The kernel operation on subsets of a T_1-space *

by

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Abstract. Every finitely generated kernel algebra of subsets of a T_1-space is finite. The structure of such algebras is fully described. Kernel and complement alone generate ten distinct set functions; kernel, complement, and union generate 64. The kernel operation is shown to be characterized by a set of six postulates.

1. Introduction. The kernel of a set A contained in a T_1-space C is the largest subset of A that is dense in itself. Given A, how many different sets can be obtained by applying the operations of kernel and complement successively? This question was raised in 1930 by Zarycki [9], who concluded, incorrectly, that at most eight different sets can be so obtained. We shall show that in general the correct answer is ten. Surprisingly, even when unions are added only a finite number of different sets are obtained (at most 64). More generally, any finitely generated kernel algebra is finite. (A kernel algebra is an algebra of subsets of C that includes the kernel of each of its elements.)

These results contrast with the corresponding facts concerning the closure operation. Kuratowski [3] (see also [10]) showed that successive application of closure and complement to a given subset of a T_1-space gives rise to at most 14 different sets, but that there exists a subset of the line that generates an infinite closure algebra.

In § 2 we determine the structure of all finitely generated kernel algebras. In § 3 we answer Zarycki's question. In § 4 we study the extent to which a topology is determined by its kernel operation, and obtain a complete postulational characterization of these set functions.

2. Kernel algebras. We denote the kernel, complement, and closure of a set $A \subset C$ by A^n, A^c (or $C-A$), and A^-, respectively, and the intersection of A and B by AB. Evidently $A^n \subset A$ and $A^{nn} = A^n$, for all A. $A^n = A$ if and only if A is dense in itself; $A^n = \emptyset$ if and only if A is scattered. The following six propositions are known to hold in any T_1-space ([4], pp. 77–79, or [8], pp. 163–165):

(2.1) $A^n = AA^{n-}$ (*A^n is a relatively closed subset of A*).

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(2.2) *The union of any family of sets dense in themselves is dense in itself.*

(2.3) *The union of any two scattered sets is scattered.*

(2.4) $A - A^n$ *is scattered.*

(2.5) *If A is dense in itself and G is open, then AG is dense in itself.*

(2.6) $p^n = \emptyset$ *for each* $p \in C$.

The following formula generalizes (2.1):

$$(2.7) \qquad \Big(\bigcup_{i=1}^{k} A_i\Big)^n = \Big(\bigcup_{i=1}^{k} A_i\Big)\Big(\bigcup_{i=1}^{k} A_i^{n-}\Big).$$

Proof. Since $A_i^n \subset A_i$ for each i, (2.2) implies that

$$\bigcup_{i=1}^{k} A_i^n \subset \Big(\bigcup_{i=1}^{k} A_i\Big)^n.$$

Taking closures and using (2.1) it follows that

$$\Big(\bigcup_{i=1}^{k} A_i\Big)\Big(\bigcup_{i=1}^{k} A_i^{n-}\Big) \subset \Big(\bigcup_{i=1}^{k} A_i\Big)\Big(\bigcup_{i=1}^{k} A_i\Big)^{n-} = \Big(\bigcup_{i=1}^{k} A_i\Big)^n.$$

On the other hand,

$$\Big(\bigcup_{i=1}^{k} A_i\Big)^n\Big(\bigcup_{i=1}^{k} A_i^{n-}\Big)^c \subset \Big(\bigcup_{i=1}^{k} A_i\Big)\Big(\bigcap_{i=1}^{k} A_i^{n-c}\Big) \subset \bigcup_{i=1}^{k} A_i A_i^{n-c} \subset \bigcup_{i=1}^{k} (A_i - A_i^n).$$

The first of these sets is dense in itself, by (2.5). The last one is scattered, by (2.4) and (2.3). Hence the first set is empty. Therefore

$$\Big(\bigcup_{i=1}^{k} A_i\Big)^n \subset \Big(\bigcup_{i=1}^{k} A_i\Big)\Big(\bigcup_{i=1}^{k} A_i^{n-}\Big),$$

and the conclusion follows.

(2.8) *Let U be the union of a finite family $\mathcal{F}$ of subsets of C such that*

(i) *each member of $\mathcal{F}$ is either scattered or dense in itself, and*

(ii) *if A in $\mathcal{F}$ is dense in itself and B in $\mathcal{F}$ is scattered, then either $B \subset A^-$ or $B \subset A^{-c}$.*

Then U^n is the union of those members of $\mathcal{F}$ that are contained in the closure of at least one member of $\mathcal{F}$ that is dense in itself.

Proof. This follows from (2.7). Under hypotheses (i) and (ii), the intersection of a set A_i in $\mathcal{F}$ with the union of the sets A_j^{n-} is equal to A_i in case $A_i \subset A_j^-$ for some $A_j = A_j^n$, and otherwise it is empty.

In the case $k = 2$, (2.7) reduces to the useful formula:

$$(2.9) \qquad (A \cup B)^n = A^n \cup B^n \cup AB^{n-} \cup BA^{n-}.$$

The following three propositions are corollaries of (2.9).

(2.10) *If A is perfect ($A = A^n = A^-$), then*

$$(A \cup B)^n = A^n \cup B^n \quad \textit{for all } B \subset C.$$

(2.11) *If B is scattered, then*

$$BA^{n-} = B(A \cup B)^n \quad \textit{and} \quad BA^{n-c} = B(A \cup B)^{nc}.$$

(2.12) *If $A^n \subset B \subset A^{n-}$, then $B = B^n$; in particular, A^{n-} is perfect.*

$$(2.13) \qquad A^{n-} = \{p\colon p \in (A^n \cup p)^n\}.$$

Proof. In (2.11), take $B = p$ and replace A by A^n.

(2.14) *A is perfect if and only if $(A \cup p)^n = A$ for all $p \in C$.*

Proof. If A is perfect, then $(A \cup p)^n = A$ by (2.10) and (2.6). If $(A \cup p)^n = A$ for all p, then $A = A^n = (A^n \cup p)^n$ for all p. Hence $A = A^{n-}$, by (2.13), and therefore A is perfect.

The following proposition generalizes (2.10).

(2.15) *In order that $(A \cup B)^n = A^n \cup B^n$ for all $B \subset C$ it is necessary and sufficient that $AC^n \cup A^{n-} \subset A^n$.*

Proof. If $AC^n \cup A^{n-}$ is not contained in A^n, let $p \in (AC^n \cup A^{n-})A^{nc}$. There are two cases: (i) If $p \in AC^nA^{nc}$, take $B = C^n - p$. Then $C^n = p \cup B \subset A \cup B$ and therefore $p \in (A \cup B)^n - (A^n \cup B^n)$. (ii) If $p \in A^{n-}A^{nc}$, take $B = p$. Then $A^n \subset A^n \cup p \subset A^{n-}$ and therefore, by (2.12),

$$(A^n \cup p)^n = A^n \cup p \subset A \cup B \quad \text{and} \quad p \in (A \cup B)^n - (A^n \cup B^n).$$

In either case, $(A \cup B)^n \neq A^n \cup B^n$.

If $AC^n \cup A^{n-} \subset A^n$, then for any set $B \subset C$ we have

$$AB^{n-} \subset AC^n \subset A^n \quad \text{and} \quad BA^{n-} \subset A^n,$$

hence $(A \cup B)^n = A^n \cup B^n$, by (2.9).

It follows from (2.15) that the converse of (2.10) is valid when and only when C is dense in itself.

(2.16) Theorem. *Let $\mathfrak{I}_0$ be a partition of C into finitely many sets $A_1, \ldots, A_k$. Let $\mathfrak{I}$ be the refinement of $\mathfrak{I}_0$ obtained by partitioning each of the sets A_j into A_j^n and the at most 2^{k-1} sets of the form $B = B_1B_2 \cdots B_k$, where $B_j = A_j - A_j^n$ and for each $i \neq j$, either $B_i = A_i^{n-}$ or $B_i = A_i^{n-c}$. Then the kernel algebra generated by $\mathfrak{I}_0$ is the algebra generated by $\mathfrak{I}$.*

Proof. Since every subset of $A_j - A_j^n$ is scattered, repeated application of (2.11) shows that the kernel algebra generated by $\mathfrak{I}_0$ must include each set of the form B. It also includes each set of the form A_j^n, and therefore every member of $\mathfrak{I}$. To show that the algebra generated by $\mathfrak{I}$ is the kernel algebra generated by $\mathfrak{F}_0$ it therefore suffices to prove that the kernel of any union of members of $\mathfrak{F}$ is a union of members of $\mathfrak{F}$. But this follows from (2.8), since (i) the non-empty members of $\mathfrak{F}$ are either scattered and of the form B for some j, or else dense in themselves and of the form A_i^n for some i, and (ii) for any i and j, either $B \subset A_i^{n-}$ or $B \subset A_i^{n-c}$, by the definition of B_i in case $i \neq j$, and by (2.1) in case $i = j$.

The atoms of the generated kernel algebra are the non-empty members of $\mathfrak{F}$. The possible combinations may be described as follows.

(2.17) *If $A_1, \ldots, A_k$ are non-empty, then the class $\mathfrak{F}_1$ of non-empty members of $\mathfrak{F}$ satisfies the following conditions*:

(1) *$\mathfrak{F}_1$ includes at least one subset of each A_i, and*

(2) *if $\mathfrak{F}_1$ includes an element of the form B with $B_i = A_i^{n-}$, then $\mathfrak{F}_1$ includes A_i^n.*

Any subfamily of $\mathfrak{F}$ (regarded as a set of formal expressions) that satisfies these two conditions is a possible candidate for $\mathfrak{F}_1$.

Proof. The necessity of (1) and (2) is clear. To show that they are sufficient we first construct an example in which all members of $\mathfrak{F}$ are non-empty. Let $Q_1, \ldots, Q_k$ be k disjoint irrational translates of the set of rational numbers. Let J be the set of integers from 1 to $k \cdot 2^{k-1}$. Define ψ on J so that for each $1 \leqslant i \leqslant k$, ψ maps the integers from $(i-1)2^{k-1}+1$ to $i \cdot 2^{k-1}$ bijectively onto the 2^{k-1} subsets of $\{Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_k\}$. Put

$$D_i = Q_i - \bigcup \{(j-\tfrac{1}{2}, j+\tfrac{1}{2})\colon\ j \in J,\ Q_i \notin \psi(j)\}$$

and

$$A_i = D_i \cup \{j \in J\colon\ (i-1)2^{k-1} < j \leqslant i \cdot 2^{k-1}\}\,.$$

Then the sets $A_1, \ldots, A_k$ constitute a partition $\mathfrak{F}_0$ of their union C. The corresponding partition $\mathfrak{F}$ consists of the sets $D_1, \ldots, D_k$ and the singleton subsets of J.

Now let $\mathfrak{F}_1$ be an arbitrary subfamily of this partition $\mathfrak{F}$ that satisfies conditions (1) and (2). Let C_1 be the union of $\mathfrak{F}_1$. Then the sets $A_i \cap C_1$ $(i = 1, \ldots, k)$ constitute a partition of C_1 into non-empty sets, and the atoms of the generated kernel algebra are the members of $\mathfrak{F}_1$.

In particular, $\mathfrak{F}_1$ may consist of $D_1, \ldots, D_k$ and the singletons contained in any subset of J. Hence the generated kernel algebra may have any number of atoms from k to $k(1+2^{k-1})$.

Let $\mathcal{F}$ be any set of atoms of a finite kernel algebra $\mathcal{A}$. Each member of $\mathcal{F}$ is either scattered or dense in itself. If $A \in \mathcal{F}$ is dense in itself and $B \in \mathcal{F}$ is scattered, then $BA^- \in \mathcal{A}$, by (2.11), and therefore $BA^- = B$ or $\emptyset$, since B is an atom. It follows from (2.8) that the kernel of each member of $\mathcal{A}$ is determined as soon as we know (i) the set D of atoms that are dense in themselves, (ii) the set S of scattered atoms, and (iii) the mapping φ that assigns to each $s \in S$ the set

$$\varphi(s) = \{d \in D\colon\ s \subset d^-\} = \{d \in D\colon\ (s \cup d)^n = s \cup d\}\,.$$

Let us define two kernel algebras to be isomorphic if there exists a Boolean isomorphism of one onto the other that commutes with the kernel operation. Then two finite kernel algebras are isomorphic if and only if there exists a one-to-one correspondence between their atoms that makes the φ-function of one correspond to that of the other.

The φ-function can be specified arbitrarily. Indeed, let $\{D, S\}$ be any partition of $\{1, 2, \ldots, k\}$, and let φ be any map of S into the power set of D. When $i \in D$, define $f(i) = D_i$. When $i \in S$, define $f(i)$ to be the unique element $j \in A_i \cap J$ such that $\psi(j) = \{Q_r\colon\ r \in \varphi(i)\}$. Then the range of f is a family $\mathfrak{F}_1$ that satisfies the conditions of (2.17); it is the set of atoms of a kernel algebra having the prescribed φ-function.

We may regard the set $D \cup S$ of atoms of a finite kernel algebra as the vertices of a bipartite graph [5] in which an element $s \in S$ is joined to an element $d \in D$ if and only if $(s \cup d)^n = s \cup d$. The problem of classifying finite kernel algebras up to isomorphism is seen to be equivalent to that of classifying all finite bipartite graphs (without multiple edges) regarded as based on an ordered partition of the vertex set into two sets.

For a somewhat analogous study of finite algebras that include the closure and the coherence of each member, see [7].

3. Kernel and complement. Let A be an arbitrary subset of a T_1-space C. Note that $A - A^n = AA^{n-c}$, by (2.1). By (2.16), the kernel algebra generated by A consists of the 64 unions that can be formed from the following six sets (each of which, by (2.11), can be expressed in terms of kernel, complement, and union):

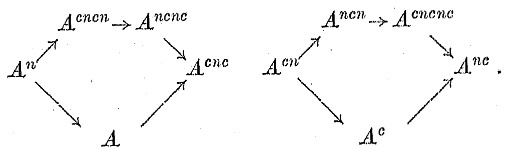
$$\begin{aligned} a &= A^n\,, & b &= A^{cn}\,,\\ c &= AA^{n-c}A^{cn-}\,, & d &= A^cA^{cn-c}A^{n-}\,,\\ e &= AA^{n-c}A^{cn-c}\,, & f &= A^cA^{cn-c}A^{n-c}\,. \end{aligned}$$

It follows from (2.8) that

$$\begin{aligned} A &= a \cup c \cup e\,, & A^c &= b \cup d \cup f\,,\\ A^n &= a\,, & A^{cn} &= b\,,\\ A^{nc} &= b \cup c \cup d \cup e \cup f\,, & A^{cnc} &= a \cup c \cup d \cup e \cup f\,,\\ A^{ncn} &= b \cup c\,, & A^{cncn} &= a \cup d\,,\\ A^{ncnc} &= a \cup d \cup e \cup f\,, & A^{cncnc} &= b \cup c \cup e \cup f\,, \end{aligned}$$

and that $A^{ncncn} = A^{cncn}$ and $A^{cncncn} = A^{ncn}$. Also $C^n = a \cup b \cup c \cup d$. Part of the information contained in these formulas is summarized in the following theorem.

(3.1) THEOREM. *Any succession of the operations of kernel and complement applied to a set $A \subset C$ leads to one of the ten sets appearing in the following diagrams:*



The arrows indicate inclusion relations of the form $\subset$. Unless a, b, c, d, or $e \cup f$ is empty, no other inclusions relate any two of these sets. The two uppermost sets in either diagram are equal to each other if and only if C is dense in itself.

In [9], as already noted by Vaidyanathaswamy ([8], p. 166), it was erroneously supposed that $(A \cup B)^n = A^n \cup B^n$ for all A and B. This and the relations $A^n \subset A$ and $A^{ncnc} \subset A^{ncncn}$ (which holds if and only if $C = C^n$) were treated as axioms. These three propositions are actually equivalent to the single proposition $A^n = A$. (The last two, with $A = \emptyset$, imply $C^{nc} = C^{ncn}$. Then the first two imply that

$$C = C^n \cup C^{nc} = C^n \cup C^{ncn} = (C \cup C^{nc})^n = C^n$$

and

$$A = A(A \cup A^c)^n = AA^n \cup AA^{cn} = A^n.)$$

Consequently, these "axioms" are never satisfied by the kernel operation. Nevertheless, the numbered propositions deduced from them in [9] are, in fact, true of the kernel operation in case $C = C^n$.

4. Postulates for the kernel operation. The closure A^- is not uniquely determined by the kernel operation A^n. In fact, using (2.9), it is easy to verify that the set function $A^\sim = A \cup A^{n-}$ also satisfies the axioms ([4], p. 38) for a closure in C. Since $A \subset A^\sim \subset A^-$ for each $A \subset C$, the new topology $\tilde{\mathfrak{T}}$ contains the given topology $\mathfrak{T}$.

Let A^d and A^δ denote the derived sets of A relative to $\mathfrak{T}$ and $\tilde{\mathfrak{T}}$, respectively. Since $\mathfrak{T} \subset \tilde{\mathfrak{T}}$, we have $A^\delta \subset A^d$ for each $A \subset C$. If $p \in A \subset A^d$, then $A = A^n$, $(A-p)^n = A-p$ by (2.5), and $p \in (A-p)^-$. Hence $p \in (A-p)^{n-} \subset (A-p)^\sim$, and therefore $p \in A^\delta$. Thus $A \subset A^d$ implies $A \subset A^\delta$. The converse follows from the fact that $A^\delta \subset A^d$. Hence A^n is the kernel function for $\tilde{\mathfrak{T}}$ as well as for $\mathfrak{T}$.

Any topology in C that has A^n for its kernel function will have the same scattered sets, the same dense in themselves sets, and also the same

perfect sets as $\mathfrak{T}$, by (2.14). Since every closed set is the union of a perfect set and a scattered set ([4], p. 79), $\tilde{\mathfrak{T}}$ is the maximal topology that has A^n for its kernel function; it is distinguished among these by the property that relative to it every scattered set is closed. $\tilde{\mathfrak{T}}$ is an example of a *topology in the sense of Hashimoto [2]; it is the *topology corresponding to $\mathfrak{T}$ and the ideal of scattered sets (cf. [8], p. 183, or [1]). (A *topology is the "P-topology" corresponding to an adherence ideal P in the sense of Vaidyanathaswamy ([8], pp. 174–177).)

According to (2.13), A^n determines $\tilde{\mathfrak{T}}$ by means of the formula

$$A^\sim = A \cup \{p\colon\ p \in (A^n \cup p)^n\}. \tag{4.1}$$

Under what conditions on the set function A^n does this formula define a closure in C that has A^n for its kernel function? This question is answered by the following theorem.

(4.2) THEOREM. *Let C be a non-empty set and let $A \to A^n$ be a mapping of the power set of C into itself. In order that there exist a T_1-topology in C having A^n for its kernel function it is necessary and sufficient that the following postulates be satisfied:*

I. $A^n \subset A$,

II. $A \subset B$ *implies* $A^n \subset B^n$,

III. $A^{nn} = A^n$,

IV. $p^n = \emptyset$,

V. $A^n = B^n = \emptyset$ *implies* $(A \cup B)^n = \emptyset$,

VI. *If* $A = \{p\colon\ p \in (X^n \cup p)^n\}$ *for some* $X \subset C$, *then* $(A \cup B)^n = A^n \cup B^n$ *for all* $B \subset C$.

Proof. The necessity of I, II, and III is clear, IV follows from (2.6), V from (2.3), and VI from (2.13) and (2.10). Their sufficiency is established by the following sequence of lemmas, in each of which it is assumed that Postulates I to VI are satisfied and that $A^\sim$ is defined by (4.1).

$$X^\sim = (X - X^n) \cup \{p\colon\ p \in (X^n \cup p)^n\}. \tag{4.3}$$

Proof. $X^n \subset (X^n \cup p)^n$ by II and III, hence

$$X^\sim \subset (X - X^n) \cup X^n \cup \{p\colon\ p \in (X^n \cup p)^n\}$$
$$= (X - X^n) \cup \{p\colon\ p \in (X^n \cup p)^n\} \subset X^\sim.$$

$$(X - X^n)^n = \emptyset. \tag{4.4}$$

Proof. $(X - X^n)^n \subset (X - X^n)X^n$, by I and II.

(4.5) $X \subset Y$ *implies* $X^\sim \subset Y^\sim$.

Proof. $X \subset Y$ implies $(X^n \cup p)^n \subset (Y^n \cup p)^n$ by II, and therefore $X^\sim \subset Y^\sim$.

(4.6) *If* $A = \{p:\ p \in (X^n \cup p)^n\}$ *for some* $X \subset C$, *then* $(A \cup Z)^n = A \cup Z^n$ *for all* $Z \subset C$.

Proof. Using I, II, and III, we have $X^n \subset A$ and $X^n \subset (X^n \cup p)^n \subset X^n \cup p$. If $p \in A$, then $p \in (X^n \cup p)^n$ and the last inclusion becomes an equality; therefore

$$(X^n \cup p)^n = X^n \cup p \subset A, \quad X^n \cup p \subset A^n, \quad \text{and} \quad p \in A^n.$$

Thus $A \subset A^n$. Hence $A = A^n$. The conclusion then follows from VI.

(4.7) $$X^{\sim\sim} = X^{\sim}.$$

Proof. Let $A = \{p:\ p \in (X^n \cup p)^n\}$. Then $X^{\sim} = (X - X^n) \cup A$, by (4.3). By (4.6), (4.4), and IV, we have $X^{\sim n} = A$ and $(X^{\sim n} \cup p)^n = A \subset X^{\sim}$ for every $p \in C$. Hence $X^{\sim\sim} = X^{\sim} \cup \{p:\ p \in (X^{\sim n} \cup p)^n\} = X^{\sim}$.

(4.8) $$(X \cup Y)^{\sim} = X^{\sim} \cup Y^{\sim}.$$

Proof. Let $A = \{p:\ p \in (X^n \cup p)^n\}$ and $B = \{p:\ p \in (Y^n \cup p)^n\}$. Then

$$X \cup Y \subset X^{\sim} \cup Y^{\sim} = A \cup B \cup (X - X^n) \cup (Y - Y^n),$$

by (4.3). Hence

$$(X \cup Y)^n \subset A \cup B \quad \text{and} \quad [(X \cup Y)^n \cup p]^n \subset A \cup B,$$

by II, (4.6), (4.4), V, and IV. Therefore

$$(X \cup Y)^{\sim} = X \cup Y \cup \{p:\ p \in [(X \cup Y)^n \cup p]^n\}$$
$$\subset X \cup Y \cup A \cup B = X^{\sim} \cup Y^{\sim}.$$

The reverse inclusion follows from (4.5).

Since the relations $X \subset X^{\sim}$, $\emptyset^{\sim} = \emptyset$, and $p^{\sim} = p$ are obvious, (4.7) and (4.8) show that $X^{\sim}$ satisfies the axioms for a closure in C. Call this topology $\mathfrak{T}$. Let X^{δ} denote the derived set of X relative to $\mathfrak{T}$.

(4.9) *If* $X \subset X^{\delta}$, *then* $X = X^n$.

Proof. Suppose $X \subset X^{\delta}$ and $p \in X$. Then $p \in X^{\delta}$ and so

$$p \in (X - p)^{\sim} = (X - p) \cup \{q:\ q \in [(X - p)^n \cup q]^n\}.$$

Hence $p \in [(X - p)^n \cup p]^n$. Since $p \in X$, we have

$$(X - p)^n \cup p \subset (X - p) \cup p = X,$$

by I. Therefore

$$[(X - p)^n \cup p]^n \subset X^n,$$

by II. Hence $p \in X^n$. Thus $X \subset X^n$, and so $X = X^n$, by I.

(4.10) *If* $X = X^n$, *then* $X \subset X^{\delta}$.

Proof. For any $X \subset C$ and $p \in C$ we have

$$(X - p)^{\sim} = (X - p) \cup A, \quad \text{where } A = \{q:\ q \in [(X - p)^n \cup q]^n\}.$$

Since $(X - p)^n \subset A$, we have

$$X \subset (X - p)^n \cup [(X - p) - (X - p)^n] \cup p$$
$$\subset A \cup [(X - p) - (X - p)^n] \cup p.$$

Therefore, by II, (4.6), (4.4), IV, and V,

$$X^n \subset A \subset (X - p)^{\sim}.$$

Hence if $X = X^n$ and $p \in X$, then $p \in X^{\delta}$.

Lemmas (4.9) and (4.10) show that X is dense in itself relative to $\mathfrak{T}$ if and only if $X = X^n$. It then follows from I, II, and III that for each $A \subset C$, A^n is the largest subset of A that is dense in itself. Thus Postulates I to VI completely characterize those set functions on the power set of C that can be identified with the kernel operation of a T_1-topology in C.

As an illustration, let C be a T_1-space without isolated points. For each $A \subset C$ let A^n denote the set of points of A at which A is of second category. A^n is not in general the kernel of A, but it is easy to verify that Postulates I to VI are satisfied. The corresponding topology $\mathfrak{T}$ can be seen to coincide with one of the *topologies considered by Hashimoto [2], namely, the *topology induced by the ideal of sets of first category in C. This topology has also been considered by Vaidyanathaswamy ([8], p. 178) and by Freud [1]. The category measure spaces constructed from lower densities in [6] can also be seen to be examples of topologies of this kind, constructed in answer to a question arising in measure theory.

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References

[1] G. Freud, *Ein Beitrag zu dem Satze von Cantor und Bendixson*, Acta Math. Acad. Sci. Hungar. 9 (1958), pp. 333–336.
[2] H. Hashimoto, *On the *topology and its applications*, (to appear).
[3] C. Kuratowski, *Sur l'opération $\bar{A}$ de l'Analysis Situs*, Fund. Math. 3 (1922), pp. 182–199.
[4] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[5] O. Ore, *Theory of Graphs*, Amer. Math. Soc., Providence, 1962.
[6] J. C. Oxtoby, *Measure and Category*, Berlin–Heidelberg–New York 1971.

[7] R. S. Pierce, *Compact zero-dimensional metric spaces of finite type*, Mem. Amer. Math. Soc. 130 (1972).
[8] R. Vaidyanathaswamy, *Treatise on Set Topology*, Part I, Indian Mathematical Society, Madras 1947.
[9] M. Zarycki, *Über den Kern einer Menge*, Jber. Deutsch. Math.-Verein. 39 (1930), pp. 154–158.
[10] Problem 5349. Amer. Math. Monthly 73 (1966), p. 1132.

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Filter characterizations of z-, C^*-, and C-embeddings

by

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To the memory of Lee W. Anderson

Abstract. The paper [6] by the author and A. W. Hager is supplemented here by a number of filter-theoretic characterizations of z-embedding, and of those conditions which must be added to z-embedding to produce C^*- or C-embedding. These lead to filter characterizations of C^*- and C-embedding which include results of J. W. Green [15].

1. Introduction. The subset S of the topological space X is *z-embedded* in X if each zero-set of S is the restriction to S of a zero-set of X. (A zero-set is the set of zeros of a real-valued continuous function.) The notion of z-embedding occurs (sometimes only implicitly) in some special contexts in the early papers [12], [16], [17], and [18]. In 1963 the author initiated the general theory of z-embedding, and at the same time introduced the term "z-embedding" itself. (See [2]; portions of [2] are incorporated in [3], [4], [5], and [6].) Subsequently, the theory has been developed by A. W. Hager and by the author (sometimes jointly), as well as by others; see [6] for a number of basic results and for a comprehensive bibliography of relevant papers.

This paper may be regarded as a sequel to both [6] and [15]: [6] is devoted to a study of z-embedding and its relation to C^*- and C-embedding, but convergence (i.e., filter-theoretic) considerations are ignored. [15], on the other hand, is devoted to filter characterizations of C^*- and C-embeddings, but with no mention of z-embedding. In the present paper we supplement both [6] and [15] by providing filter characterizations of z-embedding (see 3.1) and of those conditions which must be added to z-embedding to produce C^*- or C-embedding (see 4.1 and 4.2). The C^*- and C-embedding characterizations of [15] (as well as improvements thereon) are then deduced as consequences (see 5.1 and 5.3).

Except for 3.2(d), 3.5, 3.6, and 3.9, the results of this paper require no separation axioms.

2. Preliminaries. We assume familiarity with [11], whose notation and terminology will be used throughout.