Multi-valued maps of subsets of Euclidean spaces

by

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Abstract. Let X and Y be two metric spaces and let $p\colon Y\to X$ be a proper, surjective map. Define, for each $i\geqslant 0$, the set

$$M_p^i=\{y\in Y;\ H^i(p^{-1}(y))\neq 0\},$$

where H^i denotes the Čech cohomology functor with integer coefficients. Call p an n-*Vietoris map* if $\mathrm{rd}_X M_p^i\leqslant n-2-i$, for each $i\geqslant 0$, where $\mathrm{rd}_X M_p^i$ is the maximum covering dimension for finite covers of subsets of M_p^i which are closed in Y. In what follows a 1-Vietoris map is called simply a *Vietoris map*.

A multi-valued map $\varphi\colon X\to Z$ is called an n-*admissible map* if there exist a space Y and a pair of single-valued (continuous) maps of the form $X\overset{p}{\leftarrow}Y\overset{q}{\rightarrow}Z$ such that the following two conditions are satisfied:

(i) p is an n-Vietoris map,

(ii) $q(p^{-1}(x))\subset\varphi(x)$ for each $x\in X$.

In this case the pair of maps (p,q) is called a *selected pair of* φ (written $(p,q)\subset\varphi$). A 1-admissible map is called simple an *admissible map*. An admissible map $\varphi\colon X\to Z$ is called *strongly admissible* (s-*admissible*) if there exists a selected pair $(p,q)\subset\varphi$ such that $q(p^{-1}(x))=\varphi(x)$ for each $x\in X$.

In the present paper we prove some fixed point theorems and a theorem on the antipodes for n-admissible maps of subsets of Euclidean spaces. The proofs of these results depend on the concept of degree of n-admissible maps. Moreover, we prove a generalized version of the theorem on the antipodes for admissible maps and a theorem on the invariance of domain for s-admissible maps of subsets of Euclidean spaces.

The class of admissible multi-valued maps was first studied in [4]. A multi-valued map $\varphi\colon X\to Z$ is called *admissible* provided there exist a space Y and a pair of single-valued (continuous) maps of the form $X\overset{p}{\leftarrow}Y\overset{q}{\rightarrow}Z$ such that the following two conditions are satisfied:

(i) p is a Vietoris map,

(ii) $q\big(p^{-1}(x)\big)\subset\varphi(x)$ for each $x\in X$.

We note that every acyclic map is admissible and that the composition of admissible maps is also admissible map.

Using the generalized version of the Vietoris Mapping Theorem [8], we generalize the notion of admissible maps to n-admissible maps (comp. [3]).

In the present paper we prove some fixed point theorems and a theorem on the antipodes for n-admissible maps of subsets of Euclidean spaces. The proofs of these results depend on the concept of degree of n-admissible maps.

Moreover, we prove a generalized version of the theorem on the antipodes for admissible maps and a theorem on the invariance of domain for some admissible maps of subsets of Euclidean spaces.

We note that for acyclic maps and n-acyclic maps these facts were studied in [1, 3, 5, 6, 7].

All spaces are assumed to be metric.

1. Preliminaries. Let H denote the Čech cohomology functor with integer coefficients Z from the category of metric spaces and continuous maps to the category of graded abelian groups and homomorphisms of degree zero.

Thus, for a space X

$$H(X) = \{H^k(X)\}_{k\geqslant 0}$$

is a graded abelian group and, for a continuous map $f\colon X\to Y$, $H(f)$ is the induced homomorphism

$$H(f) = f^* = \{f^{*k}\}\colon H(Y)\to H(X),$$

where $f^{*k}\colon H^k(Y)\to H^k(X)$.

A non-empty space is called *0-acyclic* if $H^0(X) = Z$; let us call X *k-acyclic*, $k\geqslant 1$, if $H^k(X) = 0$; let us call X *acyclic* if X is k-acyclic for each $k\geqslant 0$.

Let A be a subset of a space X. Denote by $\mathrm{rd}_X A$ the relative dimension of A in X. From the definition, given in [8], we have

$$\mathrm{rd}_X A = \sup_{C\subset A} \dim C,$$

where C is a closed subset of X and by $\dim C$ we denote the topological dimension of C. We assume that $\mathrm{rd}_X A < 0$ if and only if the set A is empty and in this case we put $\mathrm{rd}_X A = -\infty$. We observe that:

(1.1) *If $A\subset X$ and $B\subset Y$, where X, Y are two compact spaces, then*

$$\mathrm{rd}_{X\times Y} A\times B \leqslant \mathrm{rd}_X A + \mathrm{rd}_Y B.$$

(1.2) *Let X_0 be a closed subset of X. Assume further that A is a subset of X_0, B is subset of X and $A\subset B$. Then we have*

$$\mathrm{rd}_{X_0} A \leqslant \mathrm{rd}_X B.$$

A continuous map $f\colon Y\to X$ is *proper* if for each compact subset $A\subset X$ the counter image $f^{-1}(A)$ of A under f is compact; f is closed if for each closed subset $B\subset Y$ the image of B under f is closed in X.

The following fact is evident:

(1.3) *If $f\colon Y\to X$ is a proper map, then f is closed.*

The Vietoris–Begle Theorem (see [8] and (1.3)) gives

(1.4) Theorem. *Let $f\colon Y\to X$ be a proper and surjective map, and let M_f^i be the set of all $x\in X$ such that $f^{-1}(x)$ fails to be i-acyclic. Let $n = 1 + \max_{i\geqslant 0}(\mathrm{rd}_X M_f^i + i)$. Then for each $k > n$ the induced homomorphism $f^{*k}\colon H^k(X)\to H^k(Y)$ is an isomorphism.*

Remark. We observe that, if M_f^i is the empty set for each $i\geqslant 0$, then $\mathrm{rd}_X M_f^i = -\infty$ and hence Theorem (1.4) implies that

$f^*\colon H(X)\to H(Y)$ is an isomorphism.

(1.5) Definition. A map $p\colon Y\to X$ is called an *n-Vietoris map* if the following two conditions are satisfied:

(i) p is a proper and surjective map,

(ii) $\mathrm{rd}_X M_p^i \leqslant n-2-i$, for each $i\geqslant 0$.

Definition (1.5) implies that if p is a 1-Vietoris map, then $\mathrm{rd}_X M_p^i < 0$, and from the above remark we deduce that $p^*\colon H(X)\to H(Y)$ is an isomorphism. In what follows a 1-Vietoris map is called simply a *Vietoris map*.

Finally, we note that if p is an n-Vietoris map, then we have:

$$1+\max_{i\geqslant 0}(\mathrm{rd}_X M_p^i + i) \leqslant 1+\max_{i\geqslant 0}[(n-2-i)+i] = n-1$$

and (1.4) implies that $p^{*k}\colon H^k(X)\to H^k(Y)$ is an isomorphism for each $k\geqslant n$.

(1.6) *If $p\colon Y\to X$ is a Vietoris map, then for each $A\subset X$, the map $\tilde{p}\colon p^{-1}(A)\to A$ is a Vietoris map, where $\tilde{p}$ is given by $\tilde{p}(y) = p(y)$ for all $y\in p^{-1}(A)$.*

From (1.6) we deduce

(1.7) *If $p_1\colon Y\to X$ and $p_2\colon X\to Z$ are Vietoris maps, then the composition $p_2\circ p_1\colon Y\to Z$ of p_1 and p_2 is also a Vietoris map.*

2. Multi-valued maps. Let X and Y be two spaces and assume that for every point $x\in X$ a non-empty subset $\varphi(x)$ of Y is given; in this case we say that φ is a *multi-valued map* from X to Y and we write $\varphi\colon X\to Y$. In what follows the symbols φ, ψ will be reserved for multi-valued maps; single-valued maps will be denoted by f, g, h, etc.

Let $\varphi\colon X\to Y$ be a multi-valued map. We associate with φ the following diagram of continuous maps:

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y$$

in which $\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$ is the *graph of* φ and *natural projections* p_φ and q_φ are given by $p_\varphi(x, y) = x$ and $q_\varphi(x, y) = y$.

The point-to-set map φ is extended to a set-to-set map by putting $\varphi(A) = \bigcup_{a \in A} \varphi(a) \subset Y$ for $A \subset X$; $\varphi(A)$ is said to be the *image* of A under φ. If $\varphi(A) \subset B \subset Y$, then the *contraction* of φ to the pair (A, B) is the multi-valued map $\varphi'\colon A \to B$ defined by $\varphi'(a) = \varphi(a)$ for each $a \in A$. A contraction of φ to the pair (A, Y) is simply the *restriction* $\varphi|_A$ of φ to A.

Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be two maps; then the composition of φ and ψ is a map $\psi \circ \varphi\colon X \to Z$ given by $(\psi \circ \varphi)(x) = \psi(\varphi(x))$.

(2.1) Definition. A multi-valued map $\varphi\colon X \to Y$ is said to be *continuous* if the grap Γ_φ of φ is closed in the product $X \times Y$; in other words, the conditions $x_n \to x$, $y_n \to y$, $y_n \in \varphi(x_n)$ imply $y \in \varphi(x)$.

(2.2) Definition. A continuous multi-valued map $\varphi\colon X \to Y$ is called *compact* if the image $\varphi(X)$ of X under φ is contained in a compact subset of Y.

(2.3) Definition. A continuous multi-valued map $\varphi\colon X \to Y$ is said to be *acyclic* if the set $\varphi(x)$ is acyclic for every point $x \in X$.

(2.4) Definition. Let $\varphi\colon X \to Y$ be a multi-valued map. A point x is called a *fixed point* for φ if $x \in \varphi(x)$.

We observe that, if $\varphi\colon X \to Y$ is a compact acyclic map, then, for every point $x \in X$, $p_\varphi^{-1}(x)$ is homeomorphic to $\varphi(x)$ and hence $p_\varphi\colon \Gamma_\varphi \to X$ is a Vietoris map.

(2.5) Definition. Let $\varphi, \psi\colon X \to Y$ be two multi-valued maps such that $\varphi(x) \subset \psi(x)$ for each $x \in X$; in this case we say that φ is a *selector* of ψ and indicate this by writing $\varphi \subset \psi$.

3. Admissible, s-admissible and n-admissible maps. In this section we introduce the classes of multi-valued maps which are of importance in our considerations.

(3.1) Definition. A map $\varphi\colon X \to Z$ is called *n-admissible*, $n \geqslant 1$, if there exist a space Y and a pair of single-valued (continuous) maps of the form

$$X \xleftarrow{p} Y \xrightarrow{q} Z$$

such that the following two conditions are satisfied:

(i) p is an n-Vietoris map,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for each $x \in X$.

In this case the pair of maps (p, q) is called a *selected pair* of φ (written $(p, q) \subset \varphi$).

The class of n-admissible maps from X to Z will be denoted by $C^n(X, Z)$.

A map $\varphi \in C^1(X, Z)$ is called simply an *admissible map*.

(3.2) Definition. An admissible map $\varphi\colon X \to Z$ is called *strongly admissible* (*s-admissible*) if there exists a selected pair $(p, q) \subset \varphi$ such that

$$q(p^{-1}(x)) = \varphi(x) \quad \text{for each } x \in X.$$

If a selected pair (p, q) of φ satisfies the above condition, then we write $(p, q) = \varphi$.

The class of s-admissible maps from X to Z we denote by $C_s^1(X, Z)$.

Remarks. 1. We observe that, if $\varphi\colon X \to Z$ is a compact acyclic map, then for example the pair (p_φ, q_φ) is a selected pair of φ and hence $(p_\varphi, q_\varphi) = \varphi$.

2. Denote by $C(X, Z)$ the class of compact acyclic maps from X to Z; then we have the following inclusions:

$$C(X, Z) \subset C_s^1(X, Z) \subset C^1(X, Z) \subset C^2(X, Z) \ldots$$

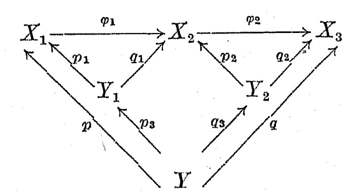
(3.3) Proposition. *If* $\varphi_1 \in C^1(X_1, X_2)$, $\varphi_2 \in C^1(X_2, X_3)$, *then* $\varphi_2 \circ \varphi_1 \in C^1(X_1, X_3)$ *and for every selected pair* $(p_1, q_1) \subset \varphi_1$, $(p_2, q_2) \subset \varphi_2$ *there exists a selected pair* $(p, q) \subset \varphi_2 \circ \varphi_1$ *such that*

$$(p^*)^{-1} q^* = (p_1^*)^{-1} q_1^* (p_2^*)^{-1} q_2^* .$$

Proof. Let $(p_1, q_1) \subset \varphi_1$ and $(p_2, q_2) \subset \varphi_2$ be selected pairs of the form

$$X_1 \xleftarrow{p_1} Y_1 \xrightarrow{q_1} Y_2, \qquad X_2 \xleftarrow{p_2} Y_2 \xrightarrow{q_2} X_3 .$$

Consider the diagram



in which $Y = \{(y_1, y_2) \in Y_1 \times Y_2;\ q_1(y_1) = p_2(y_2)\}$, $p_3(y_1, y_2) = y_1$, $q_3(y_1, y_2) = y_2$, $p(y_1, y_2) = p_1 \circ p_3(y_1, y_2)$, $q(y_1, y_2) = q_2 \circ q_3(y_1, y_2)$.

The map p as the composition of the Vietoris maps p_1 and p_2 is also a Vietoris map (see (1.7)).

Let A be a subset of Y_1. It is easy to see that $p_2^{-1} q_1(A) = q_3 p_3^{-1}(A)$. This implies that for every point $x \in X_1$ we have

$$q(p^{-1}(x)) = q_2 q_3 p_3^{-1}(p_1^{-1}(x)) = q_2 p_2^{-1}\big(q_1(p_1^{-1}(x))\big) \subset \varphi_2 \circ \varphi_1(x) ,$$

and hence we obtain $(p, q) \subset \varphi_2 \circ \varphi_1$.

Applying to the above diagram the functor H, we simply deduce

$$(p^*)^{-1}q^* = (p_1^*)^{-1}q_1^*(p_2^*)^{-1}q_2^*$$

and the proof of (3.3) is completed.

(3.4) Proposition. *If $\varphi_1 \in C_s^1(X_1, X_2)$ and $\varphi_2 \in C_s^1(X_2, X_3)$, then $\varphi_2 \circ \varphi_1 \in C_s^1(X_1, X_3)$ and for every $(p_1, q_1) = \varphi_1$, $(p_2, q_2) = \varphi_2$ there exists a pair $(p, q) = \varphi_2 \circ \varphi_1$, such that*

$$(p^*)^{-1}q^* = (p_1^*)^{-1}q_1^*(p_2^*)^{-1}q_2^* .$$

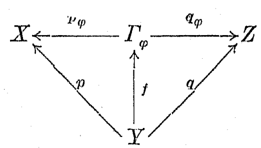
The proof of (3.4) is analogous to the proof of (3.3).

(3.5) Proposition. *If $\varphi \in C(X, Z)$, then for every selected pair $(p, q) \subset \varphi$ we have $(p^*)^{-1}q^* = (p_\varphi^*)^{-1}q_\varphi^*$.*

Proof. Let (p, q) be a selected pair of φ of the form

$$X \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} Z .$$

Consider the commutative diagram



in which $f\colon Y \to \Gamma_\varphi$ is given by $f(y) = (p(y), q(y))$. Applying to the above diagram the functor H, we obtain $(p^*)^{-1}q^* = (p_\varphi^*)^{-1}q_\varphi^*$, and the proof is completed.

(3.6) Examples. 1. Observe that if $\varphi \in C_s^1(X, Z)$ then for every point $x \in X$ the image $\varphi(x)$ of x under φ must be a connected set. Hence we simply deduce that the class of admissible maps is larger than the class of s-admissible maps.

2. Now, we give an example of an s-admissible map φ such that there are selected pairs $(p, q) = \varphi$ and $(\bar{p}, \bar{q}) = \varphi$ and $(p^*)^{-1}q^* \neq (\bar{p}^*)^{-1}\bar{q}^*$.

Let $\varphi_1\colon S^2 \to S^2$, where S^2 denotes the unit sphere in the Euclidean space R^3, be the map given by $\varphi_1(x) = \{y \in S^2, \|y - x\| \leqslant \frac{3}{2}\}$ and let $\varphi_2\colon S^2 \to S^2$ be the map given by $\varphi_2(x) = \varphi_1(-x)$ for each $x \in S^2$. Define the map $\varphi\colon S^2 \to S^2$ by putting $\varphi = \varphi_1 \circ \varphi_2$. Observe that φ_1 and φ_2 are acyclic maps. From (3.4) we deduce that φ is an s-admissible map. Using (3.5) and (3.4), we infer that there exists a selected pair $(p, q) = \varphi$ such that $(p^*)^{-1}q^* = (-\mathrm{id}_{S^2})^*$. Moreover, we conclude that $\varphi = \varphi_1 \circ \varphi_1$ and hence, in view (3.5) and (3.4), we deduce that there is a selected pair $(\bar{p}, \bar{q}) = \varphi$ such that $(\bar{p}^*)^{-1}\bar{q}^* = (\mathrm{id}_{S^2})^*$.

(3.7) Definition. Two maps $\varphi, \psi \in C^n(X, Z)$, $n \geqslant 1$, $(\varphi, \psi \in C_s^1(X, Z))$ are called *homotopic* (written $\varphi \sim \psi$) if there exists a map $\varkappa \in C^n(X \times I, Z)$, $(\varkappa \in C_s^1(X \times I, Z))$, where $I = [0, 1]$ is the unit interval, such that $\varkappa(x, 0) \subset \varphi(x)$ and $\varkappa(x, 1) \subset \psi(x)$ for each $x \in X$.

(3.8) Proposition. *If two maps $\varphi, \psi \in C^n(X, Z)$ $(\varphi, \psi \in C_s^1(X, Z))$, are homotopic, then there exist selected pairs $(p_0, q_0) \subset \varphi$ and $(p_1, q_1) \subset \psi$ such that for each $k \geqslant n$*

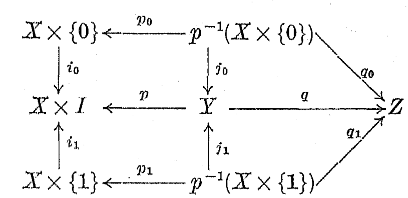
$$(p_0^{*k})^{-1}q_0^{*k} = (p_1^{*k})^{-1}q_1^{*k} ,$$

(for each $k \geqslant 0$, $(p_0^{*k})^{-1}q_0^{*k} = (p_1^{*k})^{-1}q_1^{*k}$).

Proof. Let $\varkappa \in C^n(X \times I, Z)$ $(\varkappa \in C_s^1(X \times I, Z))$ be a homotopy joining φ and ψ, and $(p, q) \subset \varkappa$ a selected pair of the form

$$X \times I \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} Z .$$

For the proof we consider the commutative diagram



in which p_0, p_1 are contractions of p, q_0, q_1 are restrictions of q and i_0, i_1, j_0, j_1 are inclusions, respectively.

Since p is an n-Vietoris map, $n \geqslant 1$, and $X \times \{0\}$, $X \times \{1\}$ are closed subsets of $X \times I$, from (1.2) we deduce that p_0, p_1 are n-Vietoris maps. From $\varkappa(x, 0) \subset \varphi(x)$ and $\varkappa(x, 1) \subset \psi(x)$ for each $x \in X$, we infer that (p_0, q_0), (p_1, q_1) are selected pairs of φ and ψ respectively. It is well known that $i_0^* = i_1^*$ and hence applying to the above diagram the functor H, in view (1.4), we obtain:

(i) $$(p_0^{*k})^{-1}q_0^{*k} = (p_1^{*k})^{-1}q_1^{*k} \quad \text{for each } k \geqslant n ,$$

(for each $k \geqslant 0$). Finally, since we may identify $X \times \{0\}$ with X and $X \times \{1\}$ with X, from (i) we deduce (3.8) and the proof is completed.

4. Degree of n-admissible maps. Let S^n be the unit sphere in the Euclidean $(n+1)$-space R^{n+1}, K^{n+1} the unit closed ball in R^{n+1} and P^{n+1} the space R^{n+1} without the point 0.

In this section we define the degree of an n-admissible map $\varphi\colon S_1^n \to S_2^n$ where S_i^n $(i = 1, 2)$ are two spaces which have the cohomology of an n-sphere S^n. We orient S_i^n by choosing the generators $\beta_i \in H^n(S_i^n)$ $(i = 1, 2)$.

Consider the diagram

$$S_1^n \xleftarrow{p} Y \xrightarrow{q} S_2^n ,$$

in which p is an n-Vietoris map and q is a continuous map. In this case we define the degree $\deg(p, q)$ of the pair (p, q) as follows: $\deg(p, q)$ is the unique integer which satisfies

$$(p^{*n})^{-1}q^{*n}(\beta_2) = \deg(p, q)\beta_1 .$$

(4.1) Definition. Let $\varphi\colon S_1^n \to S_2^n$ be an n-admissible map. We define $\operatorname{Deg}(\varphi)$ of φ as follows:

$$\operatorname{Deg}(\varphi) = \{\deg(p, q);\ (p, q) \subset \varphi\} .$$

(4.2) *Let $\varphi, \psi \in C^n(S_1^n, S_2^n)$. Then*

(i) *$\varphi \sim \psi$ implies that $\operatorname{Deg}(\varphi) \cap \operatorname{Deg}(\psi) \neq \emptyset$,*

(ii) *$\varphi \subset \psi$ implies that $\operatorname{Deg}(\varphi) \subset \operatorname{Deg}(\psi)$.*

Assertion (i) simply follows from (3.8). For the proof of (ii) we observe that if $(p, q) \subset \varphi$, then $(p, q) \subset \psi$.

If $\varphi\colon S_1^n \to S_2^n$ is an acyclic compact map, then from (3.5) we deduce that $\operatorname{Deg}(\varphi)$ is a set consisting of exactly one element, and in this case we have:

(4.3) *Let $\varphi, \psi \in C(S_1^n, S_2^n)$. Then*

(i) *$\varphi \sim \psi$ implies that $\operatorname{Deg}(\varphi) = \operatorname{Deg}(\psi)$,*

(ii) *$\varphi \subset \psi$ implies that $\operatorname{Deg}(\varphi) = \operatorname{Deg}(\psi)$.*

Let $\varphi\colon K^{n+1} \to R^{n+1}$ be an n-admissible map and assume that $\varphi(S^n) \subset R^{n+1}\setminus\{x_0\}$. By $\varphi|_{S^n}\colon S^n \to R^{n+1}\setminus\{x_0\}$ we denote the contraction of φ to the pair $(S^n, R^{n+1}\setminus\{x_0\})$.

From (1.2) we infer that $\varphi|_{S^n} \in C^n(S_1^n, R^{n+1}\setminus\{x_0\})$. In this case with every selected pair $(p, q) \subset \varphi$ we associate the pair $(p_1, q_1) \subset \varphi|_{S^n}$ as follows: let $p\colon Y \to K^{n+1}$, $q\colon Y \to R^{n+1}$ be two maps such that $(p, q) \subset \varphi$; then $p_1\colon p^{-1}(S^n) \to S^n$, $q_1\colon p^{-1}(S^n) \to R^{n+1}\setminus\{x_0\}$ are given as contractions of p and q, respectively (evidently $(p_1, q_1) \subset \varphi|_{S^n}$).

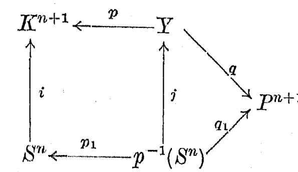
We define the degree $\operatorname{Deg}(\varphi, x_0)$ of φ by putting

$$\operatorname{Deg}(\varphi, x_0) = \{\deg(p_1, q_1);\ (p, q) \subset \varphi\} .$$

Clearly, we have $\operatorname{Deg}(\varphi, x_0) \subset \operatorname{Deg}(\varphi|_{S^n})$.

Example. Let $\varphi\colon K^{n+1} \to R^{n+1}$ be a map given by $\varphi(x) = S^n$ for each $x \in K^{n+1}$. It is easy to see that φ is an admissible map. We assert that $\operatorname{Deg}(\varphi, 0) = \{0\}$.

Indeed, let $p\colon Y \to K^{n+1}$, $q\colon Y \to R^{n+1}$ be two maps such that $(p, q) \subset \varphi$. Then $q(Y) \subset P^{n+1}$. Consider the commutative



where i, j are, respectively, inclusions.

Applying to the above diagram the functor H^n, we deduce that $(p_1^{*n})^{-1}q_1^{*n} = 0$ and hence $\deg(p_1, q_1) = 0$. Since $(p, q) \subset \varphi$ were arbitrary, we obtain $\operatorname{Deg}(\varphi, 0) = \{0\}$.

It is easy to see (compare Example (3.6), 2) that for $\varphi|_{S^n}\colon S^n \to R^{n+1}$ the set $\operatorname{Deg}(\varphi|_{S^n}) \neq \{0\}$.

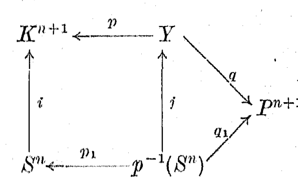
Finally, we note that for $x_0 \in R^{n+1}$, $\|x_0\| > 1$, $\operatorname{Deg}(\varphi, x_0) = \operatorname{Deg}(\varphi|_{S^n}) = \{0\}$, where $\varphi|_{S^n}$ is regarded as the map from S^n to $R^{n+1}\setminus\{x_0\}$.

(4.4) Lemma. *Let $\varphi\colon K^{n+1} \to R^{n+1}$ be an n-admissible map such that $\varphi(S^n) \subset P^{n+1}$. If $\operatorname{Deg}(\varphi, 0) \neq \{0\}$, then there exists a point $x \in K^{n+1}$ such that $0 \in \varphi(x)$.*

Proof. By assumption there exists a selected pair of φ of the form

$$K^{n+1} \xleftarrow{p} Y \xrightarrow{q} R^{n+1}$$

such that $\deg(p_1, q_1) \neq 0$ where $p_1\colon p^{-1}(S^n) \to S^n$, $q_1\colon p^{-1}(S^n) \to P^{n+1}$ and $p_1(y) = p(y)$, $q_1(y) = q(y)$ for each $y \in p^{-1}(S^n)$. Suppose that $0 \notin \varphi(x)$ for each $x \in K^{n+1}$. Then we have $q(Y) \subset P^{n+1}$ and therefore the map q may be regarded as the map from Y to P^{n+1}. We have the commutative diagram



in which i, j are inclusions.

Applying the functor H^n to the above diagram, we obtain

$$(p_1^{*n})^{-1}q_1^{*n} = i^{*n}(p^{*n})^{-1}q^{*n} ,$$

but i^{*n} is the zero homomorphism and hence $(p_1^{*n})^{-1}q_1^{*n} = 0$. This gives $\deg(p_1, q_1) = 0$ and we obtain a contradiction.

(4.5) THEOREM. *Let $\varphi\colon K^{n+1}\to R^{n+1}$ be an n-admissible map such that $\varphi(S^n)\subset K^{n+1}$. Then φ has a fixed point.*

Proof. For the proof we may assume, without loss of generality, that $x\notin\varphi(x)$ for each $x\in S^n$.

Let $(p,q)\subset\varphi$ be a selected pair of φ of the form

$$K^{n+1}\overset{p}{\leftarrow} Y\overset{q}{\rightarrow} R^{n+1}.$$

Define a continuous single-valued map $f\colon Y\to R^{n+1}$ by putting $f(y)=p(y)-q(y)$ for each $y\in Y$.

From the above assumption we infer that $f(y)\in P^{n+1}$ for each $y\in p^{-1}(S^n)$.

Let $\psi\colon K^{n+1}\to R^{n+1}$ be a map given by

$$\psi(x)=f\big(p^{-1}(x)\big)\quad\text{for each } x\in K^{n+1}.$$

The map ψ is n-admissible, since the pair (p,f) is a selected pair of ψ. It is easy to see that $\psi(S^n)\subset P^{n+1}$. We assert that

(i) $\mathrm{Deg}(\psi,0)\neq\{0\}$.

To prove (i) we define the following maps:

$$g\colon p^{-1}(S^n)\to P^{n+1},\qquad g(y)=p(y)\quad\text{for each } y\in p^{-1}(S^n),$$

$$h\colon p^{-1}(S^n)\times I\to P^{n+1},\qquad h(y,t)=p(y)-tq(y).$$

Observe that h is well defined. For $t=0$ or $t=1$ we know that $\big(p(y)-tq(y)\big)\in P^{n+1}$ for each $y\in p^{-1}(S^n)$.

Assume that there exist a number $0<t<1$ and a point $y\in p^{-1}(S^n)$ such that $p(y)-tq(y)=0$. Then we have

$$1=\|p(y)\|=t\|q(y)\|\leqslant t<1$$

and we obtain a contradiction (since $q(y)\in K^{n+1}$ for each $y\in p^{-1}(S^n)$).

Let (p_1,f_1) be the associate pair for the selected pair $(p,f)\subset\psi$ (compare the definition of degree for maps from K^{n+1} to R^{n+1}).

Since h is a homotopy joining f_1 and g and p_1^{*n} is an isomorphism, we have

$$(p_1^{*n})^{-1}f_1^{*n}=(p_1^{*n})^{-1}g^{*n},$$

but $(p_1^{*n})^{-1}g^{*n}\neq 0$ and therefore $\deg(p_1,f_1)\neq 0$.

Finally, we infer that ψ satisfies all the assumptions of Lemma (4.4). This implies that there exists a point $x\in K^{n+1}$ such that $0\in\psi(x)$. Then $x\in\varphi(x)$ and the proof of (4.5) is completed.

As an immediate consequence of (4.5) we obtain the following

(4.6) COROLLARY (Brouwer's Theorem). *Every n-admissible map, in particular every admissible map, of the ball K^{n+1} into itself has a fixed point.*

5. Theorem on antipodes for n-admissible maps. In this section we generalize the classical Borsuk–Ulam Theorem from the case of single-valued maps to n-admissible maps.

(5.1) THEOREM. *Let $\varphi\colon S^n\to P^{n+1}$ be an n-admissible map. If for every $x\in S^n$ there exists an n-subspace of R^{n+1} strictly separating $\varphi(x)$ and $\varphi(-x)$, then $0\notin\mathrm{Deg}(\varphi)$.*

Proof. Let (p,q) be a selected pair of φ of the form

$$S^n\overset{p}{\leftarrow} Y\overset{q}{\rightarrow} P^{n+1}.$$

Define the map $\psi\colon S^n\to P^{n+1}$ given by

$$\psi(x)=q\big(p^{-1}(x)\big)\quad\text{for each } x\in S^n.$$

It is easy to see that ψ is a compact multi-valued map satisfying the assumptions of (5.1). This implies that for every $y\in S^n$ the following set is open:

$$Uy=\{x\in S^n;\ (y,z)>0\ \text{for each } z\in\psi(x)\ \text{and } (y,z)<0\ \text{for each } z\in\psi(-x)\},$$

where (y,z) denote the inner product in R^{n+1}. From the assumption we deduce that for every $x\in S^n$ there is $y\in S^n$ such that $(y,z)>0$ for all $z\in\psi(x)$ and $(y,z)<0$ for all $z\in\psi(-x)$. Therefore the family $\{Uy\}_{y\in S^n}$ is an open covering of S^n. Since S^n is compact, there exists a finite subcover $\{Uy_i\}_{i=1,\dots,m}$. Let $\{g_i\}_{i=1,\dots,m}$ be a subordinated partition of unity.

Consider $g\colon S^n\to P^{n+1}$ defined by $g(x)=\sum_i\big(g_i(x)-g_i(-x)\big)y_i$. Observe that if $z\in\psi(x)$ then:

(i) $(y_i,z)>0$ for some $i=1,\dots,m$,

(ii) $(y_i,z)<0$ implies $g_i(x)=0$,

(iii) $(y_i,z)>0$ implies $g_i(-x)=0$.

Conditions (i)-(iii) imply that g is a well-defined map (i.e., $g(S^n)\subset P^{n+1}$).

The map g is odd, and thus $\deg(g)\neq 0$. (Compare the theorem on antipodes for single-valued maps.)

Define the map $\tilde g\colon Y\to P^{n+1}$ given as the composition of p and g ($\tilde g=g\cdot p$), and the map $h\colon Y\times I\to P^{n+1}$ given by $h(y,t)=tg(y)+(1-t)q(y)$ for each $y\in Y$ and $t\in I$.

We know that, for $t=0$ or $t=1$ and for each $y\in Y$, $h(y,t)\neq 0$. Assume that $h(y,t)=0$ for some $y\in Y$ and $0<t<1$. Then we have $-(1-t)q(y)=t\tilde g(y)$.

Taking the inner product of both sides of the above equality with $q(y)$, we obtain

$$-(1-t)\big(q(y),\, q(y)\big) = t\big(\tilde{g}(y),\, q(y)\big)\,.$$

We have $-(1-t)\big(q(y), q(y)\big) < 0$, but (i)-(iii), for $z = q(y)$, and the definition of $\tilde{g}$ implies that $t\big(g(y), q(y)\big) > 0$, and thus we obtain a contradiction. Since h is a homotopy joining q and $\tilde{g}$ and p^{*n} is an isomorphism, we obtain

$$(p^{*n})^{-1}q^{*n} = (p^{*n})^{-1}\tilde{g}^{*n}\,,$$

and hence we have

$$(p^{*n})^{-1}q^{*n} = (p^{*n})^{-1}p^{*n}g^{*n} = g^{*n} \neq 0\,.$$

This implies that $\deg(p, q) \neq 0$ and the proof of (5.1) is completed.

Theorems (5.1) and (4.4) give

(5.2) COROLLARY. *Suppose that $\varphi\colon K^{n+1} \to R^{n+1}$ is an n-admissible map and for each $x \in S^n$ there is an n-subspace of R^{n+1} strictly separating $\varphi(x)$ and $\varphi(-x)$. Then $0 \in \varphi(x)$ for some $x \in K^{n+1}$.*

6. Theorem on antipodes for admissible maps. In this section we denote by M a compact space which has the cohomology of the unit n-sphere S^n in R^{n+1}.

A continuous multi-valued map $\Phi\colon M \to M$ is called an *involution* if the condition $(x, y) \in \Gamma_\Phi$, for every $(x, y) \in M \times M$, implies that $(y, x) \in \Gamma_\Phi$.

In our subsequent consideration an essential use will be made of the following:

(6.1) *Let $g\colon M \to M$ be a single-valued involution and let $f\colon M \to S^n$ be a single-valued continuous map such that $f(x) \neq fg(x)$ for each $x \in M$. Then the induced homomorphism $f^{*n}\colon H^n(S^n) \to H^n(M)$ is a non-zero homomorphism.*

Theorem (6.1) clearly follows from Corollary 4, p. 299 in [1].

We prove, for admissible maps, the following general version of the Theorem on antipodes.

(6.2) THEOREM. *Let $\Phi\colon M \to M$ be an acyclic involution and let $\varphi\colon M \to P^{n+1}$ be an admissible map such that following condition is satisfied:*

every radius with origin at the zero point of R^{n+1} has an empty intersection with the set $\varphi(x)$ or $\varphi(\Phi(x))$ for each $x \in M$.

Then $0 \notin \operatorname{Deg}\varphi$.

Proof. Let (p, q) be a selected pair of φ of the form

$$M \overset{p}{\leftarrow} Y \overset{q}{\rightarrow} P^{n+1}\,.$$

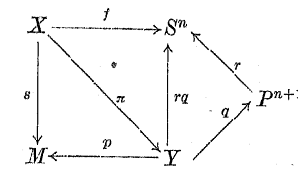
We prove that $\deg(p, q) \neq 0$.

Define the set X by putting

$$X = \{(x, x', y, y');\ x \in M,\ x' \in \Phi(x),\ y \in p^{-1}(x),\ y' \in p^{-1}(x')\} \subset M \times M \times Y \times Y\,.$$

Since p as a Vietoris map is proper, M is compact and Φ is continuous, we infer that X is a compact set.

Consider the diagram



in which

$$s(x, x', y, y') = x\,, \qquad f(x, x', y, y') = \frac{q(y)}{\|q(y)\|}\,,$$

$$\pi(x, x', y, y') = y \qquad \text{for each } (x, x', y, y') \in X$$

and

$$r(z) = \frac{z}{\|z\|} \qquad \text{for each } z \in P^{n+1}\,.$$

It is easy to see that the above diagram commutes. The map s has the following decomposition:

$$(x, x', y, y') \to (x, x', y) \to (x, x') \to x\,.$$

Since the maps given in the decomposition of s are determined by the Vietoris maps p and p_Φ, respectively, we infer that s is a Vietoris map (comp. 1.7). This implies that X has the cohomology of n-sphere S^n. Define the single-valued involution $g\colon X \to X$ by putting $g(x, x', y, y') = (x', x, y', y)$. We prove that $f(x, x', y, y') \neq f\big(g(x, x', y, y')\big)$. Indeed, we have

$$f(x, x', y, y') = \frac{q(y)}{\|q(y)\|} \quad \text{and} \quad f\big(g(x, x', y, y')\big) = \frac{q(y')}{\|q(y')\|}\,.$$

Since $q\big(p^{-1}(x)\big) \subset \varphi(x)$ for each $x \in M$, from the assumption we deduce that

$$\frac{q(y')}{\|q(y')\|} \neq \frac{q(y)}{\|q(y)\|}\,.$$

Therefore from (6.1) we find that $f^{*n}\colon H^n(S^n)\to H^n(X)$ is a non-zero homomorphism. Since p^*, s^*, r^* are isomorphisms, from the commutativity of the above diagram we have

$$(p^{*n})^{-1}q^{*n} = (s^{*n})^{-1}f^{*n}(r^{*n})^{-1} \neq 0\,,$$

and $\deg(p,q)\neq 0$. The proof of (6.2) is completed.

We now draw a few consequences from the main theorem.

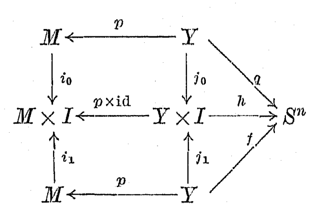
(6.3) Corollary. *Let $\Phi\colon M\to M$ be an acyclic involution and let $\varphi\colon M\to S^n$ be an admissible map which satisfies the following condition:*

$$\varphi(x)\cap\varphi(y)=\emptyset\,,\quad \textit{for each } x\in M \textit{ and for each } y\in\Phi(x)\,.$$

Then $0\notin \operatorname{Deg}\varphi$.

(6.4) Corollary. *Let $\Phi\colon M\to M$ and $\varphi\colon M\to S^n$ are as in* (6.3). *Then $\varphi(M)=S^n$.*

Proof. Let (p,q) be a selected pair of φ. From (6.3) we have $\deg(p,q)\neq 0$. Assume that there exists a point $u_0\in S^n\setminus qp^{-1}(M)$. Consider the commutative diagram



in which $f(y)=-u_0$ for all $y\in Y$,

$$h(y,t)=\frac{tq(y)+(t-1)u_0}{\|tq(y)+(t-1)u_0\|}\,,\quad \text{for each } y\in Y \text{ and } t\in I\,.$$

We have $q(y)\neq u_0$ for each $y\in Y$. This implies that the map h is well defined. Since f is a constant map, we have $f^{*n}=0$. The commutativity of the above diagram implies that

$$(p^{*n})^{-1}q^{*n}=(p^{*n})^{-1}f^{*n}=0\,,$$

and $\deg(p,q)=0$. This contradicts the fact that $\deg(p,q)\neq 0$, and the proof of (6.4) is completed.

(6.5) Corollary. *Let $\Phi\colon M\to M$ be an acyclic involution and let $\varphi\colon M\to R^n$ be an admissible map. Then there exists a point $(x,y)\in\Gamma_\Phi$ such that $\varphi(x)\cap\varphi(y)\neq\emptyset$.*

Proof. We may regard the map φ as the map from M to S^n. Then (6.5) simply follows from (6.4).

Assume that $M=S^n$ and $\Phi=-\mathrm{id}_{S^n}$; then from (6.5) we obtain

(6.6) Corollary. *For every admissible map $\varphi\colon S^n\to R^n$ there is a point $x\in S^n$ such that $\varphi(x)\cap\varphi(-x)\neq\emptyset$.*

Note that Corollary (6.6) is a generalization, to multi-valued maps, of the classical Borsuk–Ulam Theorem.

7. Theorem on the invariance of domain. In this section we show that the Brouwer Invariance of Domain Theorem may be generalized to s-admissible maps.

For a subset A of R^{n+1} we denote by $\operatorname{Int}A$ the union of all open sets in R^{n+1} which are contained in A. For a point $a_0\in R^{n+1}$ and a positive real number r we denote by $B(a_0,r)$ the open ball in R^{n+1} with centre a_0 and radius r.

Let A be a compact subset of R^{n+1}, $a_0\in A$ and U an open neighbourhood of a_0 in R^{n+1}; then by $j\colon A\setminus U\to A$ we denote the inclusion map. We note the following fact:

(7.1) *Let A be a compact subset of R^{n+1} and $a_0\in A$. The point $a_0\in\operatorname{Int}A$, if and only if there exists a positive number r_0 such that for every $0<r<r_0$ the homomorphism $j^{*n}\colon H^n(A)\to H^n(A\setminus B(a_0,r))$ induced by $j\colon A\setminus B(a_0,r)\to A$ is not an epimorphism.*

The above fact is well known; for example see ([2] p. 394). Next, we prove the following two lemmas:

(7.2) Lemma. *Let A be a compact subset of R^{n+1} and let a_0 be a point of A. The point $a_0\in\operatorname{Int}A$ if and only if there exists an s-admissible map $\varphi\colon K^{n+1}\to A$ such that*

(i) $\varphi(S^n)\subset A\setminus\{a_0\}$,

(ii) $0\notin\operatorname{Deg}(\varphi,a_0)$.

Proof. We observe that if $a_0\in\operatorname{Int}A$, then there exists a single-valued, continuous map $f\colon K^{n+1}\to A$ such that $f(S^n)\subset A\setminus\{a_0\}$ and $\deg(f|_{S^n})\neq 0$. Conversely, assume that there exists an s-admissible map $\varphi\colon K^{n+1}\to A$ such that $\varphi(S^n)\subset A\setminus\{a_0\}$ and $0\notin\operatorname{Deg}(\varphi,a_0)$. We prove that $a_0\in\operatorname{Int}A$.

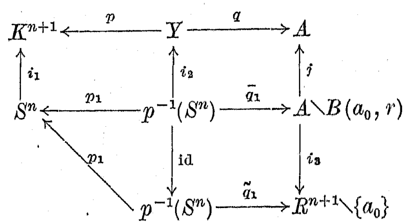
Let $(p,q)\subset\varphi$ be a selected pair of φ of the form

$$K^{n+1}\overset{p}{\leftarrow}Y\overset{q}{\to}A\,.$$

Define a map $p_1\colon p^{-1}(S^n)\to S^n$ and $q_1\colon p^{-1}(S^n)\to A\setminus\{a_0\}$ by putting $p_1(y)=p(y)$, $q_1(y)=q(y)$ for each $y\in p^{-1}(S^n)$. The set $q_1p^{-1}(S^n)$ is compact.

Let $r_0=\operatorname{dist}\big(a_0,q_1(p^{-1}(S^n))\big)$. Then r_0 is a positive real number. Consider the open ball $B(a_0,r)$ for $0<r<r_0$ and the inclusion map $j\colon A\setminus B(a_0,r)\to A$; we assert that j^{*n} is not an epimorphism.

Indeed, we have the commutative diagram



in which i_1, i_2, i_3 are inclusions, $\tilde{q}_1$, $\bar{q}_1$ are given $\bar{q}_1(y) = \tilde{q}_1(y) = q_1(y)$ for each $y \in p^{-1}(S^n)$. From the assumption we have $(p_1^{*n})^{-1}\tilde{q}_1^{*n} \neq 0$.

This implies that $(p_1^{*n})^{-1}\bar{q}_1^{*n}i_3^{*n} \neq 0$ and hence $\bar{q}_1^{*n} \neq 0$. Assume that j^{*n} is an epimorphism. Then we obtain

$$i_1^{*n}(p^{*n})^{-1}q^{*n} = (p_1^{*n})^{-1}\bar{q}_1^{*n}j^{*n} \neq 0 ,$$

which is a contradiction. Since j^{*n} is not an epimorphism, from (7.1) we obtain $a_0 \in \operatorname{Int} A$, and the proof of (7.2) is completed.

An s-admissible map $\varphi\colon X \to Z$ is called an *ε-map* if the condition $\varphi(x) \cap \varphi(x') \neq \emptyset$ implies $d(x, x') < \varepsilon$ for each $x, x' \in X$.

(7.3) LEMMA. *Let $\varphi\colon K^{n+1} \to R^{n+1}$ be a 1-map. Then*

(i) *$\varphi(S^n) \subset R^{n+1}\setminus\{z_0\}$ for each $z_0 \in \varphi(0)$.*

(ii) *$0 \notin \operatorname{Deg}(\varphi, z_0)$.*

Proof. Let $z_0 \in \varphi(0)$; we prove that $z_0 \notin \varphi(S^n)$. Assume that $z_0 \in \varphi(x)$ for some $x \in S^n$. Then we have $\varphi(0) \cap \varphi(x) \neq \emptyset$ and from the assumption we deduce that $\|x\| < 1$, which is a contradiction.

Now we prove (ii). Let $(p, q) = \varphi$ be a selected pair of φ of the form

$$K^{n+1} \overset{p}{\leftarrow} Y \to R^{n+1} .$$

Let $y_0 \in p^{-1}(0)$ be a point such that $q(y_0) = z$. Define the maps $p_1\colon p^{-1}(S^n) \to S^n$, $q_1\colon p^{-1}(S^n) \to R^{n+1}\setminus\{z_0\}$ by putting $p_1(y) = p(y)$, $q_1(y) = q(y)$ for each $y \in p^{-1}(S^n)$. For the proof it is sufficient to show that $\deg(p_1, q_1) \neq 0$.

Define the following sets:

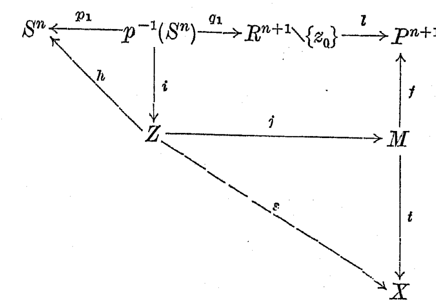
$$X = \{(x, x') \in K^{n+1} \times K^{n+1};\ \|x - x'\| = 1\} ,$$

$$M = \{(x, x', y, y')\colon (x, x') \in X,\ y \in p^{-1}(x),\ y' \in p^{-1}(x')\} ,$$

$$Z = \{(x, x', y, y')\colon (x, x', y, y') \in M,\ x' = 0\} .$$

It is easy to see that X, M, Z are compact sets.

Consider the diagram



in which

$$i(y) = (p_1(y), 0, y, y_0) , \quad h(x, 0, y, y') = x ,$$

$$j(x, 0, y, y') = (x, 0, y, y') , \quad t(x, x', y, y') = (x, x') ,$$

$$s(x, 0, y, y') = (x, 0) , \quad l(z) = z - z_0 ,$$

$$f(x, x', y, y') = q(y) - q(y') .$$

Since φ is an 1-map, we have $f(x, x', y, y') \neq 0$. It is evident, that the above diagram commutes.

As in the proof of Theorem (6.2), we deduce that h^*, s^*, t^* are isomorphisms. Hence the commutativity of the above diagram implies that j^* and i^* are isomorphisms. This implies that M has the cohomology of S^n.

Define the involution $g\colon M \to M$ by putting $g(x, x', y, y') = (x', x, y', y)$. Then $f(g(x, x', y, y)) \neq f(x, x', y, y')$. Applying Theorem (6.2) to the maps f, g, we obtain $f^{*n} \neq 0$. From the commutativity of the above diagram we have $q_1^{*n}l^{*n} \neq 0$. Finally, we obtain $q_1^{*n} \neq 0$, and this implies that $\deg(p_1, q_1) \neq 0$. The proof of (7.3) is completed.

(7.4) Remark. It is evident that Lemma (7.2) remains true for any closed ball in R^{n+1} with radius ε and for any ε-map, where ε is a positive real number.

Now we prove two theorems of the type of the Brouwer Invariance of Domain Theorem for s-admissible maps.

(7.5) THEOREM. *Let $\varepsilon > 0$ be a positive real number. If $\varphi\colon R^{n+1} \to R^{n+1}$ is an ε-map, then $\varphi(R^{n+1})$ is an open subset of R^{n+1}.*

Proof. Let $y \in \varphi(R^{n+1})$. We prove that $y \in \operatorname{Int}\varphi(R^{n+1})$. Assume that $y \in \varphi(x)$ for some $x \in R^{n+1}$. Let K_ε^{n+1} be a closed ball in R^{n+1} with the centre at x and radius ε.

Since φ is an s-admissible map, we deduce that $\varphi(K_\varepsilon^{n+1})$ is compact set. We have $y \in \varphi(K_\varepsilon^{n+1})$. Let ψ be the restriction of φ to the ball K_ε^{n+1}. Then ψ is an ε-map and hence we have $\psi(S_\varepsilon^n) \subset R^{n+1} \setminus \{y\}$, where S_ε^n denotes the boundary of K_ε^{n+1}. Therefore Lemma (7.3) (comp. Remark (7.4)) implies that $0 \notin \mathrm{Deg}(\psi, y)$ and from (7.2) we obtain $y \in \mathrm{Int}\varphi(R^{n+1})$. The proof of (7.5) is completed.

(7.6) THEOREM. *Let U be an open subset of R^{n+1} and $\varphi\colon U \to R^{n+1}$ an s-admissible map. Assume further that for every point $x_1, x_2 \in U$ the condition $x_1 \neq x_2$ implies $\varphi(x_1) \cap \varphi(x_2) = \emptyset$. Then $\varphi(U)$ is an open subset of R^{n+1}.*

Proof. From the assumption we infer that φ is an ε-map for each $\varepsilon > 0$. Let $y \in \varphi(U)$. We prove that $y \in \mathrm{Int}\varphi(U)$. Assume that $y \in \varphi(x)$ for some $x \in U$. Since U is open, there exists an $\varepsilon > 0$ that the set K_ε^{n+1} is contained in U, where K_ε^{n+1} is a closed ball with centre at x and radius ε. Let ψ be the restriction of φ to the set K_ε^{n+1}. Since ψ is an ε-map, we have $y \notin \psi(S_\varepsilon^n)$, where S_ε^n is the boundary of K_ε^{n+1}. Applying Lemma (7.3) and (7.2) as in the proof of (7.5), we obtain $y \in \mathrm{Int}\varphi(U)$. The proof of (7.6) is completed.

Remark. Theorem (7.6) is not true for admissible maps. For example, let $\varphi\colon (-1, 1) \to R$ be the map given by

$$\varphi(t) = \begin{cases} t & \text{for} \quad t \neq 0, \\ \{0, 2\} & \text{for} \quad t = 0. \end{cases}$$

Then φ is an admissible map and the following condition is satisfied: if $t_1 \neq t_2$, then $\varphi(t_1) \cap \varphi(t_2) = \emptyset$ but $\varphi(-1, 1)$ is not an open set in R. Observe (comp. (3.6)) that for every selected pair $(p, q) \subset \varphi$ we have $2 \notin qp^{-1}(-1, 1)$.

It is easy to see that Theorem (7.5) is not true for admissible maps, either.

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