Embedding Cantor sets in a manifold
III. Approximating spheres

by

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Abstract. It is shown that Cantor sets can be constructed in n-manifolds that behave with respect to lying in open n-cells exactly as embedded spheres behave. That is we can "approximate" an embedded sphere by a Cantor set. Since every Cantor set in a manifold lies on a simple closed curve we show that n-manifolds that contain nicely embedded spheres that are not homologically trivial contain simple closed curves that lie in no open n-cell.

M^n will denote a topological n-manifold, S^p the p-sphere and B^{n-p} the ball of dimension $n-p \geqslant 3$. A mapping $h\colon X \to M^n$ will be called *homologically trivial* if $h_{\#}\colon H_p(X) \to H_p(M^n)$ is trivial for $p = 1, 2, \ldots, n$. Our principal theorem is the following.

THEOREM 1. *Let $g\colon S^p \times B^{n-p} \to M^n$ be a homologically non-trivial embedding with $n-p \geqslant 3$, then $g(S^p \times B^{n-p}) \subset M^n$ contains a Cantor set that lies in no open n-cell in M^n. If $p = 1$ and $g\colon S^1 \times B^{n-1} \to M^n$ is homotopically non-trivial then $g(S^1 \times B^{n-1}) \subset M^n$ contains a Cantor set that lies in no open n-cell in M^n.*

Combining this result with Theorem 6 of [7] we get the following.

COROLLARY 1. *If $g\colon S^p \times B^{n-p} \to M^n$ is a homologically non-trivial embedding and $0 < k \leqslant n$, then M^n contains a closed k-cell and a $(k-1)$-sphere (with the exception of the 0-sphere) that lie in no open n-cell.*

COROLLARY 2. *If $p+q \geqslant 3$ and p and $q > 0$ then $S^p \times S^q$ contains an arc that lies in no open $p+q$-cell.*

Proof. If $p+q \geqslant 5$ or $p = 1$ and $q = 3$ the result follows from Corollary 1. If $p = q = 2$ the result was given in [8]. Actually, it will be easy to see how to handle this case when the methods of this paper have been studied. If $p+q = 3$ the result follows from [6, Corollary 1].

It should be mentioned that Doyle and Hocking [5, Theorem 7] argue that an n-manifold in which every Cantor set lies in an open n-cell is simply connected. However, in their proof they seem to assume that simple closed curves have trivial tubular neighborhoods. This is not even the case for the projective plane. Their argument is correct for orientable 3-manifolds.

Corollary 1 shows that engulfing without a local flatness condition is not possible even if we are only trying to engulf an arc with a ball in a 100-connected manifold!

The author wishes to thank David G. Wright for pointing out an error in a previous version of Lemma 4.1 and showing how to correct this error. The statement and proof of the present version of this lemma are essentially those suggested by Wright.

2. The construction of the generalized necklaces. In this section we give a construction of a wild Cantor set in E^n. Although the construction given is related to the construction of Blankinship in [3] it is somewhat different and we get a stronger result than Blankinship got (Theorem 2.5). Although many details of the proofs of this section have been omitted they are not difficult to fill in. The most frequently omitted arguments involve the use of the Van Kampen Theorem.

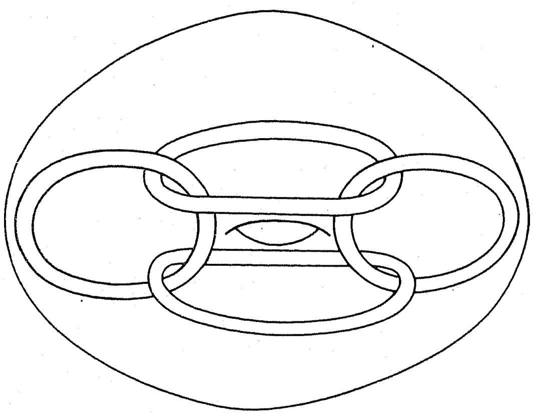

Fig. 1

Let T_0^3 be a solid torus and let $T_1^3, \ldots, T_4^3$ be four cyclically linked solid tori in T_0^3 as shown in Figure 1. If we construct four linked solid tori inside each T_i^3, construct four more linked solid tori inside each of these, etc., as in the construction of Antoine's necklace, we can prove the following theorem. In this and the following sections all unlabeled maps are taken to be those induced by inclusion.

THEOREM 2.1. *Given $\varepsilon > 0$ there exists a set $D_\varepsilon^3 \subset T_0^3$ such that*

1) *D_ε^3 is the union of disjoint solid tori each of diameter less than ε,*

2) *$\pi_1(\partial T_0^3) \to \pi_1(T_0^3 \sim D_\varepsilon^3)$ is a monomorphism,*

3) *$\pi_1(E^3 \sim T_0^3) \to \pi_1(E^3 \sim D_\varepsilon^3)$ is a monomorphism.*

Properties 2) and 3) are derived from Theorem 1 of [4].

Denote by E_+^n the positive half space of E^n given by

$$\{(x_1, x_2, \ldots, x_n) \in E^n |\ x_n > 0\}.$$

Let $X \subset E_+^{n-1}$. Define $\tilde{X} \subset E^n$ by $\tilde{X} = \{(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in E^n |$ there exists $(x_1, x_2, \ldots, x_{n+1}) \in X$ and there exists θ such that $x_i = \tilde{x}_i$ for $i = 1, 2, \ldots$ $\ldots, n-2$, $\tilde{x}_{n-1} = x_{n-1}\cos\theta$ and $\tilde{x}_n = x_{n-1}\sin\theta\}$. $\tilde{X}$ is called the rotation of X about the first $(n-2)$-coordinate axes. It is easy to show that $\tilde{X} = S^1 \times X$. (Throughout this paper we use $\cong$ to denote homeomorphism.) It is not necessary to distinguish between two homeomorphic spaces unless we are concerned with embedding properties.

The following lemma is a trivial generalization of a theorem of Artin [1].

LEMMA 2.2. *Let $X \subset E_+^{n-1}$ and assume $E_+^{n-1} \sim X$ is path connected. Let $\tilde{X} \subset E^n$ be the rotation of X about the first $(n-2)$-coordinate axes. Then $\pi_1(E_+^{n-1} \sim X) \to \pi_1(E^n \sim \tilde{X})$ is an isomorphism.*

LEMMA 2.3. *Let X_1, X_2 and X_3 be compact metric spaces and let $g\colon X_1 \to X_2$ be a homeomorphism. Define $\sigma\colon X_1 \times X_2 \times X_3 \to X_2 \times X_1 \times X_3$ by $\sigma(x_1, x_2, x_3) = (g(x_1), g^{-1}(x_2), x_3)$. Let $\varepsilon > 0$ be given. Then there exists $\delta(\varepsilon) > 0$ such that if the diameter of subsets A and B of $X_2 \times X_3$ is less than $\delta(\varepsilon)$ and $f\colon X_1 \times X_3 \to A$ then the diameter of $(1 \times f)\sigma(X_1 \times B)$ is less than ε.*

The proof of the above lemma is a routine exercise in point set topology. In fact the hypotheses could be substantially weakened without changing the conclusion.

Notation. $(S^1)^k$ denotes the k-fold product of S^1.

DEFINITION. An n-tube is a *space homeomorphic* with $(S^1)^{n-3} \times T^3$, where T^3 denotes a solid torus. T^n will henceforth denote an n-tube.

We can now proceed inductively to construct the pair (T^n, D_ε^n) in E^n.

THEOREM 2.4. *Let $\varepsilon > 0$ be given. There exists a pair (T^n, D_ε^n) in E^n, $n \geq 3$, such that*

1) *$D_\varepsilon^3 \subset T^n$ is the union of disjoint n-tubes each of diameter less than ε,*

2) *$\pi_1(\partial T^n) \to \pi_1(T^n \sim D^n)$ is a monomorphism,*

3) *$\pi_1(E^n \sim T^n) \to \pi_1(E^n \sim D^n)$ is a monomorphism.*

Proof. Suppose we have proved the theorem for dimensions less than n, that is for any choice of $\delta > 0$ we have a pair $(T^{n-1}, D_\delta^{n-1})$ satisfying 1), 2), and 3). We may assume that T^{n-1} is in E_+^{n-1}. We rotate E^n about the hyperplane determined by the first $(n-2)$-coordinate axes. This generates a pair homeomorphic with $(S_1^1 \times T^{n-1}, S_1^1 \times D_\delta^n)$. By Lemma 2.2 this pair satisfies condition 3). Clearly condition 2) is satisfied. However condition 1) is not satisfied. In order to insure that the diameters of each component are small we view $(S_1^1\times T^{n-1}, S_1^1\times D_\delta^n)$ as a product space with the usual product metric and switch coordinates as in Lemma 2.3. Note that $S_1^1\times T^{n-1} = S_1^1\times S_1^1\times T^{n-2}$ and let $g\colon S_1^1\to S_2^1$ be a homeomorphism. Let T_i^{n-1}, $i = 1, 2, \ldots, k_\delta$, be the components of D_δ^{n+1}. For each i, choose a homeomorphism $f_i\colon T^{n-1}\to T_i^{n-1}$. Now we apply Lemma 2.3 to show that each component of $(1\times f_i)\sigma(S_1^1\times D_\delta^{k-1})$ has diameter less than ε. A simple application of the Van Kampen Theorem shows that 2) and 3) are still satisfied by the set $\bigcup_{i=1}^{k_\delta}(1\times f_i)\sigma(S^1\times D_\delta^{k-1})$.

Using Theorem 2.4 it is now a routine matter to construct the Cantor set $A\subset T^n\subset E^n$. One need only map the model (T^n, D_ε^n) homeomorphically onto each of the components of D_ε^n successively, choosing ε at each step so that the diameters of the components at successive stages approaches zero.

From Theorem 2.4, 2) and 3) we use direct limits to get:

THEOREM 2.5. *$A\subset T^n\subset E^n$ is a wild Cantor set. Furthermore,*

$$\pi_1(E^n\sim T^n)\to\pi_1(E^n\sim A)$$

is a monomorphism.

COROLLARY 2.6. *$\pi_1(E^n\sim A)$ contains no non-trivial elements of finite order.*

3. The embedding of T^m in $S^p\times B^{n-p}$. Let $h'\colon (S^1)^p\times B^1\to S^p\times B^1$ be an embedding such that $S^p\times B^1\sim h'((S^1)^p\times B^1)$ has exactly two components, one containing $S^p\times\{-1\}$, the other containing $S^p\times\{1\}$. Let $h''\colon (S^1)^{n-p-2}\times B^1\to \mathrm{Int}(B^{n-p-1})$ be any embedding. Define $h = h'\times h''\colon ((S^1)^p\times B^1)\times((S^1)^{n-p-2}\times B^1)\to(S^p\times B^1)\times B^{n-p-1}$. Since $((S^1)^p\times B^1)\times((S^1)^{n-p-2}\times B^1)$ is homeomorphic with $(S^1)^{n-2}\times B^2$, we shall henceforth assume that $h\colon (S^1)^{n-2}\times B^2\to S^p\times B^{n-p}$.

LEMMA 3.1. *$h_\#\colon H_p((S^1)^p\times B^{n-p})\to H_p(S^p\times B^{n-p})$ is an epimorphism, and if $p = 1$, $h_*\colon \pi_1(S^1\times B^{n-1})\to\pi_1(S^1\times B^{n-1})$ is an isomorphism.*

Proof. Since $h'((S^1)^p\times\{0\})$ separates $S^p\times B^1$, it follows that $h'((S^1)^p\times\{0\})$ carries a cycle homologous with a generator of $H_p(S^p\times B^1)$.

4. Proof of Theorem 1. Throughout this section all unnamed maps will be assumed to be induced by inclusion. $(S^1)^m$ will be thought of as a subspace of $(S^1)^k$ obtained by the injection onto the first m factors of $(S^1)^k$. $(\tilde S^1)^m$ denotes the subspace of $(S^1)^k$ obtained by injection of $(S^1)^m$ onto the last m coordinates of $(S^1)^k$.

LEMMA 4.1. *Let $P\subset(S^1)^k$ be a polyhedron. If $\mathrm{Im}(\pi_1(P)\to\pi_1((S^1)^k))$ is a subset of $\mathrm{Im}(\pi_1((S^1)^{k-1})\to\pi_1((S^1)^k))$ then $\mathrm{Im}(H_{k-1}((S^1)^{k-1})\to H_{k-1}((S^1)^k))$ is a subset of $\mathrm{Im}(H_{k-1}((S^1)^k\sim P)\to H_{k-1}((S^1)^k))$.*

Proof. Let $\varrho\colon (S^1)^{k-1}\times E^1\to(S^1)^k$ be the covering projection that is the identity on the first $k-1$ coordinates. We may assume that P is connected (since if not we can enlarge it without changing the assumed property of its fundamental group so that it is). Our hypotheses guarantee that each component of $\varrho^{-1}(P)$ is a compact polyhedron. Let N be a regular neighborhood of $\varrho^{-1}(P)$ in $(S^1)^{k-1}\times E^1$. Let $\hat N$ be the union of all components of N which intersect $(S^1)^{k-1}\times\{0\}$ and let $N^\# = \hat N\cap((S^1)^{k-1}\times[0,\infty))$. Clearly $\partial N^\#$ determines a bounding cycle in the $(k-1)$-dimensional chain group of $(S^1)^{k-1}\times E^1$. Call this cycle b_{k-1}. Denote by C_{k-1} the generator of the $(k-1)$-dimensional homology of $(S^1)^{k-1}\times E^1$ carried by $(S^1)^{k-1}\times\{0\}$. Now with the proper orientations $C_{k-1}-b_{k-1}$ is a generating cycle for $H_{k-1}((S^1)^{k-1}\times E^1)$ and lies in $(S^1)^{k-1}\times E^1\sim\varrho^{-1}(P)$. The lemma follows.

LEMMA 4.2. *Let P be a finite polyhedron in $(S^1)^k$ and suppose $\pi_1(P)\to\pi_1((S^1)^k)$ is trivial on the last two factors, i.e., $\mathrm{Im}(\pi_1(P)\to\pi_1((S^1)^k))$ is in $\mathrm{Ker}(\pi_1((S^1)^k)\xrightarrow{\varphi_*}\pi_1((\tilde S^1)^2))$ where φ is the projection of $(S^1)^k$ onto the last two coordinates of $(S^1)^k$. Then $\pi_1((S^1)^k\sim P)\to\pi_1((S^1)^k)$ is an epimorphism on the last two factors, i.e., the injection $(S^1)^k\sim P\to(S^1)^k$ followed by φ induces an epimorphism on the fundamental groups.*

Proof. Without loss of generality we may assume that P is connected. Let $\varrho\colon (S^1)^{k-2}\times E^2\to(S^1)^k$ be the covering projection for $(S^1)^k$ that is the identity on the first $k-2$ factors. The hypotheses guarantee that each component of $\varrho^{-1}(P)$ is a compact polyhedron. Let $\hat P$ be union of all bounded components of $(S^1)^{k-2}\times E^2\sim\varrho^{-1}(P)$. Then because $H_{k-1}((S^1)^{k-2}\times E^2)$ is trivial, each component of a regular neighborhood N of $\hat P$ has a connected boundary. From this we see that $(S^1)^{k-2}\times E^2\sim N$ is path connected. Let $x_0\in(S^1)^k\sim\varrho(\hat P)$ be a base point for $\pi_1(S^1)^k$. The points in $\varrho^{-1}(x_0)$ can all be joined by paths to a base point $\tilde x_0$ in $\varrho^{-1}(x_0)$. ϱ of these paths followed by the projection onto the last two coordinates provides the desired epimorphism.

Proof of Theorem 1. Let h be as defined in Section 3. If $gh(A)$ lies in an open n-cell in M^n then it lies in a nested pair of closed cells $C_1\subset\mathrm{Int}\,C_2$. Triangulate $\partial(gh(T^m))$ so that no simplex intersects both C_1 and ∂C_2 and let K be the union of all simplexes of $\partial(gh(T^m))$ that do not intersect C_1. Suppose K contains a loop β in $\partial(gh(T^m))$ that projects non-trivially into the $gh(\partial B^2)$ factor of $gh(\partial T^m)$. This loop lies in $M^n\sim C_1$, is homotopically trivial in M^n and is linked with $gh(A)$ in $g(S^p\times B^{n-p})$. This last statement follows from Theorem 2.5 and an application of the VanKampen Theorem to $S^p\times B^{n-p}\subset E^n$. Another application of the VanKampen Theorem shows that β is homotopically linked with $gh(A)$ in M^n. (It is at this point that we use $n-p \geqslant 3$ so that $\pi_1(\partial B^{n-p}) = 1$). This contradiction leads us to conclude that K contains no loop that projects non-trivially into the $gh(\partial B^2)$ factor of $gh(\partial T^n)$. By Lemma 4.1,

$$\operatorname{Im} H_{n-2}\big(\partial\big(gh(T^{n-1})\big)\big) \to H_{n-2}\big(gh(\partial T^n)\big)$$

is contained in

$$\operatorname{Im}\Big(H_{n-2}\big(\partial\big(gh(T^n)\big) \sim K\big) \to H_{n-2}\big(\partial\big(gh(T^n)\big)\big)\Big).$$

This implies that $gh(\partial T^n)$ carries only homologically trivial cycles in M^n. By Lemma 3.1 $g(S^p)$ is homologically trivial in M^n.

If $p = 1$ we need to modify the Cantor set $h(A) \subset S^1 \times B^{n-1}$. We do this by constructing another Cantor set $\hat{A}$ in $S^1 \times B^{n-1}$ so that if γ is a loop in ∂T^n that projects non-trivially by the projection of ∂T^n onto the next-to-the-last factor and $h(\gamma)$ is homotopically trivial in $S^1 \times B^{n-1}$ then $h(\gamma)$ is linked with $\hat{A}$. Let $h^*\colon T^n \to S^1 \times B^{n-1}$ be an embedding with the property that $h(\gamma)$ is linked with $h^*(T^n)$ in $S^1 \times B^{n-1}$. Then by Theorem 2.5 and an application of the VanKampen Theorem we see that $\hat{A} = h^*(A)$ has the desired properties.

We are now ready to complete the proof of Theorem 1 in the case $p = 1$. Denote by A^* the Cantor set $g\big(h(A) \cup \hat{A}\big)$ in $g(S^1 \times B^{n-1})$. Let C_1, C_2 and K be defined as above. Suppose K contains a non-trivial loop β that is homotopic with a non-trivial loop in $\operatorname{Im}\Big(\pi_1\big(gh(\widetilde{S}^1 \times \partial \widetilde{B}^2)\big) \to \pi_1\big(gh(\partial T^n)\big)\Big)$. Then β lies in $M^n \sim C_1$ and is linked with A^*. This contradiction leads us to conclude that K satisfies the hypotheses of Lemma 4.2. It follows from this lemma that $\pi_1\big(\partial T^n \sim h^{-1}g^{-1}(K)\big)$ contains a loop δ homotopic with the generator a of $\operatorname{Im}\big(\pi_1(\widetilde{S}^1) \to \pi_1(\partial T^n)\big)$. But then $gh(\delta)$ lies in C_2. Theorem 1 follows.

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