Superpositions of transformations of bounded variation

by

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Abstract. The work deals with some classification of continuous functions transforming plane into plane. For every finite or countable ordinal number was defined a class of superposition of functions of bounded variation (in the sense of Rado, see [3]). The main results of the work are following theorems: every class of superpositions is non-empty and there exists a continuous function which does not belong to the sum of all classes. Similar results for real functions of real variable are included in the classical work [1] by Nina Bary.

Nina Bary in [1] has studied the possibility of representing arbitrary real continuous functions of a real variable only by superpositions of continuous functions of bounded variation. She has introduced the notion of superposition of class α for every finite or countable ordinal α and she has proved that all classes of superpositions are non-empty and that their sum is not equal to the class of all continuous functions. This work contains similar results for plane transformations defined on the unit square (open or closed). The notion of transformation of bounded variation is taken from [3] and [4]. The definition of superposition of class α is similar to that in [1] if α is a countable ordinal of the first kind (i.e. having a predecessor) and differs from the definition in [1] by using uniform convergence instead of ordinary convergence if α is a countable ordinal of the second kind.

The work consists of two parts. The first part contains the proof of an auxiliary theorem which explains the structure of plane transformations $F_1$, $F_2$ such that their superposition $F = F_2 \circ F_1$ is of the form $F(x, y) = (f(x), y)$ for $(x, y) \in [0, 1] \times [0, 1]$. The second part contains several theorems concerning superpositions of transformations of bounded variation. The main results of the work are Theorem 13, which states that every class of superpositions is non-empty, and Theorem 14, which gives the construction of continuous plane transformation which does not belong to the sum of all classes (both these theorems deal with transformations defined on the open unit square) and the corollary (after Theorem 14), which includes the same results for transformations defined on the closed unit square.
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Let \( \mathbb{R}^2 \) denote the Euclidean plane. Let \( K = [0, 1] \times [0, 1] \), \( p_i(x^2) = [0, 1] \times \{0, 1\} \) for \( x_i \in \{0, 1\} \), \( p_2(x^2) = \{0, 1\} \times [0, 1] \) for \( x_2 \in [0, 1] \). Suppose that \( F : K \rightarrow \mathbb{R}^2 \) is a continuous transformation such that \( F(x_1, x_2) = (f(x_1), f(x_2)) \) for \( x_1, x_2 \in K \), where \( f : [0, 1] \rightarrow [0, 1] \) fulfills the following conditions: \( f(0) = 0, f(1) = 1, 0 < f(x^2) < 1 \) for every \( x_2 \in [0, 1] \) and \( f \) has not an interval of constancy. Let \( F = F_1 \cup F_2 \), where \( F_1 : K \rightarrow \mathbb{R}^2 \), \( F_2 : K \rightarrow \mathbb{R}^2 \) are continuous functions. To prove the main theorem of this part we shall need the following lemmas:

**Lemma 1.** If \( x_1^2, x_2^2 \in [0, 1], x_1^2 \neq x_2^2 \), then \( F_1(p_1(x^2)) \cap F_2(p_1(x^2)) = \emptyset \).

**Lemma 2.** For every \( x^2 \in [0, 1] \) the reduced functions \( F_1, F_2 \) are homeomorphisms.

**Lemma 3.** For every \( x^2, y^2 \in [0, 1] \) the set \( F_1(p_0(x^2)) \cap F_2(p_0(y^2)) \) contains exactly one point \( F_1(x^2) \).

**Lemma 4.** \( F_1 \mid \text{Int}(K) \cap F_2 \mid \text{Fr}(K) = \emptyset \).

The proofs of all the above lemmas, based upon the properties of \( F \), are nearly obvious.

**Lemma 5.** Let \( L(0) \subset F_1(p_0(0)) \) be the simple arc joining \( F_1(0, 0) \) and \( F_1(1, 0) \), \( L(1) \subset F_2(p_0(1)) \) — the simple arc joining \( F_2(0, 1) \) and \( F_2(1, 1) \) (the sets \( F_1(p_0(0)) \) and \( F_2(p_0(1)) \) are arcwise connected as continuous images of \([0, 1]\); see, for example, [2], p. 245). The set \( E = L(0) \cup L(1) \cup F_1(p_0(0)) \cup F_2(p_0(1)) \) is homeomorphic with the set \( F \left( [0, 1] \times [-1, 1] \right) \) and there exists a homeomorphism \( G : E \rightarrow F \left( [0, 1] \times [-1, 1] \right) \) such that

\[
G(L(0)) = (-1, 1), \quad G(L(1)) = (-1, 1), \quad G(F_1(p_0(0))) = (1, -1) \times (1, -1), \quad G(F_2(p_0(1))) = (1, -1) \times (1, -1).
\]

**Proof.** In virtue of the definition or of Lemma 2 all the four terms of \( E \) are simple arcs, and so it suffices to show that

\[
L(0) \cap L(1) = \emptyset, \quad F_1(p_0(0)) \cap F_2(p_0(1)) = \emptyset.
\]

Let \( L(0) \cap L(1) = \emptyset, \quad L(0) \cap F_1(p_0(0)) = F_1(0, 0), \quad L(0) \cap F_2(p_0(1)) = F_2(0, 1), \quad L(1) \cap F_1(p_0(0)) = F_1(1, 0), \quad L(1) \cap F_2(p_0(1)) = F_2(1, 1).
\]

The first equality follows at once from Lemma 1. The second is a consequence of the following inclusion: \( F_1(F_1(p_0(0)) \cap F_1(p_0(1))) \subset (0, 1) \times (0, 1) \). The remaining equalities follow from Lemma 3.

**Lemma 6.** Let \( O_1 \) and \( O_2 \) be open regions into which \( E \) divides the plane according to the well-known theorem of Jordan. Then \( F_1(\text{Int}(K)) \subset O_1 \) or \( F_2(\text{Int}(K)) \subset O_2 \).

**Proof.** Suppose that neither the first nor the second inclusion is fulfilled. Then in virtue of Lemma 4 there exists a point \( x_1^2, x_2^2 \in \text{Int}(K) \) such that \( F_1(x_1^2, x_2^2) \in O_1 \) and there exists a point \( x_1^2, x_2^2 \in \text{Int}(K) \) such that \( F_1(x_1^2, x_2^2) \in O_2 \). Let \( d \) be the segment joining those points. We have \( d \subset \text{Int}(K) \) and \( F(d) \cap E = \emptyset \). This contradicts Lemma 4.

**Lemma 7.** If \( O_1 \) denotes the bounded region (with the notation of Lemma 6), then \( F_1(\text{Int}(K)) \subset O_1 \).

**Proof.** Suppose that \( F_1(\text{Int}(K)) \subset O_2 \). Let \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a homeomorphism such that \( G \mid E = G_0 \) (such a homeomorphism exists in virtue of the theorem of Schönflies; see [2], p. 280). Obviously we can choose \( G \) such that for every \( x^2 \in [0, 1] \) the ordinate of \( G(F_1(x^2, 1)) \) is less than or equal to \(-1 \) and the ordinate of \( G(F_1(x^2, 1)) \) is greater than or equal to \(-1 \). We have \( G(O_1) = \text{Int}([-1, 1] \times [-1, 1]) \) and \( G(O_2) = \mathbb{R}^2 \times [-1, 1] \times [-1, 1] \). From the assumption and from the continuity of \( G \) it follows that \( G(F_1) \subset \text{Int}([-1, 1] \times [-1, 1]) \). Hence for every \( x^2 \in [0, 1] \) we have the inclusion \( G(F_1(p_0(0))) \subset \mathbb{R}^2 \times [-1, 1] \times [-1, 1] \), and so on the simple arc \( G(F_1(p_0(0))) \) there exists a single-valued continuous argument (see [4], p. 385, Lemma 10). Let us put \( \theta(x^2) = \arg G(F_1(x^2, 1)) = \arg G(F_1(x^2, 0)) \) for \( x^2 \in [0, 1] \), where the values of the argument are taken from the same single-valued argument. It is not difficult to calculate \( \theta(0) = -2\pi \) and \( \theta(1) = 2\pi \). The function \( \theta : [0, 1] \rightarrow \mathbb{R} \) is continuous on \([0, 1]\) ([4], p. 360, Lemma 15), and so there exists an \( x^2 \in [0, 1] \), for which \( \theta(x^2) = 0 \). This is impossible, because two points \( G(F_1(x^2, 0)) \) and \( G(F_1(x^2, 1)) \) cannot lie on the same half-line issuing from \((0, 0)\). Hence \( F_1(\text{Int}(K)) \subset O_1 \).

**Lemma 8.** For every \( x_1^2 \in (0, 1) \) the set \( F_1(p_0(x_1^2)) \) is a simple arc and \( F_1(p_0(0)) \cup F_1(p_0(1)) \subset O_1 \).

**Proof.** For every \( x_2^2 \in (0, 1) \) the set \( F_1(p_0(x_2^2)) \) is an arcwise connected set as a continuous image of a segment. Let \( L(x_2^2) \subset F_1(p_0(x_2^2)) \) be a simple arc joining \( F_1(0, x_2^2) \) and \( F_1(1, x_2^2) \). We shall prove that \( L(x_2^2) = F_1(p_0(x_2^2)) \). Let

\[
E_1(x_2^2) = L(0) \cup L(x_2^2) \cup F_1((0, x^2): 0 \leq x^2 \leq x_2^2) \cup F_1((1, x^2): 0 \leq x^2 \leq x_2^2),
\]

\[
E_2(x_2^2) = L(x_2^2) \cup L(1) \cup F_1((0, x^2): x_2^2 \leq x^2 \leq 1) \cup F_1((1, x^2): x_2^2 \leq x^2 \leq 1).
\]

As in the proof of Lemma 5 one can prove that \( E_1(x_2^2) \) and \( E_2(x_2^2) \) are simple closed curves. Let \( O(x_2^2) \) denote the region bounded by \( E_i(x_2^2) \) for \( i = 1, 2 \). It is not difficult to see that.
From the above mentioned properties of $B(0)$ it follows that $E_f(0, x') \in B(0)$. Hence $F[p'(x')] \cap Fr B(0) \neq \emptyset$, so $E_f[p'(x')] \cap Fr[p'(0)] \neq \emptyset$, which contradicts Lemma 2. One can similarly prove that $E_f(\Int K) \cap \Int B(1) = \emptyset$.

Hence $E_f(\Int K) \subset \emptyset \cup \{B(0) \cup B(1) \cup E_f[p'(0)] \cup E_f[p'(1)]\}$. 

Suppose now that there exists a point $(y', y') \in F'(0, x')$ belonging to the set on the right-hand side of the last inclusion and not belonging to $E_f(\Int K)$. Consider the homeomorphism $G: E_f \to E_f$ fulfills the following conditions: $G[E_f] = E_f$ (for the notation see Lemma 7) and $G[B(0)] \subset \{[-1, 1] \times \{0\}, B(1) \subset \{[-1, 1] \times \{0\} \}$, and $G(y', y') = (y', y')$, it is not difficult to see that such a homeomorphism does exist. We have $E_f(0, x') \in \{[-1, 1] \times \{0\}\}$, and so on every simple arc $E_f[p'(x')]$ there exists a single-valued continuous argument. As in the proof of Lemma 7, let us put $\Delta(x') = argF[p'(x'), 1] - argF[p'(x'), 0]$ for $x' \in [0, 1]$. The function $\Delta: [0, 1] \to R$ is continuous on $[0, 1]$ and $\Delta(0) = -2\pi, \Delta(1) = 2\pi$, and so there exists an $a' \in (0, 1)$ for which $\Delta(x') = 0$. This is impossible, because points $G(E_f[x', 1])$ and $G(E_f[x', 0])$ cannot lie on the same half-line issuing from $(0, 0)$. So the lemma is proved.

**Lemma 10.** If $x', x', x', x', x', x' \neq 0$, then $E_f[p'(x')] \cap E_f[p'(x')]$.

Proof. Let us observe that if $F[x', x'] \in F[p'(x')]$ for some $x' \in [0, 1]$ and $x' \neq x'$, then from Lemma 1 it follows that $E_f[x', x'] = E_f[x', x']$. Suppose that $x' \neq x'$ and that there exists a number $x' \in (0, 1)$ such that $F[x', x'] = F[x', x']$. We shall now prove that $F[x', x'] = F[x', x']$ for every $x' \in (0, 1)$. Let $x', x', x', x' \neq 0$ denote the component of the set $x': F[x', x'] \neq F[x', x']$, $x' < x'$ (if this set is non-empty; otherwise there is nothing to prove). Let $C$ denote the region whose boundary is a simple closed curve consisting of the parts of the two simple arcs $F[x', x']$ and $F[x', x']$ included between $F[x', x']$ and $F[x', x']$ if $x' > 0$. If $x' = 0$, then $C$ denote the region with the boundary consisting of the parts of the simple arcs $F[x', x']$ and $F[x', x']$ included between $F[x', x']$, $F[x', x']$, and $F[x', x']$, respectively, and of a suitable subset of $F[p'(0)]$ chosen in such a way that $C \cap B(0) = \emptyset$ (in this case the boundary of $C$ need not be a simple closed curve). From Lemma 9 it follows that $C \subset E_f(\Int K)$ in both cases. Let $(x', x', x', x', x', x') \in \Int K$ be a point such that $E_f[x', x'] \in C$. From the continuity of $F$, it follows that there exists an $\varepsilon > 0$ such that for every $x' \in (x' - \varepsilon, x' + \varepsilon)$ we have $F[x', x'] \in C$. The end-points of the simple arc $F[x', x']$ belong to $F[p'(0)]$ and $F[p'(1)]$. Hence for every $x' \in (x' - \varepsilon, x' + \varepsilon)$ we have $F[p'(x')] \neq \emptyset$. From this we conclude that $F[p'(x')]$
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In the case where the sequence \( \{ \xi_n \} \) is non-increasing, the proof is analogous, and so \( \mathbb{H}^{-1} \) is a continuous function on \( \text{Int.K} \). Hence \( \mathbb{H} \) is a continuous function on \( \mathcal{F}_1(\text{Int.K}) \) (see [3], p. 31, I.2.49).

II

DEFINITION. The crude multiplicity function of continuous plane transformation \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R}^2 \) with respect to the set \( D \subset C \in E \) is a function defined as follows: for \( (y_1, y_2) \in \mathbb{R}^2 \) a number \( N((y_1, y_2); F; D) \) is equal to the number of elements of the set \( \mathbb{H}^{-1}((y_1, y_2)) \cap D \), if this set is finite, and to \( + \infty \) if this set is infinite ([3], p. 267, [4], p. 5).

DEFINITION 2. The continuous function \( F : E \to \mathbb{R} \), \( \mathbb{C} \subset \mathbb{R} \) is said to be of bounded variation in \( \mathcal{C} \in \mathcal{C} \) in the Banach sense (BV in \( D \)) if and only if \( \int_{y_1}^{y_2} N((y_1, y_2); F; D) \, dy_1 \, dy_2 < + \infty \) ([3], pp. 311-312, [4], pp. 278-280).

Let us note the following simple theorem on the variation of functions of a special type:

THEOREM 2. If \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R} \) is a continuous function and \( \mathbb{F}^1 \) and \( \mathbb{F}^2 \) are constants, then \( F \) is BV in \( \mathbb{K} \) if and only if \( \mathbb{F}^1 \) and \( \mathbb{F}^2 \) are of bounded variation in \([0, 1]\).

Proof. We have \( N((y_1, y_2); F; K) = N_1(y_2; F; K) - N_2(y_1; F; K) \), where \( N_1, N_2 \) are Banach indicatrices of \( \mathbb{F}^1 \) and \( \mathbb{F}^2 \), respectively (we assume \( 0 \leq \infty = 0 \)). Hence \( \int_{y_1}^{y_2} N \, dy_1 \, dy_2 \) and the number on the right-hand side of the equality is different from zero. The rest of the proof is obvious.

DEFINITION 3. We shall say that functions \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R} \) is a superposition of the class \( C \) on the set \( E \) \( \mathbb{E} \subset S \), if and only if \( F(x_1, x_2) = (x_1, x_2) \) for \( (x_1, x_2) \in E \). We shall say that a continuous function \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R} \) is a superposition of the class \( C \) on the set \( E \) \( \mathbb{E} \subset S \), if and only if \( F \) is BV in \( \mathbb{E} \) and \( \mathbb{E} \subset \mathbb{S}(\mathbb{E}) \). If \( \beta \) is a natural number or a countable ordinal of the first kind (i.e., its predecessor \( \beta - 1 \) exists), then we shall say that a continuous function \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R} \) is a superposition of the class \( \beta \) on the set \( E \) \( \mathbb{E} \subset S \), if and only if \( F \) is BV in \( \mathbb{E} \) and \( \mathbb{E} \subset \mathbb{S}(\mathbb{E}) \). If \( \beta \) is a countable ordinal of the second kind, then we shall say that a continuous function \( F : E \to \mathbb{R} \), \( E \subset \mathbb{R} \) is a superposition of the class \( \beta \) on the set \( E \) \( \mathbb{E} \subset \mathbb{S}(\mathbb{E}) \), if and only if there exists a sequence of functions \( \{g_n\} \) such that \( g_n \in E \), \( g_{n+1} \in \mathbb{E} \), \( g_n(\mathbb{E}_{n-1} \{\mathbb{E} \}) \to \mathbb{E} \) for \( n = 1, 2, \ldots \), and \( F(x_1, x_2) = \lim F_{n}(x_1, x_2) \) uniformly on \( \mathbb{E} \), where \( F_{n} = g_{n+1} \circ F_{n} \) for \( n = 1, 2, \ldots \), and \( g_0 \in \mathbb{S}_{\mathbb{E}}(\mathbb{E}_{n-1}, \{\mathbb{E} \}) \), \( F \in \mathbb{S}_{\mathbb{E}}(\mathbb{E}) \), ...
where $a_0 < \beta$ for $n = 1, 2, 3, ...$ and $\beta_n < \beta$ for $n = 2, 3, ...$ and $F \in \bigcup S_\infty^+(E)$. Classes of superpositions of real functions of a real variable are defined analogously (cf. [1], pp. 327-328) and will be denoted by $S_\infty(A)$ for $A \in B$.

**Definition.** Let $\beta$ be a natural number or a countable ordinal. We shall say that a continuous function $F: \text{Int}K \to \mathbb{R}^s$ is a simplified superposition of the class $\beta$ on $\text{Int}K$ such that if $f: \text{Int}K \to \mathbb{R}^s$ and only if $F(x_{i+1}, x_i) = [F(x_i), x_{i+1}]$ for $(x_{i+1}, x_i) \in \text{Int}K$. $F \in S_\infty^+(E)$ and $F$ may be continuously extended onto $[0, 1]$ and this extension belongs to $S_\infty(0, 1)$. One can define in an analogous manner a simplified superposition on an arbitrary open interval in $E$.

Almost all the functions constructed in this paper will be simplified superpositions.

**Theorem 3.** If $g: E \to \mathbb{R}^s$, $E \in \text{Int}K$ is a superposition of the class $\beta$ on $E$ and $f: g(E) \to \mathbb{R}^s$ is a superposition of the class $\alpha$ on $g(E)$, then the function $F = f \circ g$ is a superposition of the class $\gamma \leq \beta + \alpha$ on $E$.

The proof of this theorem proceeds by transfinite induction on $\gamma$ similarly to the proof of Theorem 1 in [1].

**Theorem 4.** If $g: E \to \mathbb{R}^s$ is a continuous and one-to-one function in the open region $E \in \text{Int}K$ and $f: g(E) \to \mathbb{R}^s$ is a superposition of the class $\alpha > 0$ on $g(E)$, then the function $F = f \circ g$ is a superposition of the class $\alpha$ on $E$.

**Proof.** If $g \in S_\infty^+(E)$, then the theorem is obviously fulfilled. Suppose that $g \in S_\infty^+(E)$, where $\gamma \leq 1 + \alpha$. If $\alpha$ is a countable ordinal, then $1 + \alpha = \alpha$. If $\alpha$ is a natural number, then $f = f_\alpha \ast (f_{\alpha-1} \ast \cdots \ast (f_2 \ast f_1) \cdots)$, where $f_i$ is of bounded variation on $[x_{i-1}, f_i(x_i)]$ for $i = 2, 3, ..., n$ and $f_1$ is of bounded variation on $g(E)$. Hence $F = f_\alpha \ast (f_{\alpha-1} \ast \cdots \ast (f_2 \ast f_1) \cdots)$.

**Theorem 5.** Let $\beta$ be a countable ordinal of the second kind. If $g: E \to \mathbb{R}^s$ is a superposition of the class $\beta$ on $E$, the set $g(E)$ is an open region and $f: g(E) \to \mathbb{R}^s$ is an uniformly continuous and one-to-one function, then the function $F = f \circ g$ is a superposition of the class $\beta$ on $E$.

**Proof.** We have $g = \lim g_n$, where $g_n = h \ast g_n = h \ast g_{n+1} \ast g_n$ for $n = 1, 2, 3, ..., \alpha_n$ and $\alpha_n$ are superpositions belonging to the classes $a_0$, respectively, and $a_0 < \beta$, $\beta_n < \beta$ for $n = 1, 2, 3, ...$ If $g_n = f \ast g_n$, then $g_n = f \ast g_n = f \ast (h \ast g_n) = f \ast (h \ast (f \ast g_n)) = f \ast (h \ast (f \ast g_n)) = f \ast (h \ast (f \ast g_n)) = f \ast (h \ast (f \ast g_n))$. Put $f_n = f \ast (h \ast (f \ast g_n))$. We have $E_n = f_n \ast S_\infty^+(E)$, for $n = 2, 3, ...$ and $F_1 = f \ast g_1$.

**Remark.** The inequality $\beta < \gamma$ implies that $\beta$ is a countable ordinal. The algebraic operations on $E$ are continuous. The uniqueness of $\beta$ is a consequence of the first kind, then $F = f \circ g$, where $g$ is a superposition of the class $\beta$ on $E$. If $g(E)$ is a uniformly continuous and one-to-one function, then the function $F = f \circ g$ is a superposition of the class $\beta$ on $E$.

**Theorem 6.** If $g: E \to \mathbb{R}^s$ is a superposition of the class $\beta$ on $E$, then $g(E) \to \mathbb{R}^s$ is not a linear function, then for every real numbers $a, b \neq a, b$ the function $f_a(x, x^2) = [x^2 + a - b, x^2]$ is a simplified superposition of the class $\alpha$ on $E$ and the function $f_a(x, x^2) = [x^2 + a - b, x^2]$ is a simplified superposition of the class $\alpha$ on $E$.

**Proof.** If $f$ is a natural number or a countable ordinal of the first kind, then $F = f \circ g$, where $g$ is a superposition of the class $\beta$ on $E$ and $f: g(E) \to \mathbb{R}^s$ is BV. We have $F_1 = g \ast F$, where $g(y, y^2) = (y-a+1, y^2)$; hence $F_1 = (g \ast f) \ast g$. It is not difficult to verify that $N((x, x^2); g(f_\infty^+(E))) = N((x-a+1, x^2); f_\infty^+(E))$, and so $g \ast f$ is BV on $g(\text{Int}K)$. Hence $F_1 \in S_\infty^+(E)$, where $\gamma < \beta$. To prove the equality it suffices only to observe that $F = g \ast F_1$ and $\gamma \ast g$ also transforms linearly the abscissa.

**Lemma 11.** Let $F(x, x^2) = \left(\begin{array}{c} x^2 + a - b \\ x^2 \end{array}\right)$ for $(x, x^2) \in \text{Int}K$. Let $d = (a_1, b_1)$, $d_2 = (a_2, b_2)$ be disjoint intervals included in $(0, 1)$. Suppose...
that $F$ fulfills the following conditions: \( \lim F(x) = 0 \), \( \lim F(x) = 1 \), \( x \in \mathbb{R}^n \),
\[
0 < F(x) < 1 \quad \text{for} \quad x \in (0, 1), \quad F_1(d_1) = \min\{F(a_1), F(b_1)\}, \quad \max\{F(a_1), F(b_1)\} = \max\{F(a_2), F(b_2)\}\]
and \( F(\alpha) \), \( F(\beta) \) in the interval \( (0, 1) \) if the end-point of \( d_1 \) or \( d_2 \) coincides with the end-point of \( (0, 1) \), then by the value of \( F \) we shall mean the limit. If \( F = f \) or \( g \), then the sets \( g(\alpha, x) \), \( g(\alpha, x) \) are open regions and \( g(\alpha, x) = g(\alpha, x) \) if \( \alpha \subseteq (0, 1) \) and \( \cap \). In
\[
\text{Proof. Let } H: g(\alpha, x) \to \text{Int } K = (0, 1)
\]
be a homeomorphism such that \( H(g(\alpha, x)) = (g(\alpha, x)) \) for \( x \in \alpha \) \( \text{Int } K \). Such a homeomorphism does exist in virtue of Theorem 1. We have \( F = (f + H^{-1}) \circ (H + g) \), and so \( F = f \circ f \), where \( f \) is the first coordinate of \( f \circ H^{-1} \) and \( f \) is a function of one variable. From Lemma 2 of [1], pp. 337–338 it follows that \( g(\alpha, x) \) or \( g(\alpha, x) \cap g(\alpha, x) = \emptyset \). Hence \( H(g(\alpha, x) \cap g(\alpha, x)) = \emptyset \). Furthermore the sets \( H(g(\alpha, x) \cap g(\alpha, x)) = \emptyset \) are non-degenerate open rectangles. The rest of the proof follows from the fact that \( H \) is a homeomorphism.

In the sequel we shall frequently use real functions fulfilling the following conditions: \( \lim f(x) = 0 \), \( \lim f(x) = 1 \), \( 0 < f(x) < 1 \) for \( x \in (0, 1) \). To abbreviate the notation we shall say that such a function fulfills condition (m).

**Theorem 7.** If \( F \in \mathcal{U}(\text{Int } K) \), \( F(x, x) = (F(x), x) \), where \( F \) fulfills condition (m), \( P \subseteq (0, 1) \) is a closed set and \( (a_n, b_n) \) is a sequence of components of \( (0, 1) \), then the function \( F(x, x) = (F(x), x) \) for \( x \in (0, 1) \) is a simplified superposition of the class \( a = \max(a_n, a_n) \) on \( \text{Int } K \).

**Proof.** From the fact that \( F \in \mathcal{U}(\text{Int } K) \) it follows that \( F^1 \in S(0, 1) \). In virtue of Theorem 5 from [1], pp. 358–360 it follows that \( F \in S(0, 1) \) for every natural number \( a \). Hence \( F \in S(\text{Int } K) \), where \( \gamma \in \text{Int } K \).

**Theorem 8.** Let \( P \subseteq (0, 1) \) be a closed set such that its complementary set \( (0, 1) \) has only a finite number of components \( (a_n, b_n) \). Then \( F_1 \in \mathcal{U}(\text{Int } K) \), \( F_1(x, x) = (F_1(x), x) \), and \( F_2 \) fulfills condition (m) for \( k = 1, 2, \ldots, m \), then the function \( F(x, x) = (F(x), x) \) for \( x \in (0, 1) \), \( \text{Int } K \).

**Proof.** We have \( F_1 \in S(0, 1) \) for \( k = 1, 2, \ldots, m \) (after the extension). In virtue of Theorem 6 from [1], pp. 341–344, \( F \in S(0, 1) \). Hence \( F \in S(\text{Int } K) \), where \( \gamma \subseteq a \). Simultaneously from Theorem 6 it follows that \( F \in S(\alpha, (0, 1), \gamma \subseteq a \).

**Definition 5.** If \( \beta \) is a natural number or a countable ordinal of the first kind, we shall say that the function \( F \in \mathcal{U}(\text{Int } K) \) is an irreducible superposition of the class \( \beta \) if and only if the following condition is fulfilled: if \( F = f + g \) and \( g \in S_P(\text{Int } K) \) and \( f \in \mathcal{U}(\text{Int } K \to R) \), then \( f \) is not one-to-one on \( g(\text{Int } K) \).

**Definition 6.** We shall say that the set \( E \subseteq R \) is an extraordinary set of order \( \beta \) for the continuous function \( F: \text{Int } K \to R \) if and only if the following condition is fulfilled: if \( F = f + g \) and \( g \in S_P(\text{Int } K) \) and \( f \in \mathcal{U}(\text{Int } K \to R) \), then \( f \) is not one-to-one on \( g(\text{Int } K) \).

**Theorem 9.** Let \( \beta \) be a natural number or a countable ordinal of the first kind. If there exists a function \( F \in S_P(\text{Int } K) \), \( F(x, x) = (F(x), x) \), where \( F \) fulfills condition (m), such that \( F \) is an irreducible superposition of the class \( \beta \), then there exists a function \( F \in \mathcal{U}(\text{Int } K) \), \( F(x, x) = (F(x), x) \), where \( F \) fulfills condition (m), such that \( F \) has an extraordinary set \( E \) of order \( \beta - 1 \) and \( E = \text{Int } K \times (0, 1) \), where \( E \subseteq (0, 1) \) is of type \( G_\beta \) and is dense in \( (0, 1) \).

**Proof.** Let \( p: (0, 1) \to (0, 1) \) be a function having the following properties: fulfills condition (m) and the Lipschitz condition and \( p \) is not monotone in any interval included in \( (0, 1) \). The existence of such a function was shown in [1], pp. 345–346. From the fact that \( p \) is not monotone in any interval it follows that for every open interval \( D \subseteq (0, 1) \) and \( A \subseteq (0, 1) \) such that \( p(A) = D \) there exist an open interval \( D \subseteq D \) and three disjoint open intervals \( a, b, c \) such that \( a \subseteq b \subseteq c \subseteq D \). For the closures obviously have \( p(a) = p(b) = p(c) = p(D) \). For the closures we obviously have \( p(a) = p(b) = p(c) = p(D) \) (if the end-point of \( a, b, c \) coincides with the end-point of \( (0, 1) \), then we put \( p(0) = 0, p(1) = 1 \).
Using the above property, one can construct a sequence of disjoint open intervals \((a_n^{(k)})\) and three sequences of disjoint open intervals \((a_n^{(k)})\), \((b_n^{(k)})\), \((c_n^{(k)})\) such that the set \(S_n = \bigcup_{k=1}^{\infty} c_n^{(k)}\) is dense in \((0, 1)\), \(d_n^{(k)} = p(a_n^{(k)}) = p(c_n^{(k)})\) for every natural number \(n\) and the sets \(\bigcup_{n=1}^{\infty} a_n^{(k)}, \bigcup_{n=1}^{\infty} b_n^{(k)}, \bigcup_{n=1}^{\infty} c_n^{(k)}\) are disjoint.

Repeating this construction for every \(c_n^{(k)}\) and \(d_n^{(k)}\) as \(D\) and \(A\), one can construct a sequence of disjoint open intervals \((d_n^{(k)})\) and three sequences of disjoint open intervals \((a_n^{(k)})\), \((b_n^{(k)})\), \((c_n^{(k)})\) such that \(d_n^{(k)} = p(a_n^{(k)}) = p(c_n^{(k)})\) for every natural number \(n\), the sets \(\bigcup_{n=1}^{\infty} a_n^{(k)}, \bigcup_{n=1}^{\infty} b_n^{(k)}, \bigcup_{n=1}^{\infty} c_n^{(k)}\) are disjoint and \(S_n = \bigcup_{k=1}^{\infty} d_n^{(k)}\) is dense in \((0, 1)\), for every \(k = 1, 2, \ldots, m+1\) and every \(n\), the sets \(\bigcup_{n=1}^{\infty} a_n^{(k)}, \bigcup_{n=1}^{\infty} b_n^{(k)}, \bigcup_{n=1}^{\infty} c_n^{(k)}\) are disjoint and \(S_n = \bigcup_{k=1}^{\infty} d_n^{(k)}\) is dense in \((0, 1)\), and \(S_m \supset S_{m-1} \supset \ldots \supset S_1\), then using repeatedly the above property of \(p\) for every \(d_n^{(k+1)}\) and \(d_n^{(k-1)}\) as \(D\) and \(A\) we construct four sequences of disjoint open intervals \((a_n^{(k)})\), \((b_n^{(k)})\), \((c_n^{(k)})\), \((d_n^{(k)})\) such that \(a_n^{(k)} = p(a_n^{(k)}) = p(c_n^{(k)})\) for every natural number \(n\), the sets \(\bigcup_{n=1}^{\infty} a_n^{(k)}, \bigcup_{n=1}^{\infty} b_n^{(k)}, \bigcup_{n=1}^{\infty} c_n^{(k)}, \bigcup_{n=1}^{\infty} d_n^{(k)}\) are disjoint and the set \(S_n = \bigcup_{k=1}^{\infty} d_n^{(k)}\) is dense in \((0, 1)\), for every \(k = 1, 2, \ldots, m+1\) and every \(n\).

Thus, by induction, we have constructed such sequences and sets \(S_n\) for every natural number \(n\).

Put \(E = \bigcup_{n=1}^{\infty} S_n\). Obviously \(E \subset (0, 1)\) is of type \(G_\delta\) and is dense in \((0, 1)\).

From the above construction it immediately follows that \(a_n^{(k)} \cap b_n^{(j)} = \emptyset\), \(a_n^{(k)} \cap c_n^{(j)} = \emptyset\), \(b_n^{(k)} \cap c_n^{(j)} = \emptyset\), where \(n \neq m\) or \(k \neq j\). Let \(P = \{0, 1\}^{\omega} \setminus \bigcup_{n=1}^{\infty} (a_n^{(k)} \cup b_n^{(k)} \cup c_n^{(k)}).\) Let us put

\[
F(x) = \begin{cases} 
 p(x) & \text{for } x^* \in P \cap (0, 1), \\
 p(b_n^{(k)}) + (p(a_n^{(k)}) - p(b_n^{(k)}))F((x_n - b_n^{(k)}) / (b_n^{(k)} - a_n^{(k)})) & \text{for } x^* \in (a_n^{(k)}, b_n^{(k)}), \\
 \text{linear and continuous on } b_n^{(k)} & \text{for } k, n = 1, 2, \ldots.
\end{cases}
\]

If \(0 \leq r \leq 1\) is the left or right end-point of some \(b_n^{(k)}\), then the requirement of continuity in \(0\) or \(1\) means the existence of limits: \(\lim_{x \to 0^+} F(x^*) = 0\), \(\lim_{x \to 1^-} F(x^*) = 1\).

From the assumption it follows that \(F^* \notin s_\gamma([0, 1])\) (after the extension) and \(F\) is an irreducible superposition (of one variable) of the class \(\beta\). From Theorem 7 of [1] (p. 345) we conclude that \(F^* \notin s_\gamma([0, 1])\). Hence \(F \in s_\gamma(I\mathbb{K})\), where \(\gamma \leq \beta\). Simultaneously in virtue of Theorem 6 \(F \in s_\gamma((0, 1])\) for every natural numbers \(n, k\), and so \(\gamma \leq \beta\). Hence \(F \in s_\gamma(I\mathbb{K})\) and finally \(F \in U_\gamma(I\mathbb{K})\).

Now we shall prove that the set \(E = E^* \times (0, 1)\) is an extraordinary set of order \(\beta - 1\) for \(F\). Let \(F = f + g\), where \(g \in S_{\gamma-1}(I\mathbb{K})\) and \(f: \gamma(I\mathbb{K}) \to \mathbb{R}\) is a continuous function. Let \((x^*, z^*) \in E\). We shall prove that \(f^{-1}((x^*, z^*))\) is an infinite set. We have \(x^* \in E^*\), and so there exists a descending sequence of intervals \((d_n^{(k)})\) such that \(x^* \in \bigcap_{n=1}^{\infty} d_n^{(k)}\). For every natural number \(k\) there exist two open intervals \(a_n^{(k)}\) and \(b_n^{(k)}\) such that \(F(a_n^{(k)}) = F(b_n^{(k)}) = d_n^{(k)}\). The function \(F\) fulfills for \(a_n^{(k)}\) and \(b_n^{(k)}\) all the assumptions of Lemma 11, and so \(g(a_n^{(k)} \times (0, 1)) = g(b_n^{(k)} \times (0, 1)) = \emptyset\). Suppose that the equality holds. Then the reduced function \(f|g(b_n^{(k)} \times (0, 1))\) is one-to-one, because \(F\) is one-to-one on \((b_n^{(k)} \times (0, 1))\). Hence \(F\) on the set \(a_n^{(k)} \times (0, 1)\) is a superposition of the function \(g|a_n^{(k)} \times (0, 1)\} \in S_{\gamma-1}(\mathbb{K})\) and \(f|g(b_n^{(k)} \times (0, 1))\) which is continuous with the one-to-one function \(f|g(b_n^{(k)} \times (0, 1))\). This is impossible, because \(F\) is an irreducible superposition of the class \(\gamma\) on \(I\mathbb{K}\) and \(F\) on \(a_n^{(k)} \times (0, 1)\) is a superposition of \(F\) with two linear functions, and so \(F\) is an irreducible superposition of the class \(\gamma\) on \(a_n^{(k)} \times (0, 1)\). Hence \(g(a_n^{(k)} \times (0, 1)) \neq \emptyset\). From the construction it follows that \(a_n^{(k)}\), \(b_n^{(k)}\) and \(c_n^{(k)}\) are included in \(d_n^{(k)}\), because \(a_n^{(k)} \subset d_n^{(k)}\). So \(g(a_n^{(k)} \times (0, 1)) \neq \emptyset\). Similarly \(F((a_n^{(k)} \times (0, 1)) = F((a_n^{(k)} \times (0, 1)) = F((d_n^{(k)} \times (0, 1)) = \emptyset\). Consequently \(F((a_n^{(k)} \times (0, 1)) = \emptyset\). Simultaneously \(F((a_n^{(k)} \times (0, 1)) = F((a_n^{(k)} \times (0, 1)) = \emptyset\). Let denote it by \(G_\delta\). We obtain a sequence of disjoint non-empty sets \(G_\delta\). We have \(f(G_\delta) = d_n^{(k)} \times (0, 1)\) and \((x^*, z^*) \in d_n^{(k)} \times (0, 1)\), and so \(f^{-1}((x^*, z^*)) \neq \emptyset\).
THEOREM 10. Let $\beta$ be a natural number or a countable ordinal of the first kind. If there exists a function $F \in U_{\beta}(IntK)$, $F(x^i, x^j) = \{F(x^i), x^j\}$, where $F^i$ fulfills condition (m), such that $F$ is an irreducible superposition of the class $\beta$, then there exists a function $\tilde{F} \in U_{\beta+1}(IntK)$, $\tilde{F}(x^i, x^j) = \{\tilde{F}(x^i), x^j\}$, where $\tilde{F}$ fulfills condition (m) and $\tilde{F}$ is an irreducible superposition of the class $\beta+1$.

Proof. From Theorem 9 it follows that there exists a function $F \in U_{\beta}(IntK)$, $F(x^i, x^j) = \{F(x^i), x^j\}$, where $F$ fulfills condition (m), having as an extraordinary set of order $\beta-1$ the set $H = F^j \times (0, 1)$, where $F^j : C(0, 1)$ is of type $G_0$ and is dense in $(0, 1)$. Let $g_1 : (0, 1) \rightarrow (0, 1)$ and $g_2 : (0, 1) \rightarrow (0, 1)$ be increasing functions such that $g_1(F^j) \subset g_2(F^j) = (0, 1)$. Such functions do exist (see [1], p. 352). Let us put $F_1(x^i, x^j) = \{(g_1 \circ F^j)(x^i), x^j\}$ for $i = 1, 2$. From Theorem 3 it follows that $F_1, F_2 \in S_{\beta}(IntK)$, where $\gamma \leq \beta+1$. Now we shall prove that the set $g_1(F^j) \times (0, 1)$ is a superposition of order $\beta-1$ for $F_1$. If $G_1 : IntK \rightarrow IntK$ is defined as $G_1(x^i, x^j) = (g_1(x^i), x^j)$, then $G_1$ is a homeomorphism onto $g_1(F^j) \times (0, 1)$ and $F_1 = G_1 \circ F$. Hence $F_1 = G_1^{-1} \circ F_2$. Let the functions $h_1 : IntK \rightarrow IntK$ and $f_1 : h_1(IntK) \rightarrow IntK$ fulfill the following conditions: $h_1 \in S_{\beta-1}(IntK)$, $f_1$ is a continuous function and $F_2 = f_1 \circ h_1$. The set $E^j = (0, 1)$ is an extraordinary set of order $\beta-1$ for $F_1$ and so for every $(x^i, x^j) \in F^j \times (0, 1)$ the set $(E^j \times f_1^{-1} \{(x^i, x^j)\})$ is infinite; thus for every $(y^j, y^k) \in g_1(F^j) \times (0, 1)$, the set $f_1^{-1}(y^j, y^k) \subseteq (0, 1)$ is a real extraordinary set of order $\beta-1$ for $F_1$. Similarly one can prove that the set $g_1(F^j) \times (0, 1)$ is an extraordinary set of order $\beta-1$ for $F_2$.

Now let us put $\tilde{F}(x^i, x^j) = \{\tilde{F}(x^i), x^j\}$ for $(x^i, x^j) \in IntK$, where $\tilde{F}(x^i, x^j) = \{\tilde{F}(x^i), x^j\}$, and the function $\tilde{F} : ((-1, 1) \rightarrow IntK)$ is defined in the following way:

$$\tilde{F}(x^i) = \begin{cases} \alpha^i & \text{for } -1 < \alpha^i \leq 0, \\ (g_1 \circ \tilde{F})(x^i) & \text{for } 0 < \alpha^i < 1, \\ 2-x^i & \text{for } 1 \leq \alpha^i \leq 2, \\ (g_2 \circ \tilde{F})(x^i-2) & \text{for } 2 < \alpha^i < 3, \\ \alpha^i-2 & \text{for } 3 \leq \alpha^i \leq 4. \end{cases}$$

We have $\tilde{F} \in S_{\beta+1}(0, 1)$ and $\tilde{F}$ is an irreducible superposition of the class $\beta+1$ after the extension (see [1], pp. 353–355). Hence $\tilde{F} \in S_{\beta}(IntK)$, where $\gamma \leq \beta+1$. To prove that $\tilde{F} \in S_{\beta+1}(IntK)$ we shall show that the set $IntK$ is an extraordinary set of order $\beta-1$ for the function $\tilde{F}(x^i, x^j)$
If for some \( n \) the equality holds, then \( r_n \) is one-to-one on \( F_4((2,5^{-1}, 3, 5^{-1}) \times \{0, 1\}) \), because \( F \) is one-to-one on \( (2,5^{-1}, 3, 5^{-1}) \times \{0, 1\} \). From this it follows that \( F \in S_{\beta}(\text{Int}K) \) in virtue of Theorem 6. This is a contradiction, for \( \beta = 1 < \beta \). Hence, for every natural number \( n \), \( F_n((2,5^{-1}, 3, 5^{-1}) \times \{0, 1\} \cap F_n((2,5^{-1}, 3, 5^{-1}) \times \{0, 1\}) = 0 \). This also leads to a contradiction, because from Theorem 1 it follows that the set \( F_n((2,5^{-1}, 3, 5^{-1}) \times \{0, 1\} \) is a region and \( \lim F_n(x^2, x^2) = F(x^2, x^2) \) for \( x^2 \in \text{Int}K \) and

If \( \gamma / \neq \beta \) so \( \gamma = \beta + 1 \) and \( F \in U_{\beta}(\text{Int}K) \).

**Theorem 13.** For every ordinal \( \beta < \Omega \) the \( S_{\beta}(\text{Int}K) \) is non-empty.

**Proof.** We shall prove that for every ordinal \( \beta < \Omega \) the smaller class \( U_{\beta}(\text{Int}K) \) is non-empty and if \( \beta \) is a natural number or a countable ordinal of the first kind, there exists an irreducible superposition of the class \( \beta \). For \( \beta = 1 \) this is obvious. If for every ordinal \( \alpha < \beta \) the class \( U_{\alpha}(\text{Int}K) \) is non-empty, then the class \( U_{\beta}(\text{Int}K) \) is non-empty:

a) In virtue of Theorem 10, if \( \beta \) is a natural number or a countable ordinal of the first kind such that \( \beta - 1 \) is a countable ordinal of the first kind.

b) In virtue of Theorem 11, if \( \beta \) is a countable ordinal of the second kind.

c) In virtue of Theorem 12, if \( \beta \) is a countable ordinal of the first kind and \( \beta - 1 \) is a countable ordinal of the second kind. The fact that the function \( F \) constructed in the proof of Theorem 12 is an irreducible superposition follows immediately from Theorem 5.

Hence, in virtue of transfinite induction, \( U_{\beta}(\text{Int}K) \neq \emptyset \) for every ordinal \( \beta < \Omega \).

**Definition 7.** The function \( F : \text{Int} K \to \mathbb{R}^2 \) is called monotone if and only if for every \( (y^1, y^2) \in F(\text{Int} K) \) the set \( F^{-1}([(y^1, y^2)]) \) is a connected set (see [3], II, 1.1, p. 4).
Suppositions of transformations of bounded variation

and for every integer n. We shall say that the function $h$ on the interval $(a, b)$ has been replaced by a polygonal line of type $P$. It is not difficult to verify that $h^*$ is a continuous function on $(a, b)$ and $h^*|_{(a, b)} \to h(a), h(b)$. If we write $b_m = 2^{-3^m}a_m + 3^{-3^m}a_m + 2^{-3^m}a_m$, then $h^*$ is a linear function on every interval $(b_m, b_{m+1})$ and it is increasing on $(b_m, b_{m+1})$ and decreasing on $(b_m, b_{m+1})$. At every $b_m$, $h^*$ has a local maximum and at every $b_{m+1}$ — a local minimum. It is easy to see that $h^*$ fulfills condition b).

In the case where $h$ is a decreasing linear function the function $h^*$ is defined similarly.

Let us observe that from the condition $a_{m+1} - a_m < 2^{-m}(b-a)$ it follows that, for every $t \in (0, 1)$, $|h^*(t) - h(t)| < 2^{-m}(b-a)$.

Now we shall construct the function $F^*$. Let $f_0(x) = x$ for $x \in (0, 1)$. To construct $f_k$ we replace $f_{k-1}$ on $(0, 1)$ by a polygonal line of type $P$ constructed for the sequence $(a_n)$ fulfilling all the conditions mentioned. Suppose that we have already constructed functions $f_0, f_1, \ldots, f_{k-1}$ such that $|f_{k-1}(x) - f_{k-2}(x)| < 2^{-k+1}$ for every $x \in (0, 1)$, the sum of intervals on which $f_{k-1}$ is a linear function is dense in $(0, 1)$ and the oscillation of $f_{k-1}$ on every such interval is not greater than $2^{-k+1}$, and $f_{k-1}$ fulfills condition b).

To construct $f_k$ we replace $f_{k-1}$ on every interval of linearity (a maximal interval, of course) by a polygonal line of type $P$ constructed for the sequence $(a_n)$ fulfilling all the conditions mentioned and the following additional one: for every natural number $n$ $f_{k-1}(a_n)$ is different from all the values taken by $f_{k-1}$ at those points where $f_{k-1}$ has an extremum. If $(a', b')$ and $(a'', b'')$ are two different intervals of linearity of $f_{k-1}$, then we choose sequences $(a'_m), (a''_m)$ in these intervals in such a way that $f_{k-1}(a'_m) \neq f_{k-1}(a''_m)$ for every integers $m, p$. The function $f_k$ constructed in this manner is a continuous function, it has a dense set of intervals of linearity, its oscillation on every such interval is not greater than $2^{-k}, f_{k-1}$ fulfills condition b) and $|f_k(x) - f_{k-1}(x)| < 2^{-k}$ for every $x \in (0, 1)$. From the construction it follows also that the length of every interval of linearity of $f_k$ is less than $2^{-k}$.

Hence, by induction, we have defined a sequence of continuous functions $\{f_k\}$, $k = 0, 1, \ldots$ This sequence is uniformly convergent. Let us put $F^*(x) = \lim_{k \to \infty} f_k(x)$ for $x \in (0, 1)$. In [1], pp. 367–368 it was proved that $F^*$ fulfills conditions a) and b). We shall prove that $F^*$ fulfills also condition c). Let $(c, d) \subset (0, 1)$ be an arbitrary interval. From the construction it follows that there is a natural number $k_0$ and the interval $(c, d) \subset (c, d)$ such that $f_{k_0}$ is linear on $(a, b)$ and not linear on any greater interval. We have $F(a, b) = f_{k_0}(a, b)$. Denote by $B$ the set of values of $F$ taken by $F^*$ at those points where $F^*$ has an extremum. For $y^* \in (0, 1)$ — B the set of values of $F$ fulfills the conditions required.
Hence for \( y^* \in \mathcal{F}^1(\{a, b\}) \) we have \((\mathcal{F}_1)^{-1}(y^*) \cap (a, b) \neq \emptyset\), and so the set \((\mathcal{F}_1)^{-1}(y^*) \cap (a, b)\) is infinite. Suppose now that \( y^* \in \mathcal{F}^1(\{a, b\}) \) and 0. There exists a point \( x^* \in (0, 1)\), at which \( \mathcal{F}^1 \) has an extremum and \( F(x) = y^*\). If neither of the points having these properties (in the first \( F^1 \) a maximum and in the second — a minimum) is in \((a, b)\), then, in the same way as in [1], pp. 368–369, one can prove that the set \((\mathcal{F}_1)^{-1}(y^*) \cap (a, b)\) is non-empty and perfect in \((a, b)\), and hence infinite. If at least one of those points belongs to \((a, b)\), then from the construction of \( F(x) \), it follows that there exists an interval \([b_k, b_{k+1}] \subset (a, b)\) such that \( F^1 \) has no extremum at any point of the set \((\mathcal{F}_1)^{-1}(y^*) \cap (b_k, b_{k+1})\). On the same way as in [1], one can prove that the set \((\mathcal{F}_1)^{-1}(y^*) \cap (b_k, b_{k+1})\) is non-empty and perfect in \([b_k, b_{k+1}]\), and so the set \((\mathcal{F}_1)^{-1}(y^*) \cap (a, b)\) is infinite. Hence \( F \) fulfills condition c).

Let \( F(x^*, y^*) = (\mathcal{F}_1(x^*), y^*) \) for \((x^*, y^*) \in \text{Int}K\). Suppose that there exists an ordinal number \( \beta < \Omega \) such that \( F \in S_0(\text{Int}K) \). It is easy to see that \( F \) fulfills all assumptions of Theorem 14. Hence there exist \( g: \text{Int}K \rightarrow \mathbb{R}^2, f: g(\text{Int}K) \rightarrow \mathbb{R}^2 \) such that \( F = f \circ g \) and \( g \) is BV in \( \text{Int}K \) and not monotone in \( \text{Int}K \) and \( f \) is continuous on \( g(\text{Int}K) \) in \( \text{Int}K \). In virtue of Theorem 1 there exists a homeomorphism \( H: g(\text{Int}K) \rightarrow \text{Int}K \) such that \((H \circ g)(x^*, y^*) = (f(x^*), y^*) \). We have also \((f \circ H^{-1})(y^*, y^*') = (f(x^*), y^*) \) for \((y^*, y^*) \in \text{Int}K\), because \( F = (f \circ H^{-1}) \circ (H \circ g) \). Hence \( F^1 = f' \circ g \).

From Lemma 3 of [1], p. 363 it follows that \( f_1 \) is one-to-one on \((0, 1)\) or there exists an open interval \( d_0 \subset f_1((0, 1)) \) on which \( f_1 \) is one-to-one. The supposition that \( f_1 \) is one-to-one leads to a contradiction, because \( g \) is not monotone on \( \text{Int}K \). Hence the second possibility ought to be fulfilled. Let \((e, d) \subset (0, 1)\) be a component of the open set \( f_1^{-1}(d_0) \) and \((a, b) \subset (e, d)\) (the interval chosen for \( F^1 \) in virtue of condition c). Let \( d_1 = f_1((a, b)) \). Obviously \( d_1 \subset d_0 \), the function \( f_2 \) is one-to-one on \( d_1 \) and it is not difficult to see that \( d_1 \) is an open interval. We have \( F^1((a, b)) = f_2(d_1) \) and for every \( y^* \in \mathcal{F}^1(\{a, b\}) \) the set \((\mathcal{F}_1)^{-1}(y^*) \cap (a, b)\) is infinite, thus for every \( x^* \in d_1 \) the set \((\mathcal{F}_1)^{-1}(x^*) \cap (a, b)\) is infinite. Hence \( \mathcal{N}((a, b); H \circ g, (a, b) \times (0, 1)) = +\infty \) for \((x^*, y^*) \in (0, 1) \times (0, 1)\) and \( \mathcal{N}((y^*, y^*); g, (a, b) \times (0, 1)) = +\infty \) for \((e, d) \times H^{-1}(d_1 \times (0, 1))\). Since \( H^{-1}(d_1 \times (0, 1)) \) is an open region, \( g \) is not BV in \((a, b) \times (0, 1)\) — a contradiction.

The supposition that \( F \in S_0(\text{Int}K) \) leads to a contradiction, and the theorem is proved.

**Corollary.** For every ordinal number \( \beta < \Omega \) the class \( S_0(\text{Int}K) \) is non-empty. There exists a function \( F: K \rightarrow \mathbb{R}^2 \) such that \( F \in S_0(\text{Int}K) \).

**Proof.** Let \( \beta < \Omega \) and let \( F: \text{Int}K \rightarrow \mathbb{R}^2 \) be a superposition of the class \( \beta \) constructed in the proof of Theorems 10, 11 or 12. The function \( F \) may be extended on \( K \). Let \( \tilde{F}: K \rightarrow \mathbb{R}^2 \) denote this extension. We have

\[
\tilde{F}(x^*, y^*) = \begin{cases} 
F(x^*, y^*) & \text{for } (x^*, y^*) \in K, \\
(F(x^*, y^*), y^*) & \text{for } (x^*, y^*) \in \{x^* \in (0, 1) \mid \mathcal{F}^1(x^*) = y^*\}. 
\end{cases}
\]

From Theorems 8, 10 or 11 of [1] (or from the proof of Theorems 10, 11 or 12 in this paper) it follows that \( F \in S_0(\text{Int}K) \), where \( \gamma \leq \beta \). The opposite inequality follows from the fact that \( F \in S_0(\text{Int}K) \) and \( \text{Int}K \subset K \). Finally \( F \in S_0(\text{Int}K) \).

Let \( F: \text{Int}K \rightarrow \mathbb{R}^2 \) be the function constructed in the proof of Theorem 15 and let \( F \) denote a continuous extension of \( F \) on \( K \). If \( F \in S_0(\text{Int}K) \) for some \( \beta < \Omega \), then \( F \in S_0(\text{Int}K) \), where \( \gamma \leq \beta \) — a contradiction. Hence \( F \not\in S_0(\text{Int}K) \).

**References**


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