ) *holds.*

(2) *If $\mathcal{F}$ is a maximal completely regular filter on S, and if $\mathcal{U}$ is the unique z-ultrafilter on X finer than the unique maximal completely regular filter on X coarser than $\mathcal{F}$, then $\mathcal{U}$ meets S.*

(3) *If $\mathcal{G}$ is any z-filter on X which meets S, then there exists a z-ultrafilter $\mathcal{U}$ on X which meets $\mathcal{G}|S$.*

Proof. A. (1) ⇒ (2): If (2) fails, then there is a maximal completely regular filter $\mathcal{F}$ on X with the following property: there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $\mathcal{F}|S$ meets Z_1 and Z_2, but $Z_1 \cap Z_2 \cap S = \emptyset$. Since $\mathcal{F}$ meets both $Z_1 \cap S$ and $Z_2 \cap S$, it follows from 2.1(a) that $Z_1 \cap S$ and $Z_2 \cap S$ are not completely separated in X. Thus (β) fails.

(2) ⇒ (1): Let $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $Z_1 \cap Z_2 \cap S = \emptyset$. For $i = 1, 2$, let

$$\mathcal{B}_i = \{f^{-1}[0, r]:\ r > 0,\ f \in C(X),\ f \geqslant 0,\ \text{and}\ Z_i \cap S \subset Z(f)\}.$$

Then $\mathcal{B}_i$ is a completely regular filter base on X. Suppose that $\mathcal{B}_1$ meets $\mathcal{B}_2$. Then there exists a maximal completely regular filter $\mathcal{F}$ on X finer than both $\mathcal{B}_1$ and $\mathcal{B}_2$. As in the proof of (g) ⇒ (a) of 3.1, it follows that $\mathcal{F}$ meets both $Z_1 \cap S$ and $Z_2 \cap S$. Then $\mathcal{F}|S$ meets both Z_1 and Z_2, so $Z_1 \cap Z_2 \cap S \neq \emptyset$ by (2). This is a contradiction, and we conclude that there exist $Z_i' \in \mathcal{B}_i$ $(i = 1, 2)$ with $Z_1' \cap Z_2' = \emptyset$. Since $Z_1', Z_2' \in \mathfrak{Z}(X)$ and $Z_i \cap S \subset Z_i'$, it follows that $Z_1 \cap S$ and $Z_2 \cap S$ are completely separated in X.

B. (1) ⇒ (2): Assume (γ) and suppose that (2) fails. Then there is $Z \in \mathcal{U}$ with $Z \cap S = \emptyset$, so by (γ) there is $f \in C(X)$ with $f = 0$ on Z, $f = 1$ on S, and $f \geqslant 0$. Then $\mathcal{B} = \{f^{-1}[0, r]:\ r > 0\}$ is a completely regular filter base on X such that $\mathcal{B} \subset \mathcal{U}$. Now if $\mathcal{F}^*$ is the (unique) maximal completely regular filter on X coarser than $\mathcal{F}$, then $\mathcal{B}$ meets $\mathcal{F}^*$ (by 2.1(b)). By maximality of $\mathcal{F}^*$, it follows that $\mathcal{B} \subset \mathcal{F}^*$. But since $\mathcal{F}^* \leqslant \mathcal{F}$, we then have $f^{-1}[0, \frac{1}{2}] \cap S \neq \emptyset$, a contradiction.

(2) ⇒ (3): Let $\mathcal{G}$ be a z-filter on X which meets S. Then $\mathcal{G}|S \subset \mathcal{U}'$ for some z-ultrafilter $\mathcal{U}'$ on S. Let $\mathcal{F}$ be the maximal completely regular filter on S such that $\mathcal{F} \leqslant \mathcal{U}'$, let $\mathcal{F}^*$ be the maximal completely regular filter on X such that $\mathcal{F}^* \leqslant \mathcal{F}$, and let $\mathcal{U}$ be the z-ultrafilter on X such that $\mathcal{F}^* \leqslant \mathcal{U}$ (see 2.1). Now since $\mathcal{F}^* \leqslant \mathcal{U}'$ and $\mathcal{G}|S \leqslant \mathcal{U}'$, $\mathcal{G}$ meets $\mathcal{F}^*$ and hence $\mathcal{G} \subset \mathcal{U}$ (2.1(b)). But $\mathcal{U}$ meets S by (2), so it follows that $\mathcal{U}$ meets $\mathcal{G}|S$.

(3) ⇒ (1): If (1) fails, then there is a zero-set $Z \in \mathfrak{Z}(X)$ with $Z \cap S = \emptyset$ such that S and Z are not completely separated. Let

$$\mathcal{B} = \{f^{-1}[0, r]:\ r > 0,\ f \in C(X),\ f \geqslant 0, \text{ and } Z \subset Z(f)\},$$

and note that $\mathcal{B}$ is a base for a z-filter $\mathcal{G}$ on X. Since S and Z are not completely separated, it is clear that $\mathcal{B}$ (and, *a fortiori*, $\mathcal{G}$) meets S. Now if (3) holds, there is a z-ultrafilter $\mathcal{U}$ on X which meets $\mathcal{G}|S$. Then $Z \notin \mathcal{U}$ (since $Z \cap S = \emptyset$), so there is $Z' \in \mathcal{U}$ with $Z \cap Z' = \emptyset$. Since disjoint zero-sets are completely separated, there is $f \in C(X)$ with $f = 1$ on Z', $f = 0$ on Z, and $f \geqslant 0$. But then $f^{-1}[0, \frac{1}{2}] \cap S \in \mathcal{G}|S$, and hence $f^{-1}[0, \frac{1}{2}] \cap Z' \neq \emptyset$, a contradiction. Thus (3) fails, and the proof is complete.

The following condition on the embedding $S \subset X$ is considered by Green ([15], pp. 103 ff):

(γ') Each z-ultrafilter on S is finer than some z-ultrafilter on X.

We conclude this section with a brief study of (γ') and its relationship to (γ).

4.3. Proposition. *If $S \subset X$, then these are equivalent*:

(a) *(γ') holds.*

(b) *If $\mathcal{F}$ is any z-ultrafilter on S, then the z-filter on X generated by $\mathcal{F}$ is a z-ultrafilter on X.*

Proof. Let φ: $S \to X$ be the canonical injection and let $\mathcal{F}$ be a z-ultrafilter on S. If (γ') holds, then there is a z-ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \leqslant \mathcal{F}$. Then $\mathcal{U} \subset \varphi^{\#}(\mathcal{F})$, so $\varphi^{\#}(\mathcal{F}) = \mathcal{U}$. Conversely, if (b) holds, then $\varphi^{\#}(\mathcal{F})$ is a z-ultrafilter on X, and clearly $\varphi^{\#}(\mathcal{F}) \leqslant \mathcal{F}$.

4.4. Proposition. *For any embedding $S \subset X$, (γ') implies (γ).*

Proof. We verify 4.2B(3): Let $\mathcal{G}$ be a z-filter on X which meets S. Then $\mathcal{G}|S \subset \mathcal{F}$ for some z-ultrafilter $\mathcal{F}$ on S; and then by (γ') there is a z-ultrafilter $\mathcal{U}$ on X such that $\mathcal{U} \leqslant \mathcal{F}$. Clearly $\mathcal{U}$ meets $\mathcal{G}|S$.

4.5. Remarks. (a) The converse of 4.4 is false: The x-axis S of the tangent circle space Γ is a zero-set in Γ, and hence the embedding $S \subset \Gamma$ automatically satisfies (γ). But [15], Lemma 8 asserts that (γ') fails.

(b) There are two other hypotheses on the embedding $S \subset X$ that automatically ensure (γ): (i) If S is G_δ-dense in X (i.e., each nonempty G_δ-set in X meets S), then (γ) holds (trivially). (ii) (γ) holds for *every* embedding of S if (and only if) S is pseudocompact ([6], 4.3). We show now that if (γ) is replaced by (γ'), then (i) fails, but (ii) still holds.

4.6. Example. There is a G_δ-dense embedding $S \subset X$ for which (γ') fails.

Proof. Let N and W be disjoint copies of the spaces of natural numbers and countable ordinals, respectively. Let S be the topological sum of N and W, let $X = S \cup \{\infty\}$ be the one-point compactification of S, and let φ: $S \to X$ be the canonical injection. Clearly S is G_δ-dense in X. For each $n \in N$, let $B_n = \{m \in N:\ m \geqslant n\}$, and let $\mathcal{B} = \{B_n:\ n \in N\}$. Then $\mathcal{B}$ is a base for a z-filter on S, so $\mathcal{B} \subset \mathcal{F}$ for some z-ultrafilter $\mathcal{F}$ on S. By 4.3, it suffices to show that $\varphi^{\#}(\mathcal{F})$ is not a z-ultrafilter on X; and this is true because: $W \cup \{\infty\} \in \mathfrak{Z}(X)$, $\infty \in Z$ for all $Z \in \varphi^{\#}(\mathcal{F})$, and $W \cup \{\infty\} \notin \varphi^{\#}(\mathcal{F})$.)

4.7. Proposition. *These conditions on a space S are equivalent*:

(a) *S is pseudocompact.*

(b) *For every embedding $S \subset X$, (γ') holds.*

Proof. (a) ⇒ (b): Let $\mathcal{F}$ be a z-ultrafilter on S and let φ: $S \to X$ be the canonical injection. As in the proof of 3.3, $\mathcal{F}$ is real, so $\varphi^{\#}(\mathcal{F})$ is a z-ultrafilter on X by 2.3(b). Now apply 4.3.

(b) ⇒ (a): By (b) and 4.4, (γ) holds for every embedding of S. Hence S is pseudocompact by [6], 4.3.

It is noted in [15], p. 103 that (γ') does not imply C^*-embedding. In fact, (γ') does not even imply ϑ-embedding. (S is *v-embedded* in the Tychonoff space X in case $vS \subset vX$. This is an embedding condition weaker even than z-embedding; see [3].) To see this, let S be the topological sum of two copies of the space of countable ordinals, and let X be the one-point compactification of S. Then S is pseudocompact, and hence the embedding $S \subset X$ satisfies (γ'); but S is not v-embedded in X (see the proof of [3], 6.2).

Green has asked whether (γ') implies C-embedding when S is a zero-set in a Tychonoff space X ([15], p. 104). In view of 4.7, any pseudocompact zero-set which is not C-embedded provides a negative answer to this question. We are indebted to the referee for the following example of such a zero-set:

4.8. Example. There is a Tychonoff space X and a pseudocompact zero-set S of X such that S is not C-embedded in X.

Proof. Let W^* be the space of ordinals $\leqslant \omega_1$, $W = W^* - \{\omega_1\}$, and $Y = W \times W^*$. Let $A = \{(\alpha, \omega_1):\ \alpha < \omega_1\}$ and $B = \{(\alpha,\alpha):\ \alpha < \omega_1\}$. By a theorem of Engelking (1), there is a Tychonoff space M containing Y as a subspace and such that $M-Y$ is countable, open, and dense in M. Let $X = A \cup \cup B \cup (M-Y)$ and let $S = A \cup B$. Since S is closed in X and $X-S$ is countable, $S \in \mathfrak{Z}(X)$; and since A and B are homeomorphic to W, S is pseudocompact. Now if S were C-embedded in X, then the characteristic function on S of the set A would have a continuous extension over X, so A and B would be separated by disjoint open sets in X. But then (because $M-Y$ is dense in M) A and B would be separated by disjoint open sets in Y. As is well-known, this last is impossible.

We remark in passing that the zero-set S of 4.8 is not even v-embedded in X (since, as noted in [3], § 3, a pseudocompact v-embedded subset is necessarily C-embedded).

5. Characterizations of C^*-and C-embeddings. Here we apply the results of §§ 3 and 4 to obtain filter characterizations of C^*- and C-embeddings. These include, and somewhat improve, characterizations given by Green in [14] and [15].

In 5.1, the equivalence of (a), (b), and (c) is due to Green (see [15], Theorem 2 and [14], Theorem 2).

5.1. THEOREM. *If $S \subset X$, then these are equivalent:*

(a) *S is C^*-embedded in X.*

(b) *If $\mathcal{F}$ is a maximal completely regular filter on X which meets S, then $\mathcal{F}|S$ is a maximal completely regular filter on S.*

(c) *Every maximal completely regular filter on S is the trace on S of some maximal completely regular filter on X.*

(d) *Every maximal completely regular filter on S is coarser than the trace on S of some (maximal) completely regular filter on X which meets S.*

(e) *If $\mathcal{F}$ is a maximal completely regular filter on S, and if $\mathcal{F}^*$ is the unique maximal completely regular filter on X coarser than $\mathcal{F}$, then $\mathcal{F} \leqslant \mathcal{F}^*|S$.*

Proof. (a) ⇒ (b): Let $A_1, A_2 \in \mathfrak{Z}(S)$ and suppose that $\mathcal{F}|S$ meets both A_1 and A_2. By 2.1(a), it suffices to show that $A_1 \cap A_2 \neq \emptyset$. If $A_1 \cap \cap A_2 = \emptyset$, then by (a) there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $A_i \subset Z_i$ and $Z_1 \cap \cap Z_2 = \emptyset$. But $\mathcal{F}$ meets both Z_1 and Z_2, which is contrary to 2.1(a).

(b) ⇒ (c): Let $\mathcal{G}$ be a maximal completely regular filter on S. By 2.1(d), there is a maximal completely regular filter $\mathcal{F}$ on X with $\mathcal{F}|S \leqslant \mathcal{G}$. Then $\mathcal{F}|S = \mathcal{G}$ by (b).

(c) ⇒ (d) is trivial.

(d) ⇒ (e): Let $\mathcal{F}$ and $\mathcal{F}^*$ be as in (e). By (d), there is a completely regular filter $\mathcal{G}$ on X such that $\mathcal{G}$ meets S and $\mathcal{F} \leqslant \mathcal{G}|S$. As in the proof of 2.2, $\mathcal{G} \subset \mathcal{F}^*$, and hence $\mathcal{F} \leqslant \mathcal{F}^*|S$.

(1) See Theorem 1 of R. Engelking, *On the double circumference of Alexandroff*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 629–634.

(e) ⇒ (a): By 4.1A, it suffices to show that S is z-embedded and satisfies (β).

By (e) and 2.2, every maximal completely regular filter on S is z-embedded in X; hence S is z-embedded by (g) ⇒ (a) of 3.1.

For (β), we verify 4.2A(2): Let $\mathcal{G}$ be a maximal completely regular filter on X, let $Z_1, Z_2 \in \mathfrak{Z}(X)$, and assume that $\mathcal{G}|S$ meets Z_1 and Z_2. Then $\mathcal{G}$ meets S, so $\mathcal{G}|S \subset \mathcal{F}$ for some maximal completely regular filter $\mathcal{F}$ on S. By (e) we have $\mathcal{F} \leqslant \mathcal{F}^*|S$. Then $\mathcal{F}^*$ meets $\mathcal{G}$, so $\mathcal{F}^* = \mathcal{G}$. It follows that $\mathcal{F}$ meets $Z_1 \cap S$ and $Z_2 \cap S$, and hence $Z_1 \cap Z_2 \cap S \neq \emptyset$ by 2.1(a). The proof is therefore complete.

Following [15], we say that two filter bases $\mathcal{B}$ and $\mathcal{B}'$ on X are *completely separated* in X in case some set in $\mathcal{B}$ is completely separated in X from some set in $\mathcal{B}'$. Green notes that S is C^*-embedded in X if and only if any two distinct maximal completely regular filters on S are completely separated in X ([15], Theorem 7). In the same vein:

5.2. COROLLARY. *If $S \subset X$, then these are equivalent:*

(a) *S is C^*-embedded.*

(b) *Distinct z-ultrafilters on S determine distinct z-ultrafilters on X* (see § 2).

(c) *Any two distinct z-ultrafilters on S are completely separated in X.*

Proof. (a) ⇒ (b): This follows easily from Urysohn's Extension Theorem.

(b) ⇒ (c): Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be distinct z-ultrafilters on S, let $\varphi\colon S \to X$ be the canonical injection, and let $\mathcal{U}_i$ be the (unique) z-ultrafilter on X with $\varphi^\#(\mathcal{G}_i) \subset \mathcal{U}_i$. By hypothesis, $\mathcal{U}_1 \neq \mathcal{U}_2$, so it follows that $\varphi^\#(\mathcal{G}_1)$ does not meet $\varphi^\#(\mathcal{G}_2)$. Hence $\mathcal{G}_1$ and $\mathcal{G}_2$ are completely separated in X.

(c) ⇒ (a): We verify 5.1(b): Let $\mathcal{F}$ be a maximal completely regular filter on X which meets S, and suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are z-ultrafilters on S such that $\mathcal{F}|S \leqslant \mathcal{G}_i$ $(i = 1, 2)$. To show that $\mathcal{F}|S$ is maximal, it suffices by [15], Lemma 3 to show that $\mathcal{G}_1 = \mathcal{G}_2$. If $\mathcal{G}_1 \neq \mathcal{G}_2$, then by (c) there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $Z_1 \cap Z_2 = \emptyset$ and $Z_i \cap S \in \mathcal{G}_i$. But then Z_1 and Z_2 meet $\mathcal{F}$, which contradicts 2.1(a).

Green notes (without proof) that a zero-set S in X is C-embedded if and only if S satisfies 5.2(c) ([15], p. 104). This is now clear in view of 5.2 and 4.1B.

If in 5.2(b), the phrase "distinct z-ultrafilters" is replaced by "distinct real z-ultrafilters", then one obtains a necessary and sufficient condition for S to be v-embedded in X (see [3], 3.2).

In 5.3, the equivalence (a) ⇔ (b) is due to Green. This is proved in [14], Theorem 4 (for X Tychonoff), and improved in [15], Theorem 5 (for X arbitrary) as follows: S is C-embedded if and only if every z-ultrafilter on S is equivalent to the trace on S of some z-ultrafilter on X. This last is a consequence of our (a) $\Leftrightarrow$ (c).

5.3. THEOREM. *If $S \subset X$, then these are equivalent:*

(a) *S is C-embedded in X.*

(b) *Every z-ultrafilter on S is the trace on S of some z-ultrafilter on X.*

(c) *Every z-ultrafilter on S is coarser than the trace on S of some z-ultrafilter on X which meets S.*

(d) *Every maximal completely regular filter on S is coarser than the trace on S of some z-ultrafilter on X.*

(e) *If $\mathcal{F}$ is a maximal completely regular filter on S, if $\mathcal{F}^*$ is the unique maximal completely regular filter on X coarser than $\mathcal{F}$, and if $\mathcal{U}$ is the unique z-ultrafilter on X finer than $\mathcal{F}^*$, then $\mathcal{F} \leqslant \mathcal{F}^*|S$ and $\mathcal{U}$ meets S.*

Proof. (a) $\Rightarrow$ (b): Let $\varphi\colon S \to X$ be the canonical injection, and let $\mathcal{F}$ be a z-ultrafilter on S. Then $\varphi^\#(\mathcal{F}) \subset \mathcal{U}$ for some z-ultrafilter $\mathcal{U}$ on X. We claim that $\mathcal{U}$ meets S. If not, there exists $Z \in \mathcal{U}$ with $S \cap Z = \emptyset$. Then S and Z are completely separated (4.1B), so there is $Z' \in \mathfrak{Z}(X)$ with $S \subset Z'$ and $Z \cap Z' = \emptyset$. But then $Z' \in \varphi^\#(\mathcal{F}) \subset \mathcal{U}$, a contradiction. Thus $\mathcal{U}$ meets S, so $\mathcal{U}|S$ is a z-filter on S by (a) $\Rightarrow$ (c) of 3.1. But by (a) $\Rightarrow$ (b) of 3.1 we also have $\mathcal{F} \subset \varphi^\#(\mathcal{F})|S \subset \mathcal{U}|S$, and hence $\mathcal{F} = \mathcal{U}|S$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d): If $\mathcal{F}$ is a maximal completely regular filter on S, then $\mathcal{F} \leqslant \mathcal{G}$ for some z-ultrafilter $\mathcal{G}$ on S (2.1(b)); and, by (c), $\mathcal{G} \leqslant \mathcal{U}|S$ for some z-ultrafilter $\mathcal{U}$ on X which meets S. Then $\mathcal{F} \leqslant \mathcal{U}|S$.

(d) $\Rightarrow$ (e): Let $\mathcal{F}$, $\mathcal{F}^*$, and $\mathcal{U}$ be as in (e). By (d), there is a z-ultrafilter $\mathcal{U}'$ on X which meets S such that $\mathcal{F} \leqslant \mathcal{U}'|S$. Since $\mathcal{F}^* \leqslant \mathcal{F}$, we have $\mathcal{F}^* \leqslant \mathcal{U}'|S$, and it follows that $\mathcal{U}'$ meets $\mathcal{F}^*$. Hence $\mathcal{U}' \subset \mathcal{U}$ (2.1(b)), so $\mathcal{U}' = \mathcal{U}$.

It remains to show that $\mathcal{F} \leqslant \mathcal{F}^*|S$. If this is false, then there is $F \in \mathcal{F}$ such that $(A \cap S) - F \neq \emptyset$ for every $A \in \mathcal{F}^*$. Choose $F' \in \mathcal{F}$ and $f \in C(S)$ such that $f = 1$ on F', $f = 0$ on $S - F$, and $f \geqslant 0$. Then $\mathcal{B} = \{f^{-1}[0, r]\colon r > 0\}$ is a completely regular filter base on S which meets $\mathcal{F}^*|S$, so by (d) there is a z-ultrafilter $\mathcal{U}''$ on X such that $\mathcal{U}''$ meets S, $\mathcal{U}''|S \geqslant \mathcal{B}$, and $\mathcal{U}''|S \geqslant \mathcal{F}^*|S$. This last inequality implies that $\mathcal{U}''$ meets $\mathcal{F}^*$, and hence (once again) $\mathcal{U}'' = \mathcal{U}$. Thus $\mathcal{U}''|S \geqslant \mathcal{F}$. But since $\mathcal{U}''|S \geqslant \mathcal{B}$, we then have $F' \cap f^{-1}[0, \frac{1}{2}] \neq \emptyset$, a contradiction.

(e) $\Rightarrow$ (a): By 5.1, S is C^*-embedded (or by 2.2 and 3.1, S is z-embedded). Moreover, 4.2B holds, so (γ) holds. Thus S is C-embedded by 4.1B, and the proof is complete.

5.4. Remark. It is perhaps worth noting that (b) $\Rightarrow$ (a) of 5.3 can be proved quite directly: If (b) holds, then S satisfies (γ'), and hence (γ) (see 4.4). To verify z-embedding, let $\mathcal{G}$ be a z-ultrafilter on S; by (b), $\mathcal{G} = \mathcal{U}|S$ for some z-ultrafilter $\mathcal{U}$ on X. Then $\mathcal{U} \subset \varphi^\#(\mathcal{G})$ (where $\varphi\colon S \to X$ is the canonical injection), and hence $\mathcal{G} = \mathcal{U}|S \subset \varphi^\#(\mathcal{G})|S$. Thus S is z-embedded by the (virtually trivial) implication (b) $\Rightarrow$ (a) of 3.1. Hence S is C-embedded by 4.1B.

5.5. COROLLARY (Green [15], p. 103). *Let $S \subset X$. S is C-embedded in X if and only if S satisfies (γ') and ($*$) (see § 3).*

This is an immediate consequence of the equivalence of (a), (b), and (c) of 5.3 and (a) $\Leftrightarrow$ (c) of 3.1. 5.5 bears a somewhat curious relationship to 4.1B: Of the two conditions (γ') and ($*$), one is (in general) strictly stronger than (γ) and the other strictly weaker than z-embedding (see 4.5(a) and 3.4); but, together, they imply (γ) and z-embedding.

5.6. COROLLARY. *If S is a pseudocompact subset of X, then these are equivalent:*

(a) *S is C-embedded in X.*

(b) *S is z-embedded in X.*

(c) *If $\mathcal{F}$ is any z-ultrafilter on X which meets S, then $\mathcal{F}|S$ is a base for a z-ultrafilter on S (i.e., ($*$) holds).*

Proof. (a) $\Rightarrow$ (b) is trivial, (b) $\Rightarrow$ (c) by 3.1, and (c) $\Rightarrow$ (a) by 5.5 and 4.7.

The equivalence (a) $\Leftrightarrow$ (b) of 5.6 is given a different proof in [6], 4.4

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