References

[1] P. T. Church and E. Hemmingsen, *Light open maps on n-manifolds*, Duke Math. J. 27 (1960), pp. 527–536.

[2] E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. 72 (1952), pp. 138–147.

[3] S. Stoilow, *Sur les transformations continues et la topologie des fonctions analytiques*, Annales Scientifiques de l'Ecole Normale Superieure 63 (1928), pp. 347–382.

[4] — Annales de l'Institut Henri Poincaré 2 (1932), pp. 233–266.

[5] C. J. Titus and G. S. Young, *The extension of interiority with some applications*, Trans. Amer. Math. Soc. 103 (1962), pp. 329–340.

[6] G. T. Whyburn, *Analytic Topology*, 2nd ed. Providence: Amer. Math. Soc. (1963).

WORTHINGTON SCRANTON CAMPUS
PENNSYLVANIA STATE UNIVERSITY
Dunmore, Pennsylvania

Accepté par la Rédaction le 8. 4. 1974

Souslin-Kleene does not imply Beth

by

Finn V. Jensen (Warszawa)

Abstract. It is proved that $L_{\min}$ (first-order logic without quantifiers) fulfils the Craig interpolation theorem. An extension of $L_{\min}$ is given, which satisfies the Souslin–Kleene interpolation theorem, but not the Beth definability theorem.

Comparing with [3] we then have shown that for extensions of $L_{\min}$, the only valid implications between Craig, Beth, S-K and weak Beth, are the following:

Craig

⇙ ⇘

S-K Beth

⇘ ⇙

weak Beth

For basic definitions, readers are refered to [3].

Definition 1. Let $\mathfrak{B}$ be a structure of type τ, let A be a set of constant-symbols, and let $\{b_a|\ a\in A\}\subseteq|\mathfrak{B}|$. Then $\mathfrak{B}(\langle b_a|\ a\in A\rangle)$ denotes the structure $\mathfrak{B}'$ of type $\tau\cup A$, s.t. $\mathfrak{B}'\upharpoonright\tau=\mathfrak{B}$, and $a^{\mathfrak{B}'}=b_a$ for all $a\in A$. The *open formula* $K(X/A)$ with $X=\{x_a|\ a\in A\}$ as free variables, is the following class of sets:

$$\{K(X/A)^{\mathfrak{B}}|\ \mathfrak{B} \text{ a structure of type } \tau\},$$

where

$$K(X/A)^{\mathfrak{B}}=\{B\subseteq|\mathfrak{B}||\ B=\{b_a|\ a\in A\} \text{ and } \mathfrak{B}(\langle b_a|\ a\in A\rangle)\in K\},$$

i.e. $K(X/A)^{\mathfrak{B}}$ is the set of tuples from $|\mathfrak{B}|$ satisfying K.

Definition 2. A logic L has

1) *Souslin-Kleene* property if for any PC_L, P, if $\overline{P}$ is PC_L then P is EC_L.

2) *Beth property* if the following holds: Let R be an n-ary relation symbol and let K be an EC_L of type $\tau\cup\{R\}$ s.t. for each structure $\mathfrak{A}$ of type τ there exists at most one $\mathfrak{B}\in K$ s.t. $\mathfrak{A}=\mathfrak{B}\upharpoonright\tau$, then there exists an open formula F of type τ with n free variables s.t. for each $\mathfrak{A}\in K$ we have $R^{\mathfrak{A}}=F^{\mathfrak{A}\upharpoonright\tau}$.

DEFINITION 3. Let L be a logic. Then $\Delta(L)$ is the family of classes of structures, obtained by adding to L all PC_L's, P, where $\overline{P}$ is PC_L too.

THEOREM 4 (Friedman [3]). *$\Delta(L)$ is a logic satisfying Souslin–Kleene.*

The proof is straight-forward.

DEFINITION 5. Let $\mathfrak{A}$ be a structure of finite type τ. $\Psi_{\mathfrak{A}}$ is the $L_{\min}$-sentence of type τ "describing" $\mathfrak{A}$. That is: $\Psi_{\mathfrak{A}}$ is the intersection of all atomic and negated atomic sentences which hold in $\mathfrak{A}$ (we allow $\emptyset$ to be an $L_{\min}$-sentence, i.e. if τ does not contain any constant-symbol $\Psi_{\mathfrak{A}}$ is $\emptyset$).

Note. If $\mathfrak{B} \models \Psi_{\mathfrak{A}}$ then $\mathfrak{B} \equiv_{L_{\min}} \mathfrak{A}$, and hence $\mathfrak{A} \models \Psi_{\mathfrak{B}}$ if and only if $\mathfrak{B} \models \Psi_{\mathfrak{A}}$.

Let C be any class of structures, each of the same finite type. Then

$$\Psi_C = \bigvee_{\mathfrak{A} \in C} \Psi_{\mathfrak{A}}.$$

This disjunction is finite, so Ψ_C is an $L_{\min}$-sentence.

PROPOSITION 6. *If K is $EC_{L_{\min}}$ then $\mathfrak{A} \in K \leftrightarrow \mathfrak{A} \models \Psi_K$.*

Proof. If $\mathfrak{A} \models \Psi_K$, then $\mathfrak{A} \models \Psi_{\mathfrak{B}}$ for some $\mathfrak{B} \in K$. Hence $\mathfrak{B} \equiv_{L_{\min}} \mathfrak{A}$.

DEFINITION 7. Let $\mathfrak{A}$ be a structure of type $\{R_1, \ldots, R_n, c_2, \ldots, c_m\}$. The substructure of $\mathfrak{A}$ formed by $\{c_1^{\mathfrak{A}}, \ldots, c_m^{\mathfrak{A}}\}$ is denoted by $c(\mathfrak{A})$. Obviously $c(\mathfrak{A}) \equiv_{L_{\min}} \mathfrak{A}$.

LEMMA 8. *For every projective $L_{\min}$-class P of finite type, we have:*

a) $\mathfrak{A} \in P \to \mathfrak{A} \models \Psi_P$,

b) *If P_1 is $PC_{L_{\min}}$ then, if for some $\mathfrak{A} \in P_1$ $\mathfrak{A} \models \Psi_P$, then $P \cap P_1 \neq \emptyset$.* (*So $\{\mathfrak{A} \mid \mathfrak{A} \models \Psi_P\}$ can be called the $L_{\min}$-elementary closure of P*).

Proof. Let $P = M \restriction \tau$, where M is $EC_{L_{\min}}$ of type $\tau \cup \tau'$, and let $P_1 = N \restriction \tau$, where N is $EC_{L_{\min}}$ of type $\tau \cup \tau''$.

Since all $EC_{L_{\min}}$ of infinite type are free expansions of some $EC_{L_{\min}}$ of finite type, we can assume without loss of generality, that all types are finite.

Suppose that for some $\mathfrak{A} \in P_1$, $\mathfrak{A} \models \Psi_P$. Then for some $\mathfrak{B} \in M$ and $\mathfrak{B}' \in N$ we have

$$\mathfrak{B} \models \Psi_{\mathfrak{A}} \quad \text{and} \quad \mathfrak{B}' \models \Psi_{\mathfrak{A}}.$$

We can assume W.l.o.g. that $|\mathfrak{B}| \cap |\mathfrak{B}'| = |c(\mathfrak{A})|$. We have $c(\mathfrak{B}) \in M$, $c(\mathfrak{B}') \in N$.

Let $c(\mathfrak{B}) \cup c(\mathfrak{B}')$ be the structure of type $\tau \cup \tau' \cup \tau''$ s.t.

$$|c(\mathfrak{B}) \cup c(\mathfrak{B}')| = |c(\mathfrak{B})| \cup |c(\mathfrak{B}')|$$

and the relations in $c(\mathfrak{B}) \cup c(\mathfrak{B}')$ are the unions of the relations in $c(\mathfrak{B})$ and $c(\mathfrak{B}')$. Then $c(\mathfrak{B}) \cup c(\mathfrak{B}') \in M \cap N$; so $P \cap P_1 \neq \emptyset$.

COROLLARY 9. *$L_{\min}$ satisfies Craig.*

Proof. $PC_{L_{\min}}$'s of infinite type are free expansions of $PC_{L_{\min}}$'s of finite type. Any two disjoint $PC_{L_{\min}}$'s of finite type are separated by their $L_{\min}$-elementary closure, and so will their free expansions be too.

EXAMPLE 10. Let $<$ be a binary and R a unary relation symbol. K is the following class of structures of type $\{<, R\}$.

$$K = \{\mathfrak{A} \mid \mathfrak{A} \simeq \langle \omega, <, \mathrm{Odd}\rangle\},$$

where Odd is the set of odd numbers. L is the logic obtained by adding K to $L_{\min}$, and close under boolean operations, free expansions and isomorphism of types.

THEOREM 11. *$\Delta(L)$ does not satisfy the Beth definability property.*

Proof. All structures $\mathfrak{A}$ of type $\{<\}$, have at most one expansion $\mathfrak{A}'$ to the type $\{<, R\}$ s.t. $\mathfrak{A}' \in K$. Put

$$F = \{\mathfrak{A} \mid \mathfrak{A} \text{ of type } \{<, c\} \text{ s.t. } \mathfrak{A} \restriction \{<\} \simeq \langle \omega, < \rangle \text{ and } c^{\mathfrak{A}} \text{ is odd}\},$$

$$F_1 = \{\mathfrak{A} \mid \mathfrak{A} \text{ of type } \{<, c\} \text{ s.t. } \mathfrak{A} \restriction \{<\} \simeq \langle \omega, < \rangle \text{ and } c^{\mathfrak{A}} \text{ is even}\}.$$

We shall prove, that there is no $EC_{\Delta(L)}$, M, s.t. $F \subseteq M$ and $F_1 \subseteq \overline{M}$.

So we shall prove, that there is no PC_L P, s.t. $\overline{P}$ is PC_L, $F \subseteq P$ and $F_1 \subseteq \overline{P}$.

Any EC_L is of the form $\bigcup_{i=1}^{j} (M_i \cap N_i)$, where M_i is $EC_{L_{\min}}$ and N_i is purely non-standard (i.e. a boolean combination of free expansions of K).

The only purely non-standard EC_L's which have nontrivial projections to the type $\{<, c\}$ are the free expansions of K.

Let K_1 be a free expansion of $\overline{K}$, M an $EC_{L_{\min}}$ of the same type. Obviously

$$(M \cap K_1) \restriction \{<, c\} \subseteq M \restriction \{<, c\} \cap K_1 \restriction \{<, c\} = M \restriction \{<, c\}.$$

Now let $\mathfrak{A} \in M$, and suppose $\mathfrak{A} \restriction \{<, R\} \in K$. Let $a \in R^{\mathfrak{A}}$ not be an interpretation of a constant-symbol. Form $\mathfrak{A}'$ from $\mathfrak{A}$ by putting $R^{\mathfrak{A}'} = R^{\mathfrak{A}} \setminus \{a\}$. Then $\mathfrak{A}' \in K_1$ and $\mathfrak{A} \restriction \{<, c\} = \mathfrak{A}' \restriction \{<, c\}$. Hence

$$M \cap K_1 \restriction \{<, c\} = M \restriction \{<, c\}.$$

Since we are concerned with EC_L's which are projected to the type $\{<, c\}$, it is enough to consider EC_L's of the form $M_1 \cup (M_2 \cap K_1)$, where M_1, M_2 are $EC_{L_{\min}}$'s and K_1 is a free expansion of K.

CLAIM. *Let M_1 be $EC_{L_{\min}}$ of type τ s.t. $\{<, c\} \subseteq \tau$. If M_1 contains any structure $\mathfrak{A}$ s.t. $\mathfrak{A} \restriction \{<\} \simeq \langle \omega, < \rangle$ then $M_1 \restriction \{<, c\} \cap F \neq \emptyset$ and $M_1 \restriction \{<, c\} \cap F_1 \neq \emptyset$.*

Proof of the claim. Let $\mathfrak{A} \in M_1$ s.t. $\mathfrak{A} \restriction \{<\} \simeq \langle \omega, < \rangle$ and suppose $c^{\mathfrak{A}}$ is odd.

Form the new structure $\mathfrak{A}'$ by adding a new element x in the beginning of $<_{\mathfrak{A}}$. Then $c^{\mathfrak{A}'}$ is even. Further $\mathfrak{A}' \in M_1$. Hence

$$\mathfrak{A}' \restriction \{<, c\} \in M_1 \restriction \{<, c\} \cap F_1 .$$

Now let P be PC_L s.t. $F \subseteq P$, $\overline{P}$ is PC_L and $F_1 \subseteq \overline{P}$. Let

$$P = M_1 \restriction \{<, c\} \cup (M_2 \cap K_1) \restriction \{<, c\} .$$

Take the structure $\langle \omega_1, <, o \rangle$. Assume $\langle \omega_1, <, o \rangle \in P$. Then

$$\langle \omega_1, <, o \rangle \in M_1 \restriction \{<, c\} .$$

Let $\mathfrak{A}' \in M_1$ s.t.

$$\mathfrak{A}' \restriction \{<, c\} = \langle \omega_1, <, o \rangle .$$

Take $c(\mathfrak{A}')$. Extend $c(\mathfrak{A}')$ by adding an infinite countable set of new elements. Call the result $\mathfrak{A}''$.

Now in $\mathfrak{A}''$ define $<$ on $|\mathfrak{A}''|$ s.t. $<$ is a well-ordering of $|\mathfrak{A}''|$ of type ω. Call this new structure $\mathfrak{A}'''$. Then

$$\Psi_{\mathfrak{A}'} = \Psi_{c(\mathfrak{A}')} = \Psi_{\mathfrak{A}''} = \Psi_{\mathfrak{A}'''}$$

and hence $\mathfrak{A}' \equiv_{L_{\min}} \mathfrak{A}'''$, so $\mathfrak{A}''' \in M_1$. But

$$\mathfrak{A}''' \restriction \{<\} \simeq \langle \omega, < \rangle ;$$

contradicting the claim and the fact that

$$F_1 \cap M_1 \restriction \{<, c\} = \emptyset . \ \blacksquare$$

The results here presented are definitely not the last word concerning interpolation and definability in abstract logics.

The counterexamples presented here and in [3] are very unnatural, and they lack some basic properties which might be thought of as necessary for to accept to call a family of classes of structures a logic.

To prove positive results (as Theorem II.3 in [3] and Theorem 2 in [1]) we should have as few axioms as possible. (Example 10 shows that the condition in Theorem 2 [1], that $L_{\omega\omega} \leqslant L$ is essential, since L and hence $\Delta(L)$ satisfies ω-compactness and downward Skolem–Löwenheim).

For a further analysis, the following distinction could be helpful:

A *model-theoretic axiom* is an axiom considering existence of models, such as compactness, elementary chain condition, omitting types "theorem" etc.

A *structural axiom* is an axiom concerning existence of EC's, such as "closed under the quantifier Q", "extension of $L_{\omega\omega}$", "closed under substitution" etc.

Results concerning logics with model-theoretic axioms are of some interest.

E.g. "compactness → (Robinsons joint consistency theorem → Craig)".

Or: "some ω-compact logic has Souslin–Kleene, but not Beth".

On the other hand, for a deeper philosophical understanding of Craig and Beth (say), structural axioms are of more interest. Since Craig and Beth are both separation properties (where Beth apparently claims separation of fewer PC's) it might be possible that some structural axioms on the logic forces, that if the logic can separate PC's of the "Beth-type", then it can separate all disjoint PC's. Such structural axioms might be found in "having a syntax".

So, a goal for further investigation should be either to find some structural axiom A. s.t. all syntactically defined logics satisfy A, and prove A → (Beth → Craig); or to find a syntactically given logic having Beth, but not Craig.

References

[1] P. Lindström, *On extensions of elementary logic*, Theoria 35 (1969), pp. 1–11

[2] H. Friedman, *On the Δ-operation on semantic systems* (mimeographed).

[3] F. V. Jensen, *Interpolation and definability in abstract logics*, Synthese 27 (1974), pp. 251–257.

Accepté par la Rédation le 9. 4. 1974