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Superpositions of transformations of bounded variation

by

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Abstract. The work deals with some classification of continuous functions transforming plane into plane. For every finite or countable ordinal number was defined a class of superposition of functions of bounded variation (in the sense of Rado, see [3]). The main results of the work are following theorems: every class of superpositions is non-empty and there exists a continuous function which does not belong to the sum of all classes. Similar results for real functions of real variable are included in the classical work [1] by Nina Bary.

Nina Bary in [1] has studied the possibility of representing arbitrary real continuous functions of a real variable only by superpositions of continuous functions of bounded variation. She has introduced the notion of superposition of class α for every finite or countable ordinal α and she has proved that all classes of superpositions are non-empty and that their sum is not equal to the class of all continuous functions. This work contains similar results for plane transformations defined on the unit square (open or closed). The notion of transformation of bounded variation is taken from [3] and [4]. The definition of superposition of class α is similar to that in [1] if α is a countable ordinal of the first kind (i.e. having a predecessor) and differs from the definition in [1] by using uniform convergence instead of ordinary convergence if α is a countable ordinal of the second kind.

The work consists of two parts. The first part contains the proof of an auxiliary theorem which explains the structure of plane transformations F_1, F_2 such that their superposition $F = F_2 \circ F_1$ is of the form $F(x^1, x^2) = \big(f(x^1), x^2\big)$ for $(x^1, x^2) \in [0,1]\times[0,1]$. The second part contains several theorems concerning superpositions of transformations of bounded variation. The main results of the work are: Theorem 13, which states that every class of superpositions is non-empty, and Theorem 14, which gives the construction of continuous plane transformation which does not belong to the sum of all classes (both these theorems deal with transformations defined on the open unit square) and the corollary (after Theorem 14), which includes the same results for transformations defined on the closed unit square.

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Let R^2 denote the Euclidean plane. Let $K = [0,1]\times[0,1]$, $p^1(x^1) = \{x^1\}\times[0,1]$ for $x^1 \in [0,1]$, $p^2(x^2) = [0,1]\times\{x^2\}$ for $x^2 \in [0,1]$. Suppose that $F\colon K \xrightarrow[\text{onto}]{} K$ is a continuous transformation such that $F(x^1, x^2) = (f(x^1), x^2)$ for $(x^1, x^2) \in K$, where $f\colon [0,1] \xrightarrow[\text{onto}]{} [0,1]$ fulfills the following conditions: $f(0) = 0$, $f(1) = 1$, $0 < f(x^1) < 1$ for every $x^1 \in (0,1)$ and f has not an interval of constancy. Let $F = F_2 \circ F_1$, where $F_1\colon K \to R^2$, $F_2\colon F_1(K) \to R^2$ are continuous functions. To prove the main theorem of this part we shall need the following lemmas:

LEMMA 1. *If* $x_1^2, x_2^2 \in [0,1]$, $x_1^2 \neq x_2^2$, *then* $F_1(p^2(x_1^2)) \cap F_1(p^2(x_2^2)) = \emptyset$.

LEMMA 2. *For every* $x^1 \in [0,1]$ *the reduced functions* $F_1|p^1(x^1)$ *and* $F_2|F_1(p^1(x^1))$ *are homeomorphisms.*

LEMMA 3. *For every* $x^1, x^2 \in [0,1]$ *the set* $F_1(p^1(x^1)) \cap F_1(p^2(x^2))$ *contains exactly one point* $F_1(x^1, x^2)$.

LEMMA 4. $F_1(\operatorname{Int} K) \cap F_1(\operatorname{Fr} K) = \emptyset$.

The proofs of all the above lemmas, based upon the properties of F, are nearly obvious.

LEMMA 5. *Let* $L(0) \subset F_1(p^2(0))$ *be the simple arc joining* $F_1(0,0)$ *and* $F_1(1,0)$, $L(1) \subset F_1(p^2(1))$ — *the simple arc joining* $F_1(0,1)$ *and* $F_1(1,1)$ *(the sets* $F_1(p^2(0))$ *and* $F_1(p^2(1))$ *are arcwise connected as continuous images of* $[0,1]$*; see, for example,* [2], p. 245). *The set* $E = L(0) \cup L(1) \cup F_1(p^1(0)) \cup F_1(p^1(1))$ *is homeomorphic with the set* $\operatorname{Fr}([-1,1]\times[-1,1])$ *and there exists a homeomorphism* $G\colon E \xrightarrow[\text{onto}]{} \operatorname{Fr}([-1,1]\times[-1,1])$ *such that*

$$G_0(L(0)) = [-1,1]\times\{-1\}, \qquad G_0(L(1)) = [-1,1]\times\{1\},$$
$$G_0(F_1(p^1(0))) = \{-1\}\times[-1,1], \qquad G_0(F_1(p^1(1))) = \{1\}\times[-1,1].$$

Proof. In virtue of the definition or of Lemma 2 all the four terms of E are simple arcs, and so it suffices to show that

$$L(0)\cap L(1) = \emptyset, \qquad F_1(p^1(0)) \cap F_1(p^1(1)) = \emptyset,$$
$$L(0) \cap F_1(p^1(0)) = \{F_1(0,0)\}, \qquad L(0)\cap F_1(p^1(1)) = \{F_1(1,0)\},$$
$$L(1) \cap F_1(p^1(0)) = \{F_1(0,1)\}, \qquad L(1)\cap F_1(p^1(1)) = \{F_1(1,1)\}.$$

The first equality follows at once from Lemma 1. The second is a consequence of the following inclusion: $F_2(F_1(p^1(0)) \cap F_1(p^1(1))) \subset p^1(0) \cap p^1(1) = \emptyset$. The remaining equalities follow from Lemma 3.

LEMMA 6. *Let* O_1 *and* O_2 *be open regions into which* E *divides the plane according to the well-known theorem of Jordan. Then* $F_1(\operatorname{Int} K) \subset O_1$ *or* $F_1(\operatorname{Int} K) \subset O_2$.

Proof. Suppose that neither the first nor the second inclusion is fulfilled. Then in virtue of Lemma 4 there exists a point $(x_1^1, x_1^2) \in \operatorname{Int} K$ such that $F_1(x_1^1, x_1^2) \in O_1$ and there exists a point $(x_2^1, x_2^2) \in \operatorname{Int} K$ such that $F_1(x_2^1, x_2^2) \in O_2$. Let d be the segment joining those points. We have $d \subset \operatorname{Int} K$ and $F_1(d) \cap E \neq \emptyset$. This contradicts Lemma 4.

LEMMA 7. *If* O_1 *denotes the bounded region (with the notation of Lemma 6), then* $F_1(\operatorname{Int} K) \subset O_1$.

Proof. Suppose that $F_1(\operatorname{Int} K) \subset O_2$. Let $G\colon R^2 \xrightarrow[\text{onto}]{} R^2$ be a homeomorphism such that $G/E = G_0$ (such a homeomorphism exists in virtue of the theorem of Schonflies; see [2], p. 280). Obviously we can choose G such that for every $x^1 \in [0,1]$ the ordinate of $G(F_1(x^1, 0))$ is less than or equal to -1 and the ordinate of $G(F_1(x^1, 1))$ is greater than or equal to 1. We have $G(O_1) = \operatorname{Int}([-1,1]\times[-1,1])$ and $G(O_2) = R^2 - [-1,1]\times[-1,1]$. From the assumption and from the continuity of $G \circ F_1$ it follows that $G(F_1(K)) \subset R^2 - \operatorname{Int}([-1,1]\times[-1,1])$. Hence for every $x^1 \in [0,1]$ we have the inclusion $G(F_1(p^1(x^1))) \subset R^2 - \operatorname{Int}([-1,1]\times[-1,1])$, and so on the simple arc $G(F_1(p^1(x^1)))$ there exists a single-valued continuous argument (see [4], p. 385, Lemma 10). Let us put $\Delta(x^1) = \arg G(F_1(x^1, 1)) - \arg G(F_1(x^1, 0))$ for $x^1 \in [0,1]$, where the values of the argument are taken from the same single-valued argument. It is not difficult to calculate $\Delta(0) = -2^{-1}\pi$ and $\Delta(1) = 2^{-1}\pi$. The function $\Delta\colon [0,1] \to R$ is continuous on $[0,1]$ ([4], p. 390, Lemma 15), and so there exists an $x_0^1 \in [0,1]$, for which $\Delta(x_0^1) = 0$. This is impossible, because two points $G(F_1(x_0^1, 0))$ and $G(F_1(x_0^1, 1))$ cannot lie on the same half-line issuing from $(0,0)$. Hence $F_1(\operatorname{Int} K) \subset O_1$.

LEMMA 8. *For every* $x_0^2 \in (0,1)$ *the set* $F_1(p^2(x_0^2))$ *is a simple arc and* $F_1(p^2(0)) \cup F_1(p^2(1)) \subset \bar{O}_1$.

Proof. For every $x_0^2 \in (0,1)$ the set $F_1(p^2(x_0^2))$ is an arcwise connected set as a continuous image of a segment. Let $L(x_0^2) \subset F_1(p^2(x_0^2))$ be a simple arc joining $F_1(0, x_0^2)$ and $F_1(1, x_0^2)$. We shall prove that $L(x_0^2) = F_1(p^2(x_0^2))$. Let

$$E_1(x_0^2) = L(0) \cup L(x_0^2) \cup F_1(\{(0, x^2)\colon 0 \leqslant x^2 \leqslant x_0^2\}) \cup F_1(\{(1, x^2)\colon 0 \leqslant x^2 \leqslant x_0^2\}),$$

$$E_2(x_0^2) = L(x_0^2) \cup L(1) \cup F_1(\{(0, x^2)\colon x_0^2 \leqslant x^2 \leqslant 1\}) \cup F_1(\{(1, x^2)\colon x_0^2 \leqslant x \leqslant 1\}).$$

As in the proof of Lemma 5 one can prove that $E_1(x_0^2)$ and $E_2(x_0^2)$ are simple closed curves. Let $O_i(x_0^2)$ denote the region bounded by $E_i(x_0^2)$ for $i = 1, 2$. It is not difficult to see that

$$\big(O_1(x_0^2) \cup O_2(x_0^2) \cup L(x_0^2)\big) - \{F_1(0, x_0^2), F_1(1, x_0^2)\} = O_1$$

and $O_1(x_0^2) \cap O_2(x_0^2) = \emptyset$. Suppose that $F_1(p^2(x_0^2)) - L(x_0^2) \neq \emptyset$. Let $(y^1, y^2) \in F_1(p^2(x_0^2)) - L(x_0^2)$. For every $x^2 \in (0, 1)$ we have

$$F_1(p^2(x^2)) \cap E = \{F_1(0, x^2), F_1(1, x^2)\}$$

(E has the same meaning as in Lemma 5). Hence $(y^1, y^2) \in O_1(x_0^2) \cup O_2(x_0^2)$. From Lemmas 1, 3 and 7 and the last equality it follows that for $0 < x^2 < x_0^2$ we have

$$F_1(p^2(x^2)) \subset O_1(x_0^2) \cup \{F_1(0, x^2), F_1(1, x^2)\}$$

and for $x_0^2 < x^2 < 1$ we have

$$F_1(p^2(x^2)) \subset O_2(x_0^2) \cup \{F_1(0, x^2), F_1(1, x^2)\}.$$

If $(y^1, y^2) \in O_1(x_0^2)$, then from the continuity of F_1 it follows that for $x^2 \in (x_0^2, 1]$ sufficiently close to x_0^2 we have $F_1(p^2(x^2)) \cap O_1(x_0^2) \neq \emptyset$, which is impossible. The assumption that $(y^1, y^2) \in O_2(x_0^2)$ similarly leads to a contradiction, so $L(x_0^2) = F_1(p^2(x_0^2))$.

The proof of the second part of the lemma is quite analogous.

Remark. It is not difficult to construct an example showing that the sets $F_1(p^2(0))$ and $F_1(p^2(1))$ need not be simple arcs.

Now we shall define two sets, $B(0)$ and $B(1)$. A point (y^1, y^2) belongs to $B(0)$ if and only if $(y^1, y^2) \in F_1(p^2(0))$ or if there exists an open set $A \subset R^2$ such that $(y^1, y^2) \in A$ and $\operatorname{Fr} A \subset F_1(p^2(0))$. A point (y^1, y^2) belongs to $B(1)$ if and only if $(y^1, y^2) \in F_1(p^2(1))$ or if there exists an open set $A \subset R^2$ such that $(y^1, y^2) \in A$ and $\operatorname{Fr} A \subset F_1(p^2(1))$. It is not difficult to verify that $B(0)$ and $B(1)$ are continua,

$$\operatorname{Fr} B(0) \subset F_1(p^2(0)), \qquad \operatorname{Fr} B(1) \subset F_1(p^2(1))$$

and

$$B(0) \cap F_1(p^1(0)) = \{F_1(0, 0)\}, \qquad B(0) \cap F_1(p^1(1)) = \{F_1(1, 0)\},$$

$$B(1) \cap F_1(p^1(0)) = \{F_1(0, 1)\}, \qquad B(1) \cap F_1(p^1(1)) = \{F_1(1, 1)\}$$

(the last equalities follow from the proof of Lemma 5).

LEMMA 9. $F_1(\operatorname{Int} K) = \bar{O}_1 - \big(B(0) \cup B(1) \cup F_1(p^1(0)) \cup F_1(p^1(1))\big)$.

Proof. From Lemmas 4 and 7 it follows that

$$F_1(\operatorname{Int} K) \subset \bar{O}_1 - \big(F_1(p^1(0)) \cup F_1(p^1(1)) \cup F_1((p^2(0)) \cup F_1(p^2(1))\big).$$

We shall prove that $F_1(\operatorname{Int} K) \cap \operatorname{Int} B(0) = \emptyset$. Suppose that there exists a point (x^1, x^2) such that $x^1 > 0$, $x^2 > 0$ and $F_1(x^1, x^2) \in F_1(\operatorname{Int} K) \cap \operatorname{Int} B(0)$.

From the above mentioned properties of $B(0)$ it follows that $F_1(0, x^2) \notin B(0)$. Hence $F_1(p^2(x^2)) \cap \operatorname{Fr} B(0) \neq \emptyset$, so $F_1(p^2(x^2)) \cap F_1(p^2(0)) \neq \emptyset$, which contradicts Lemma 2. One can similarly prove that $F_1(\operatorname{Int} K) \cap \operatorname{Int} B(1) = \emptyset$. Hence

$$F_1(\operatorname{Int} K) \subset \bar{O}_1 - \big(B(0) \cup B(1) \cup F_1(p^1(0)) \cup F_1(p^1(1))\big).$$

Suppose now that there exists a point (y^1, y^2) belonging to the set on the right-hand side of the last inclusion and not belonging to $F_1(\operatorname{Int} K)$. Consider the homeomorphism $G\colon R^2 \xrightarrow[\text{onto}]{} R^2$ fulfilling the following conditions: $G|E = G_0$ (for the denotation see Lemma 7) and $G(B(0)) \subset [-1, 1] \times [-1, -2^{-1}]$, $G(B(1)) \subset [-1, 1) \times [2^{-1}, 1]$ and $G(y^1, y^2) = (0, 0)$. It is not difficult to see that such a homeomorphism does exist. We have $(0, 0) \in [-1, 1] \times [-1, 1] - G(F_1(K))$, and so on every simple arc $G(F_1(p^1(x^1)))$ there exists a single-valued continuous argument. As in the proof of Lemma 7, let us put $\Delta(x^1) = \arg G(F_1(x^1, 1)) - \arg G(F_1(x^1, 0))$ for $x^1 \in [0, 1]$. The function $\Delta\colon [0, 1] \to R$ is continuous on $[0, 1]$ and $\Delta(0) = -2^{-1}\pi$, $\Delta(1) = 2^{-1}\pi$, and so there exists an $x_0^1 \in [0, 1]$ for which $\Delta(x_0^1) = 0$. This is impossible, because points $G(F_1(x_0^1, 0))$ and $G(F_1(x_0^1, 1))$ cannot lie on the same half-line issuing from $(0, 0)$. So the lemma is proved.

LEMMA 10. *If* $x_1^1, x_2^1 \in [0, 1]$, $x_1^1 \neq x_2^1$, *then* $F_1(p^1(x_1^1)) \cap F_1(p^1(x_2^1)) \subset \{F_1(x_1^1, 0), F_1(x_1^1, 1)\}$ *or* $F_1(p^1(x_1^1)) = F_1(p^1(x_2^1))$.

Proof. Let us observe that if $F_1(x_1^1, x^2) \in F_1(p^1(x_2^1))$ for some $x^2 \in [0, 1]$ and $x_1^1 \neq x_2^1$, then from Lemma 1 it follows that $F_1(x_1^1, x^2) = F_1(x_2^1, x^2)$. Suppose that $x_1^1 \neq x_2^1$ and that there exists a number $x_0^2 \in (0, 1)$ such that $F_1(x_1^1, x_0^2) = F_1(x_2^1, x_0^2)$. We shall now prove that $F_1(x_1^1, x^2) = F_1(x_2^1, x^2)$ for every $x^2 \in (0, x_0^2)$. Let (x_2^2, x_1^2) (or $[x_2^2, x_1^2)$, if $x_2^2 = 0$) denote the component of the set $\{x^2\colon F_1(x_1^1, x^2) \neq F_1(x_2^1, x^2), x^2 < x_0^2\}$ (if this set is non-empty; otherwise there is nothing to prove). Let C denote the region whose boundary is a simple closed curve consisting of the parts of two simple arcs $F_1(p^1(x_1^1))$ and $F_1(p_1(x_2^1))$ included between $F_1(x_1^1, x_2^2)$ and $F_1(x_1^1, x_1^2)$ if $x_2^2 > 0$. If $x_2^2 = 0$, then C denote the region with the boundary consisting of the parts of the simple arcs $F_1(p^1(x_1^1))$ and $F_1(p^1(x_2^1))$ included between $F_1(x_1^1, 0)$, $F_1(x_1^1, x_1^2)$ and $F_1(x_2^1, 0)$, $F_1(x_2^1, x_1^2)$, respectively, and of a suitable subset of $F_1(p^2(0))$ chosen in such a way that $C \cap B(0) = \emptyset$ (in this case the boundary of C need not be a simple closed curve). From Lemma 9 it follows that $C \subset F_1(\operatorname{Int} K)$ in both cases. Let $(x_3^1, x_3^2) \in \operatorname{Int} K$ be a point such that $F_1(x_3^1, x_3^2) \in C$. From the continuity of F_1 it follows that there exists an $\varepsilon > 0$ such that for every $x^1 \in (x_3^1 - \varepsilon, x_3^1 + \varepsilon)$ we have $F_1(x^1, x_3^2) \in C$. The end-points of the simple arc $F_1(p^1(x^1))$ belong to $F_1(p^2(0))$ and $F_1(p^2(1))$; hence for every $x^1 \in (x_3^1 - \varepsilon, x_3^1 + \varepsilon)$ we have $F_1(p^1(x^1)) \cap \big(F_1(p^1(x_1^1)) \cup F_1(p^1(x_2^1))\big) \neq \emptyset$. From this we conclude that $F(p^1(x^1))$

$=F(p^1(x_1^1))=F(p^1(x_2^1))$ for $x^1\in(x_3^1-\varepsilon,x_3^1+\varepsilon)$, and so f has an interval of constancy — a contradiction. Hence $F_1(x_1^1,x^2)=F_1(x_2^1,x^2)$ for $x^2\in(0,x_0^2)$. Similarly one can prove that this equality holds for $x^2\in(x_0^2,1)$. The lemma is proved.

THEOREM 1. *If the continuous function* $F\colon K\xrightarrow[\text{onto}]{}K$ *is of the form* $F(x^1,x^2)=(f(x^1),x^2)$ *for* $(x^1,x^2)\in K$, *where* $f\colon[0,1]\xrightarrow[\text{onto}]{}[0,1]$ *fulfils the following conditions:* $f(0)=0$, $f(1)=1$, $0<f(x^1)<1$ *for every* $x^1\in(0,1)$ *and* f *has no interval of constancy and if* $F=F_2\circ F_1$, *where* $F_1\colon K\to R^2$, $F_2\colon F_1(K)\xrightarrow[\text{onto}]{}K$ *are continuous functions, then there exists a homeomorphism* $H\colon F_1(\operatorname{Int}K)\xrightarrow[\text{onto}]{}\operatorname{Int}K$ *such that* $(t^1,t^2)=(H\circ F_1)(x^1,x^2)=(f^1(x^1),x^2)$ *for* $(x^1,x^2)\in\operatorname{Int}K$.

Proof. Let $(y^1,y^2)\in F_1(\operatorname{Int}K)$. Consider the set $F_1^{-1}(\{(y^1,y^2)\})$. In virtue of Lemma 1 if $(x_1^1,x_1^2),(x_2^1,x_2^2)\in F_1^{-1}(\{(y^1,y^2)\})$, then $x_1^2=x_2^2$. Let $t^2(y^1,y^2)$ be equal to the common value of the ordinates of all points belonging to $F_1^{-1}(\{(y^1,y^2)\})$. To define the function t^1 we shall use the fact that for every $x^2\in(0,1)$ the set $F_1(p^2(x^2))$ is a simple arc (Lemma 8). Let us fix a number $x_0^2\in(0,1)$ and let $g\colon[0,1]\xrightarrow[\text{onto}]{}F_1(p^2(x_0^2))$ be a homeomorphism such that $g(0)\in F_1(p^1(0))$. There exists at least one number $x_0^1\in(0,1)$ such that $(y^1,y^2)\in F_1(p^1(x_0^1))$. From Lemma 3 it follows that the set $F_1(p^1(x_0^1))\cap F_1(p^2(x_0^2))$ has exactly one point (y_0^1,y_0^2). Let us put $t^1(y^1,y^2)=g^{-1}(y_0^1,y_0^2)$. The function t^1 is defined unambiguously in virtue of Lemma 10. Let $H(y^1,y^2)=(t^1(y^1,y^2),t^2(y^1,y^2))$ for $(y^1,y^2)\in F_1(\operatorname{Int}K)$. It is not difficult to see that $(H\circ F_1)(x^1,x^2)=(f^1(x^1),x^2)$ for $(x^1,x^2)\in\operatorname{Int}K$, where f^1 is some function.

Using Lemmas 1, 3 and 10 one can easily verify that H is a one-to-one transformation. Also H transforms $F_1(\operatorname{Int}K)$ onto $\operatorname{Int}K$. To finish the proof we shall prove that H and H^{-1} are continuous functions. Let $\{(t_n^1,t_n^2)\}$, $n=1,2,\ldots$, be a sequence of points of $\operatorname{Int}K$ converging to the point $(t_0^1,t_0^2)\in\operatorname{Int}K$. Suppose additionally that the sequence $\{t_n^1\}$ is non-decreasing. Let

$$x_0^1=\inf\{x^1\colon g(t_0^1)\in F_1(p^1(x^1))\}\quad\text{and}\quad x_n^1=\sup\{x^1\colon x^1<x_0^1,\ g(t_n^1)\in F_1(p^1(x^1))\}$$

for $n=1,2,\ldots$ We obviously have $x_n^1\leqslant x_0^1$ for $n=1,2,\ldots$ and the sequence $\{x_n^1\}$ is non-decreasing. Hence $\lim\limits_{n\to\infty}x_n^1\leqslant x_0^1$. If $\lim\limits_{n\to\infty}x_n^1<x_0^1$, then from the continuity of g and F_1 we have $g(t_0^1)\in F_1(p^1(\lim\limits_{n\to\infty}x_n^1))$, which contradicts the definition of x_0^1. Hence $\lim\limits_{n\to\infty}x_n^1=x_0^1$. From the definition of H it follows that $H^{-1}(t_n^1,t_n^2)=F_1(x_n^1,t_n^2)$ for $n=1,2,\ldots$ and $H^{-1}(t_0^1,t_0^2)=F_1(x_0^1,t_0^2)$. Since $\lim\limits_{n\to\infty}(x_n^1,t_n^2)=(x_0^1,t_0^2)$ and F_1 is a continuous function, we have $\lim\limits_{n\to\infty}H^{-1}(t_n^1,t_n^2)=H^{-1}(t_0^1,t_0^2)$.

In the case where the sequence $\{t_n^1\}$ is non-increasing the proof is analogous, and so H^{-1} is a continuous function on $\operatorname{Int}K$. Hence H is a continuous function on $F_1(\operatorname{Int}K)$ (see [3], p. 31, I.2.49).

II

DEFINITION. The *crude multiplicity function of continuous plane transformation* $F\colon E\to R^2$, $E\subset R^2$ with respect to the set $D\subset E$ is a function defined as follows: for $(y^1,y^2)\in R^2$ a number $N((y^1,y^2);F;D)$ is equal to the number of elements of the set $F^{-1}(\{(y^1,y^2)\})\cap D$, if this set is finite, and to $+\infty$ if this set is infinite ([3], p. 267, [4], p. 5).

DEFINITION 2. The continuous function $F\colon E\to R^2$, $E\subset E^2$ is said to be of *bounded variation in* $D\subset E$ *in the Banach sense* (BVB in D) if and only if $\iint\limits_{R^2}N((y^1,y^2);F;D)\,dy^1dy^2<+\infty$ ([3], pp. 311–312, [4], pp. 278–280).

Let us note the following simple theorem on the variation of functions of a special type:

THEOREM 2. *If* $F(x^1,x^2)=(F^1(x^1),F^2(x^2))$ *for* $(x^1,x^2)\in K$, F *is a continuous function and neither* F^1 *nor* F^2 *is a constant, then* F *is* BVB *in* K *if and only if* F^1 *and* F^2 *are of bounded variation in* $[0,1]$.

Proof. We have $N((y^1,y^2);F;K)=N_1(y^1)N_2(y^2)$, where N_1, N_2 are Banach indicatrices of F^1 and F^2, respectively (we assume $0\cdot\infty=0$). Hence $\iint\limits_{R^2}N=\int\limits_R N_1\cdot\int\limits_R N_2$ and the number on the right-hand side of the equality is different from zero. The rest of the proof is obvious.

DEFINITION 3. We shall say that a function $F\colon E\to R^2$, $E\subset R^2$ is a *superposition of the class* 0 on the set E ($F\in S_0(E)$) if and only if $F(x^1,x^2)=(x^1,x^2)$ for $(x^1,x^2)\in E$. We shall say that a continuous function $F\colon E\to R^2$, $E\subset R^2$ is a *superposition of the class* 1 on the set E ($F\in S_1(E)$) if and only if F is BVB in E and $F\notin S_0(E)$. If β is a natural number or a countable ordinal of the first kind (i.e., its predecessor $\beta-1$ exists), then we shall say that a continuous function $F\colon E\to R^2$, $E\subset R^2$ is a *superposition of the class* β on the set E ($F\in S_\beta(E)$) if and only if there exist two functions f and g such that $g\colon E\to R^2$, $f\colon g(E)\to R^2$, $g\in S_{\beta-1}(E)$, $f\in S_1(g(E))$, $F=f\circ g$ and $F\notin\bigcup\limits_{\alpha<\beta}S_\alpha(E)$. If β is a countable ordinal of the second kind, then we shall say that a continuous function $F\colon E\to R^2$, $E\subset R^2$ is a *superposition of the class* β on the set E if and only if there exists a sequence of functions $\{g_n\}$ such that $g_1\colon E\to R^2$, $g_{n+1}\colon g_n(g_{n-1}(\ldots(g_1(E))\ldots))\to R^2$ for $n=1,2,\ldots$ and $F(x^1,x^2)=\lim\limits_{n\to\infty}F_n(x^1,x^2)$ uniformly on E, where $F_1=g_1$, $F_{n+1}=g_{n+1}\circ F_n$ for $n=1,2,\ldots$ and $g_n\in S_{\alpha_n}(g_{n-1}(\ldots(g_1(E))\ldots))$, $F_n\in S_{\beta_n}(E)$, where $\alpha_n < \beta$ for $n = 1, 2, \ldots$ and $\beta_n < \beta$ for $n = 2, 3, \ldots$ and $F \notin \bigcup_{\alpha<\beta} S_\alpha(E)$.

Classes of superpositions of real functions of a real variable are defined analogously (cf. [1], pp. 327–328) and will be denoted by $s_\beta(A)$ for $A \subset R$.

DEFINITION. Let β be a natural number or a countable ordinal. We shall say that a continuous function F: $\operatorname{Int} K \to R^2$ is a *simplified superposition of the class* β *on* $\operatorname{Int} K \big(F \in U_\beta(\operatorname{Int} K)\big)$ if and only if $F(x^1, x^2) = \big(F^1(x^1), x^2\big)$ for $(x^1, x^2) \in \operatorname{Int} K$, $F \in S_\beta(\operatorname{Int} K)$ and F^1 may be continuously extended onto $[0, 1]$ and this extension belongs to $s_\beta([0, 1])$. One can define in an analogous manner a simplified superposition on an arbitrary open interval in R^2.

Almost all the functions constructed in this paper will be simplified superpositions.

THEOREM 3. *If* g: $E \to R^2$, $E \subset R^2$ *is a superposition of the class* β *on* E *and* f: $g(E) \to R^2$ *is a superposition of the class* α *on* $g(E)$, *then the function* $F = f \circ g$ *is a superposition of the class* $\gamma \leqslant \beta + \alpha$ *on* E.

The proof of this theorem proceeds by transfinite induction on α similarly to the proof of Theorem 1 in [1].

THEOREM 4. *If* g: $E \to R^2$ *is a continuous and one-to-one function in the open region* $E \subset R^2$ *and* f: $g(E) \to R^2$ *is a superposition of the class* $\alpha > 0$ *on* $g(E)$, *then the function* $F = f \circ g$ *is a superposition of the class* α *on* E *or* $F \in S_0(E)$.

Proof. If $g \in S_0(F)$, then the theorem is obviously fulfilled. Suppose that $g \in S_1(E)$. In virtue of Theorem 3 we have $F \in S_\gamma(E)$, where $\gamma \leqslant 1 + \alpha$. If α is a countable ordinal, then $1 + \alpha = \alpha$. If $\alpha = n$ is a natural number, then $f = f_n \circ \big(f_{n-1} \circ (\ldots \circ (f_2 \circ f_1)\ldots)\big)$, where f_i is of bounded variation on $f_{i-1}\big(\ldots(f_1(E)\ldots)\big)$ for $i = 2, 3, \ldots, n$ and f_1 is of bounded variation on $g(E)$. Hence $F = f_n \circ \big(f_{n-1} \circ (\ldots \circ (f_1 \circ g)\ldots)\big)$. From the obvious equality

$$N\big((y^1, y^2); f_1 \circ g; E\big) = N\big((y^1, y^2); f_1; g(E)\big)$$

it follows that $f_1 \circ g$ is of bounded variation on E. Hence $F \in S_k(E)$, where $k \leqslant n = \alpha$. So in both cases $F \in S_\gamma(E)$, where $\gamma \leqslant \alpha$. The inverse inequality follows from the fact that $f = F \circ g^{-1}$ and g^{-1} is continuous and one-to-one ([3], p. 31). In the proofs of these inequalities the fact that α and γ are greater than zero plays an essential role. If f is a superposition of the class 1 and is one-to-one, then for $g = f^{-1}$ the function $F = f \circ g$ belongs to the class 0.

THEOREM 5. *Let* β *be a countable ordinal of the second kind. If* g: $E \to R^2$ *is a superposition of the class* β *on* E, *the set* $g(E)$ *is an open region and* f: $\overline{g(E)} \to R^2$ *is an uniformly continuous and one-to-one function, then the function* $F = f \circ g$ *is a superposition of the class* β *on* E.

Proof. We have $g = \lim_{n\to\infty} g_n$, where $g_1 = h_1$, $g_{n+1} = h_{n+1} \circ g_n$ for $n = 1, 2, \ldots$, h_n and g_n are superpositions belonging to the classes α_n and β_n, respectively and $\alpha_n < \beta$, $\beta_n < \beta$ for $n = 1, 2, \ldots$ If $F_n = f \circ g_n$, then $g_n = f^{-1} \circ F_n$ for $n = 1, 2, \ldots$, and so $F_n = f \circ g_n = f \circ (h_n \circ g_{n-1}) = f \circ \big(h_n \circ (f^{-1} \circ F_{n-1})\big)$ for $n = 2, 3, \ldots$ Put $f_n = f \circ (h_n \circ f^{-1})$. We have $F_n = f_n \circ F_{n-1}$ for $n = 2, 3, \ldots$ and $F_1 = f \circ g_1$. Hence $F_1 \in S_{\gamma_1}(E)$, where $\gamma_1 < \beta_1 + 1 < \beta$. Also, we have $h_n \circ f^{-1} \in S_{\alpha_n}\big(F(E)\big)$ or $h_n \circ f^{-1} \in S_0\big(F(E)\big)$, and so $f_n \in S_{\delta_n}\big(E(E)\big)$, where $\delta_n \leqslant \alpha_n + 1 < \beta$ for $n = 2, 3, \ldots$ Simultaneously $\lim_{n\to\infty} F_n(x^1, x^2) = \lim_{n\to\infty} f\big(g_n(x^1, x^2)\big) = f\big(g(x^1, x^2)\big) = F(x^1, x^2)$ and the sequence is uniformly convergent, and, in virtue of Theorem 3, $F_n \in S_{\gamma_n}(E)$, where $\gamma_n \leqslant \beta_n + 1 < \beta$; hence $F \in S_\gamma(E)$, where $\gamma \leqslant \beta$.

The inequality $\beta \leqslant \gamma$ follows from the fact that $g = f^{-1} \circ F$ and f^{-1} is a continuous and one-to-one function. Thus finally $\gamma = \beta$.

THEOREM 6. *If* F: $\operatorname{Int} K \to R^2$ *is a simplified superposition of the class* β *on* $\operatorname{Int} K$, $F(x^1, x^2) = \big(F^1(x^1), x^2\big)$ *and* F^1 *is not a linear function, then for every real numbers* $0 \neq a, b$ *the function* $F_1(x^1, x^2) = \big(aF^1(x^1) + b, x^2\big)$ *is a simplified superposition of the class* β *on* $\operatorname{Int} K$ *and the function* $F_2(x^1, x^2) = \big(F^1(ax^1 + b), x^2\big)$ *is a simplified superposition of the class* β *on* $\big(-b \cdot a^{-1}, (1-b) \cdot a^{-1}\big) \times (0, 1)$.

Proof. If β is a natural number or a countable ordinal of the first kind, then $F = f \circ g$, where g is a superposition of the class $(\beta - 1)$ on $\operatorname{Int} K$ and f: $g(\operatorname{Int} K) \to R^2$ is BVB on $g(\operatorname{Int} K)$. We have $F_1 = G \circ F$, where $G(y^1, y^2) = (ay^1 + b, y^2)$; hence $F_1 = (G \circ f) \circ g$. It is not difficult to verify that $N\big((z^1, z^2); G \circ f; g(\operatorname{Int} K)\big) = N\Big(\big((z^1 - b) \cdot a^{-1}, z^2\big); f; g(\operatorname{Int} K)\Big)$, and so $G \circ f$ is BVB on $g(\operatorname{Int} K)$. Hence $F_1 \in S_\gamma(\operatorname{Int} K)$, where $\gamma \leqslant \beta$. To prove the equality it suffices only to observe that $F = G^{-1} \circ F_1$ and G^{-1} also transforms linearly the abscissa.

If β is a countable ordinal of the second kind, then from Theorem 5 we have $F_1 \in S_\beta(\operatorname{Int} K)$.

Simultaneously the function $aF^1(x^1) + b$ (after extension) is a superposition of the class β in virtue of a theorem from [1], § 4, pp. 333–334. Hence $F_1 \in U_\beta(\operatorname{Int} K)$.

The second part of the theorem concerning F_2 is a consequence of Theorem 4.

LEMMA 11. *Let* $F(x^1, x^2) = \big(F^1(x^1), x^2\big)$ *for* $(x^1, x^2) \in \operatorname{Int} K$. *Let* $d_1 = (a_1, b_1)$, $d_2 = (a_2, b_2)$ *be disjoint intervals included in* $(0, 1)$. *Suppose* *that* F^1 *fulfils the following conditions:* $\lim_{x^1\to 0^+} F^1(x^1) = 0$, $\lim_{x^1\to 1^-} F^1(x^1) = 1$, $0 < F^1(x^1) < 1$ *for* $x^1 \in (0, 1)$, $F^1(d_1) = F^1(d_2) = \big(\min(F^1(a_1), F^1(b_1)), \max(F^1(a_1), F^1(b_1))\big) = \big(\min(F^1(a_2), F^1(b_2)), \max(F^1(a_2), F^1(b_2))\big)$ *(if the end-point of* d_1 *or* d_2 *coincides with the end-point of* $(0,1)$, *then by the value of* F^1 *we shall mean the limit). If* $F = f\circ g$, *then the sets* $g(d_1\times(0,1))$, $g(d_2\times(0,1))$ *are open regions and* $g(d_1\times(0,1)) = g(d_2\times(0,1))$ *or* $g(d_1\times(0,1)) \cap g(d_2\times(0,1)) = \emptyset$.

Proof. Let H: $g(\operatorname{Int}K) \xrightarrow[\text{onto}]{} \operatorname{Int}K$ be a homeomorphism such that $H(g(x^1, x^2)) = (g^1(x^1), x^2)$ for $(x^1, x^2) \in \operatorname{Int}K$. Such a homeomorphism does exist in virtue of Theorem 1. We have $F = (f\circ H^{-1})\circ(H\circ g)$, and so $F^1 = f^1\circ g^1$, where f^1 is the first coordinate of $f\circ H^{-1}$ and f is a function of one variable. From Lemma 2 of [1], pp. 337–338 it follows that $g^1(d_1) = g^1(d_2)$ or $g^1(d_1)\cap g^1(d_2) = \emptyset$. Hence $H(g(d_1\times(0,1))) = H(g(d_2\times(0,1)))$ or $H(g(d_1\times(0,1))) \cap H(g(d_2\times(0,1))) = \emptyset$. Furthermore the sets $H(g(d_1\times(0,1)))$ and $H(g(d_2\times(0,1)))$ are non-degenerate open rectangles. The rest of the proof follows from the fact that H is a homeomorphism.

In the sequel we shall frequently use real functions fulfilling the following conditions: $\lim_{x^1\to 0^+} f(x^1) = 0$, $\lim_{x^1\to 1^-} f(x^1) = 1$, $0 < f(x^1) < 1$ for $x^1 \in (0,1)$. To abbreviate the notation we shall say that such a function fulfils condition (m).

THEOREM 7. *If* $F \in U_\alpha(\operatorname{Int}K)$, $F(x^1, x^2) = (F^1(x^1), x^2)$, *where* F^1 *fulfils condition* (m), $P \subset [0,1]$ *is a closed set and* $\{(a_n, b_n)\}$ *is a sequence of components of* $[0,1]-P$, *then the function* $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$ *for* $(x^1, x^2) \in \operatorname{Int}K$, *where*

$$\overline{F}^1(x^1) = \begin{cases} x^1 & \text{for } x^1 \in P\cap(0,1), \\ a_n + (b_n - a_n)F^1\left(\dfrac{x^1-a_n}{b_n-a_n}\right) & \text{for } a_n < x^1 < b_n, \end{cases}$$

is also a simplified superposition of the class α *on* $\operatorname{Int}K$.

Proof. From the fact that $F\in U_\alpha(\operatorname{Int}K)$ it follows that $F^1 \in s_\alpha([0,1])$. In virtue of Theorem 5 from [1] (pp. 338–340) $\overline{F}^1 \in s_\alpha([0,1])$. Hence $\overline{F} \in S_\gamma(\operatorname{Int}K)$, where $\gamma \leqslant \alpha$. Simultaneously from Theorem 6 it follows that $\overline{F} \in S_\alpha((a_n, b_n)\times(0,1))$ for every natural number n. Hence $\gamma = \alpha$, because $(a_n, b_n)\times(0,1) \subset \operatorname{Int}K$. Thus $\overline{F} \in U_\alpha(\operatorname{Int}K)$.

THEOREM 8. *Let* $P \subset [0,1]$ *be a closed set such that its complementary set* $[0,1]-P$ *has only a finite number of components* (a_k, b_k), $k = 1, 2, \ldots, m$. *If* $F_k \in U_{\alpha_k}(\operatorname{Int}K)$, $F_k(x^1, x^2) = (F_k^1(x^1), x^2)$ *and* F_k^1 *fulfils condition* (m) *for* $k = 1, 2, \ldots, m$, *then the function* $F(x^1, x^2) = (F^1(x^1), x^2)$ *for* $(x^1, x^2) \in \operatorname{Int}K$, *where*

$$F^1(x^1) = \begin{cases} x^1 & \text{for } x^1 \in P\cap(0,1), \\ a_k + (b_k - a_k)F_k^1\left(\dfrac{x^1-a_k}{b_k-a_k}\right) & \text{for } x^1 \in (a_k, b_k),\ k = 1, 2, \ldots, m, \end{cases}$$

is a simplified superposition of the class $\alpha = \max(\alpha_1, \alpha_2, \ldots, \alpha_m)$ *on* $\operatorname{Int}K$.

Proof. We have $F_k^1 \in s_{\alpha_k}([0,1])$ for $k = 1, 2, \ldots, m$ (after the extension). In virtue of Theorem 6 from [1], pp. 341–344, $F^1 \in s_\alpha([0,1])$. Hence $F \in S_\gamma(\operatorname{Int}K)$, where $\gamma \leqslant \alpha$. Simultaneously from Theorem 6 it follows that $F \in S_{\alpha_k}((a_k, b_k)\times(0,1))$, and so $\gamma \geqslant \alpha_k$ for $k = 1, 2, \ldots, m$. Hence $\alpha = \gamma$ and $F \in U_\alpha(\operatorname{Int}K)$.

DEFINITION 5. If β is a natural number or a countable ordinal of the first kind, we shall say that the function $F \in U_\beta(\operatorname{Int}K)$ is an *irreducible superposition of the class* β if and only if the following condition is fulfilled: if $F = f\circ g$ and $g \in S_{\beta-1}(\operatorname{Int}K)$ and f: $g(\operatorname{Int}K) \to R^2$ is a continuous function, then f is not one-to-one on $g(\operatorname{Int}K)$.

DEFINITION 6. We shall say that the set $E \subset R^2$ is an *extraordinary set of order* β for the continuous function F: $\operatorname{Int}K \to R^2$ if and only if the following condition is fulfilled: if $F = f\circ g$, where $g \in S_\beta(\operatorname{Int}K)$ and f: $g(\operatorname{Int}K) \to R^2$ is a continuous function, then for every $(y^1, y^2) \in E$ the set $f^{-1}(\{(y^1, y^2)\})$ is infinite.

One can similarly define irreducible superpositions and extraordinary sets for real functions of a real variable (see [1], pp. 344–345).

THEOREM 9. *Let* β *be a natural number or a countable ordinal of the first kind. If there exists a function* $F \in U_\beta(\operatorname{Int}K)$, $F(x^1, x^2) = (F^1(x^1), x^2)$, *where* F^1 *fulfils condition* (m), *such that* F *is an irreducible superposition of the class* β, *then there exists a function* $\overline{F} \in U_\beta(\operatorname{Int}K)$, $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$, *where* $\overline{F}^1$ *fulfils condition* (m), *such that* $\overline{F}$ *has an extraordinary set* E *of order* $\beta-1$ *and* $E = E^1\times(0,1)$, *where* $E^1 \subset (0,1)$ *is of type* G_δ *and is dense in* $(0,1)$.

Proof. Let p: $(0,1)\to(0,1)$ be a function having the following properties: fulfils condition (m) and the Lipschitz condition and p is not monotone in any interval included in $(0,1)$. The existence of such a function was shown in [1], pp. 345–346. From the fact that p is not monotone in any interval it follows that for every open interval $D \subset (0,1)$ and $A \subset (0,1)$ such that $p(A) = D$ there exist an open interval $d \subset D$ and three disjoint open intervals a, b, c such that $a\cup b\cup c \subset A$ and $p(a) = p(b) = p(c) = d$ ([1], p. 346). For the closures we obviously have $p(\bar{a}) = p(\bar{b}) = p(\bar{c}) = \bar{d}$ (if the end-point of a, b or c coincides with the end-point of $(0,1)$, then we put $p(0) = 0$, $p(1) = 1$).

Using the above property, one can construct a sequence of disjoint open intervals $\{d_n^{(1)}\}$ and three sequences of disjoint open intervals $\{a_n^{(1)}\}$, $\{b_n^{(1)}\}$, $\{c_n^{(1)}\}$ such that the set $S_1 = \bigcup_{n=1}^{\infty} d_n^{(1)}$ is dense in $(0, 1)$, $d_n^{(1)} = p(a_n^{(1)}) = p(b_n^{(1)}) = p(c_n^{(1)})$ for every natural number n and the sets $\bigcup_{n=1}^{\infty} a_n^{(1)}$, $\bigcup_{n=1}^{\infty} b_n^{(1)}$, $\bigcup_{n=1}^{\infty} c_n^{(1)}$ are disjoint.

Repeating this construction for every $d_n^{(1)}$ and $c_n^{(1)}$ as D and A, one can construct a sequence of disjoint open intervals $\{d_n^{(2)}\}$ and three sequences of disjoint open intervals $\{a_n^{(2)}\}$, $\{b_n^{(2)}\}$, $\{c_n^{(2)}\}$ such that $d_n^{(2)} = p(a_n^{(2)}) = p(b_n^{(2)}) = p(c_n^{(2)})$ for every natural number n, the sets $\bigcup_{n=1}^{\infty} a_n^{(2)}$, $\bigcup_{n=1}^{\infty} b_n^{(2)}$, $\bigcup_{n=1}^{\infty} c_n^{(2)}$ are disjoint and the set $S_2 = \bigcup_{n=1}^{\infty} d_n^{(2)} \subset S_1$ is dense in $(0, 1)$.

If for $k = 1, 2, \ldots, m-1$ we have constructed sequences of disjoint open intervals $\{a_n^{(k)}\}$, $\{b_n^{(k)}\}$, $\{c_n^{(k)}\}$ and $\{d_n^{(k)}\}$ such that $d_n^{(k)} = p(a_n^{(k)}) = p(b_n^{(k)}) = p(c_n^{(k)})$ for $k = 1, 2, \ldots, m-1$ and every n, the sets $\bigcup_{n=1}^{\infty} a_n^{(k)}$, $\bigcup_{n=1}^{\infty} b_n^{(k)}$, $\bigcup_{n=1}^{\infty} c_n^{(k)}$ are disjoint and $S_k = \bigcup_{n=1}^{\infty} d_n^{(k)}$ is dense in $(0, 1)$ for $k = 1, 2, \ldots, m-1$, and $S_1 \supset S_2 \supset \ldots \supset S_{m-1}$, then using repeatedly the above property of p for every $d_n^{(m-1)}$ and $c_n^{(m-1)}$ as D and A we construct four sequences of open disjoint intervals $\{a_n^{(m)}\}$, $\{b_n^{(m)}\}$, $\{c_n^{(m)}\}$, $\{d_n^{(m)}\}$ such that $d_n^{(m)} = p(a_n^{(m)}) = p(b_n^{(m)}) = p(c_n^{(m)})$ for every natural number n, the sets $\bigcup_{n=1}^{\infty} a_n^{(m)}$, $\bigcup_{n=1}^{\infty} b_n^{(m)}$, $\bigcup_{n=1}^{\infty} c_n^{(m)}$ are disjoint and the set $S_m = \bigcup_{n=1}^{\infty} d_n^{(m)} \subset S_{m-1}$ is dense in $(0, 1)$. Thus, by induction, we have constructed such sequences and sets S_m for every natural number m.

Put $E^1 = \bigcap_{n=1}^{\infty} S_m$. Obviously $E^1 \subset (0, 1)$ is of type G_δ and is dense in $(0, 1)$.

From the above construction it immediately follows that $a_n^{(k)} \cap a_m^{(j)} = \emptyset$, $a_n^{(k)} \cap b_m^{(j)} = \emptyset$, $b_n^{(k)} \cap b_m^{(j)} = \emptyset$, where $n \neq m$ or $k \neq j$. Let $P = [0, 1] - \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (a_n^{(k)} \cup b_n^{(k)})$. Let us put

$$\bar{F}^1(x^1) = \begin{cases} p(x^1) & \text{for } x^1 \in P \cap (0, 1), \\ p(l_n^{(k)}) + \big(p(r_n^{(k)}) - p(l_n^{(k)})\big) F^1\left(\dfrac{x^1 - l_n^{(k)}}{r_n^{(k)} - l_n^{(k)}}\right) & \text{for } x^1 \in (l_n^{(k)}, r_n^{(k)}) = a_n^{(k)}, \\ & \text{for } k, n = 1, 2, \ldots \\ \text{linear and continuous on } \bar{b}_n^{(k)} & \text{for } k, n = 1, 2, \ldots \end{cases}$$

If 0 or 1 is the left or right end-point of some $b_n^{(k)}$, then the requirement of continuity in 0 or 1 means the existence of limits: $\lim_{x^1 \to 0+} \bar{F}^1(x^1) = 0$, $\lim_{x^1 \to 1-} \bar{F}^1(x^1) = 1$.

From the assumption it follows that $F^1 \in s_\beta([0, 1])$ (after the extension) and F^1 is an irreducible superposition (of one variable) of the class β. From Theorem 7 of [1] (p. 345) we conclude that $\bar{F}^1 \in s_\beta([0, 1])$. Hence $\bar{F} \in S_\gamma(\operatorname{Int} K)$, where $\gamma \leqslant \beta$. Simultaneously in virtue of Theorem 6 $\bar{F} \in S_\beta(a_n^{(k)} \times (0, 1))$ for every natural numbers n, k, and so $\gamma \geqslant \beta$. Hence $\bar{F} \in S_\beta(\operatorname{Int} K)$ and finally $\bar{F} \in U_\beta(\operatorname{Int} K)$.

Now we shall prove that the set $E = E^1 \times (0, 1)$ is an extraordinary set of order $\beta - 1$ for $\bar{F}$. Let $\bar{F} = f \circ g$, where $g \in S_{\beta-1}(\operatorname{Int} K)$ and $f\colon g(\operatorname{Int} K) \to R^2$ is a continuous function. Let $(z^1, z^2) \in E$. We shall prove that $f^{-1}(\{(z^1, z^2)\})$ is an infinite set. We have $z^1 \in E^1$, and so there exists a descending sequence of intervals $\{d_{n_k}^{(k)}\}$ such that $z^1 \in \bigcap_{k=1}^{\infty} d_{n_k}^{(k)}$. For every natural number k there exist two open intervals $a_{n_k}^{(k)}$ and $b_{n_k}^{(k)}$ such that $\bar{F}^1(a_{n_k}^{(k)}) = \bar{F}^1(b_{n_k}^{(k)}) = d_{n_k}^{(k)}$. The function $\bar{F}$ fulfills for $a_{n_k}^{(k)}$ and $b_{n_k}^{(k)}$ all the assumptions of Lemma 11, and so $g(a_{n_k}^{(k)} \times (0, 1)) = g(b_{n_k}^{(k)} \times (0, 1))$ or $g(a_{n_k}^{(k)} \times (0, 1)) \cap g(b_{n_k}^{(k)} \times (0, 1)) = \emptyset$. Suppose that the equality holds. Then the reduced function $f | g(b_{n_k}^{(k)} \times (0, 1))$ is one-to-one, because $\bar{F}$ is one-to-one on $b_{n_k}^{(k)} \times (0, 1)$. Hence $\bar{F}$ on the set $a_{n_k}^{(k)} \times (0, 1)$ is a superposition of the function $g | a_{n_k}^{(k)} \times (0, 1) \in S_\gamma(a_{n_k}^{(k)} \times (0, 1))$, where $\gamma \leqslant \beta - 1$ with the one-to-one function $f | g(b_{n_k}^{(k)} \times (0, 1))$. This is impossible, because F is an irreducible superposition of the class β on $\operatorname{Int} K$ and $\bar{F}$ on $a_{n_k}^{(k)} \times (0, 1)$ is a superposition of F with two linear functions, and so $\bar{F}$ is an irreducible superposition of the class β on $a_{n_k}^{(k)} \times (0, 1)$. Hence $g(a_{n_k}^{(k)} \times (0, 1)) \cap g(b_{n_k}^{(k)} \times (0, 1)) = \emptyset$. From the construction it follows that $a_{n_k}^{(k)}$, $b_{n_k}^{(k)}$ and $c_{n_k}^{(k)}$ are included in $c_{n_{k-1}}^{(k-1)}$, because $d_{n_k}^{(k)} \subset d_{n_{k-1}}^{(k-1)}$. So $g(a_{n_k}^{(k)} \times (0, 1)) \cup g(b_{n_k}^{(k)} \times (0, 1)) \subset g(c_{n_{k-1}}^{(k-1)} \times (0, 1))$. Simultaneously $\bar{F}^1(a_{n_{k-1}}^{(k-1)}) = \bar{F}^1(b_{n_{k-1}}^{(k-1)}) = \bar{F}^1(c_{n_{k-1}}^{(k-1)})$ (see [1], pp. 350-351) and $\bar{F}$ fulfills on these intervals all the assumptions of Lemma 11; thus $g(c_{n_{k-1}}^{(k-1)} \times (0, 1))$ either is disjoint with $g(a_{n_{k-1}}^{(k-1)} \times (0, 1))$ and $g(b_{n_{k-1}}^{(k-1)} \times (0, 1))$ or coincides with exactly one of them. Hence sets $g(a_{n_k}^{(k)} \times (0, 1))$, $g(b_{n_k}^{(k)} \times (0, 1))$ are both disjoint with at least one of the sets $g(a_{n_{k-1}}^{(k-1)} \times (0, 1))$, $g(b_{n_{k-1}}^{(k-1)} \times (0, 1))$. For every natural number k choose from the pair of sets $g(a_{n_k}^{(k)} \times (0, 1))$, $g(b_{n_k}^{(k)} \times (0, 1))$ that one which is disjoint with both of the sets $g(a_{n_{k+1}}^{(k+1)} \times (0, 1))$ and $g(b_{n_{k+1}}^{(k+1)} \times (0, 1))$ and denote it by G_k. We obtain a sequence of disjoint non-empty sets $\{G_k\}$. We have $f(G_k) = d_{n_k}^{(k)} \times (0, 1)$ and $(z^1, z^2) \in d_{n_k}^{(k)} \times (0, 1)$, and so $f^{-1}(\{(z^1, z^2)\}) \cap$

$\cap G_k \neq \emptyset$ for every natural number k. Hence $f^{-1}(\{(z^1, z^2)\})$ is infinite, and so E is an extraordinary set of order $\beta-1$ for $\overline{F}$.

THEOREM 10. *Let β be a natural number or a countable ordinal of the first kind. If there exists a function $F \in U_\beta(\operatorname{Int} K)$, $F(x^1, x^2) = (F^1(x^1), x^2)$, where F^1 fulfils condition* (m), *such that F is an irreducible superposition of the class β, then there exists a function $\widetilde{F} \in U_{\beta+1}(\operatorname{Int} K)$, $\widetilde{F}(x^1, x^2) = (\widetilde{F}^1(x^1), x^2)$, where $\widetilde{F}^1$ fulfils condition* (m) *and $\widetilde{F}$ is an irreducible superposition of the class $\beta+1$.*

Proof. From Theorem 9 it follows that there exists a function $\overline{F} \in U_\beta(\operatorname{Int} K)$, $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$, where $\overline{F}^1$ fulfils condition (m), having as an extraordinary set of order $\beta-1$ the set $E = E^1 \times (0,1)$, where $E^1 \subset (0,1)$ is of type G_δ and is dense in $(0,1)$. Let $g_1\colon (0,1) \xrightarrow[\text{onto}]{} (0,1)$ and $g_2\colon (0,1) \xrightarrow[\text{onto}]{} (0,1)$ be increasing functions such that $g_1(E^1) \cup g_2(E^1) = (0,1)$. Such functions do exist (see [1], p. 352). Let us put $F_i(x^1, x^2) = \big((g_i \circ \overline{F}^1)(x^1), x^2\big)$ for $i = 1, 2$. From Theorem 3 it follows that $F_1, F_2 \in S_\gamma(\operatorname{Int} K)$, where $\gamma \leqslant \beta+1$. Now we shall prove that the set $g_1(E^1) \times (0,1)$ is an extraordinary set of order $\beta-1$ for F_1. If $G_1\colon \operatorname{Int} K \xrightarrow[\text{onto}]{} \operatorname{Int} K$ is defined as $G_1(x^1, x^2) = \big(g_1(x^1), x^2\big)$, then G_1 is a homeomorphism and $F_1 = G_1 \circ \overline{F}$. Hence $\overline{F} = G_1^{-1} \circ F_1$. Let the functions $h_1\colon \operatorname{Int} K \to R^2$ and $f_1\colon h_1(\operatorname{Int} K) \xrightarrow[\text{onto}]{} \operatorname{Int} K$ fulfil the following conditions: $h_1 \in S_{\beta-1}(\operatorname{Int} K)$, f_1 is a continuous function and $F_1 = f_1 \circ h_1$. We have $\overline{F} = (G_1^{-1} \circ f_1) \circ h_1$. The set $E^1 \times (0,1)$ is an extraordinary set of order $\beta-1$ for $\overline{F}$, and so for every $(z^1, z^2) \in E^1 \times (0,1)$ the set $(G_1^{-1} \circ f_1)^{-1}\big(\{(z^1, z^2)\}\big)$ is infinite; thus for every $(y^1, y^2) \in G_1\big(E^1 \times (0,1)\big) = g_1(E^1) \times (0,1)$ the set $f_1^{-1}\big(\{(y^1, y^2)\}\big)$ is infinite and $g_1(E^1) \times (0,1)$ is really an extraordinary set of order $\beta-1$ for F_1. Similarly one can prove that the set $g_2(E^1) \times (0,1)$ is an extraordinary set of order $\beta-1$ for F_2.

Now let us put $\widetilde{F}(x^1, x^2) = \big(\widetilde{F}^1(x^1), x^2\big)$ for $(x^1, x^2) \in \operatorname{Int} K$, where $\widetilde{F}^1(x^2) = 3^{-1} + 3^{-1} \cdot \hat{F}^1(5x^1 - 1)$ for $x^1 \in (0,1)$ and the function $\hat{F}^1\colon (-1, 4) \to R$ is defined in the following way:

$$\hat{F}^1(x^1) = \begin{cases} x^1 & \text{for } -1 < x^1 \leqslant 0, \\ (g_1 \circ \overline{F}^1)(x^1) & \text{for } 0 < x^1 < 1, \\ 2 - x^1 & \text{for } 1 \leqslant x^1 \leqslant 2, \\ (g_2 \circ \overline{F}^1)(x^1 - 2) & \text{for } 2 < x^1 < 3, \\ x^1 - 2 & \text{for } 3 \leqslant x^1 < 4. \end{cases}$$

We have $\widetilde{F}^1 \in s_{\beta+1}([0,1])$ and $\widetilde{F}^1$ is an irreducible superposition of the class $\beta+1$ after the extension (see [1], pp. 353–355). Hence $\widetilde{F} \in S_\gamma(\operatorname{Int} K)$, where $\gamma \leqslant \beta+1$. To prove that $\widetilde{F} \in S_{\beta+1}(\operatorname{Int} K)$ we shall show that the set $\operatorname{Int} K$ is an extraordinary set of order $\beta-1$ for the function $\hat{F}(x^1, x^2) = \big(\hat{F}^1(x^1), x^2\big)$, so $(3^{-1}, 2\cdot 3^{-1}) \times (0,1)$ is an extraordinary set of order $\beta-1$ for $\widetilde{F}$. Let $\hat{F} = h \circ g$, where $g \in S_{\beta-1}\big((-1, 4) \times (0,1)\big)$ and $h\colon g\big((-1,4) \times (0,1)\big) \to R^2$ is a continuous function. We have $\hat{F}^1(x^1) = (g_1 \circ \overline{F}^1)(x^1)$ for $x^1 \in (0,1)$ and $\hat{F}^1(x^1) = (g_2 \circ \overline{F}^1)(x^1 - 2)$ for $x^1 \in (2,3)$. If $(y^1, y^2) \in \operatorname{Int} K$, then $(y^1, y^2) \in g_1(E^1) \times (0,1)$ or $(y^1, y^2) \in g_2(E^1) \times (0,1)$. In the first case the set $h^{-1}\big(\{(y^1, y^2)\}\big)$ is infinite, because (y^1, y^2) belongs to the extraordinary set of order $\beta-1$ for F_1, in the second case it is infinite because (y^1, y^2) belongs to the extraordinary set of order $\beta-1$ for F_2.

Suppose now that $\widetilde{F} = f_1 \circ f_2$, where $f_2 \in S_\beta(\operatorname{Int} K)$ and $f_1\colon f_2(\operatorname{Int} K) \to R^2$ is a continuous and one-to-one function. We have $f_2 = f_3 \circ f_4$, where $f_4 \in S_{\beta-1}(\operatorname{Int} K)$ and $f_3\colon f_4(\operatorname{Int} K) \to R^2$ is BVB in $f_4(\operatorname{Int} K)$. Hence $\widetilde{F} = (f_1 \circ f_3) \circ f_4$. We have $f_4 \in S_{\beta-1}(\operatorname{Int} K)$, and so from the previous part of the proof it follows that $N\big((y^1, y^2); (f_1 \circ f_3); f_4(\operatorname{Int} K)\big) = +\infty$ for $(y^1, y^2) \in (3^{-1}, 2\cdot 3^{-1}) \times (0,1)$. Hence $N\big((z^1, z^2); f_3; f_4(\operatorname{Int} K)\big) = +\infty$ for $(z^1, z^2) \in f_1^{-1}\big((3^{-1}, 2\cdot 3^{-1}) \times (0,1)\big)$. The set $f_1^{-1}\big((3^{-1}, 2\cdot 3^{-1}) \times (0,1)\big)$ is open and has positive measure, thus f_3 is not BVB in $f_4(\operatorname{Int} K)$ — a contradiction. Hence $\widetilde{F}$ cannot be represented as a superposition of a function belonging to the class β on $\operatorname{Int} K$ with a one-to-one continuous function; so $\widetilde{F} \in S_\gamma(\operatorname{Int} K)$, where $\gamma \geqslant \beta+1$. Finally $\widetilde{F} \in S_{\beta+1}(\operatorname{Int} K)$ and $\widetilde{F}$ is an irreducible superposition of the class $\beta+1$ on $\operatorname{Int} K$. Simultaneously $\widetilde{F}^1 \in s_{\beta+1}([0,1])$; thus $\widetilde{F} \in U_{\beta+1}(\operatorname{Int} K)$.

THEOREM 11. *Let β be a countable ordinal of the second kind and let $\beta_1 < \beta_2 < \ldots < \beta_n < \ldots$ be a sequence of natural numbers or countable ordinals such that $\lim_{n\to\infty} \beta_n = \beta$ (i.e., there is no ordinal number $\gamma < \beta$ for which $\beta_n \leqslant \gamma$ for every n. If for every natural number n there exists a function $F_n \in U_{\beta_n}(\operatorname{Int} K)$, $F_n(x^1, x^2) = (F_n^1(x^1), x^2)$, where F_n^1 fulfils condition* (m), *then there exists a function $F \in U_\beta(\operatorname{Int} K)$, $F(x^1, x^2) = (F^1(x^1), x^2)$, where F^1 fulfils condition* (m).

Proof. Let $\{a_n\}$ be a sequence of numbers such that $a_0 = 0$, $\lim_{n\to\infty} a_n = 1$ and $a_n < a_{n+1}$ for every n. Put

$$F^1(x^1) = \begin{cases} x^1 & \text{for } x^1 = a_n,\ n = 1, 2, \ldots, \\ a_{n-1} + (a_n - a_{n-1}) F_n^1\left(\dfrac{x^1 - a_{n-1}}{a_n - a_{n-1}}\right) & \text{for } x^1 \in (a_{n-1}, a_n),\ n = 1, 2, \ldots \end{cases}$$

and $F(x^1, x^2) = \big(F^1(x^1), x^2\big)$ for $(x^1, x^2) \in \operatorname{Int} K$. We shall prove that $F \in U_\beta(\operatorname{Int} K)$. Let

$$g_n^1(x^1) = \begin{cases} x^1 & \text{for } x^1 \in (0, a_{n-1}] \cup [a_n, 1), \\ a_{n-1} + (a_n - a_{n-1}) F_n^1\left(\dfrac{x^1 - a_{n-1}}{a_n - a_{n-1}}\right) & \text{for } x^1 \in (a_{n-1}, a_n) \end{cases}$$

and $g_n(x^1, x^2) = (g_n^1(x^1), x^2)$ for $(x^1, x^2) \in \mathrm{Int}K$ and for every n. If $\overline{F}_1 = g_1$ and $\overline{F}_{n+1} = g_{n+1} \circ \overline{F}_n$ for $n = 1, 2, \ldots$, then it is not difficult to see that $\overline{F}_n(x^1, x^2) = (\overline{F}_n^1(x^1), x^2)$, where

$$\overline{F}_n^1(x^1) = \begin{cases} x^1 & \text{for } x^1 \in [a_n, 1), \\ a_{k-1} + (a_k - a_{k-1}) F_k^1\left(\dfrac{x^1 - a_{k-1}}{a_k - a_{k-1}}\right) & \text{for } x^1 \in (a_{k-1}, a_k],\ k = 1, 2, \ldots, n. \end{cases}$$

In virtue of Theorem 8, $\overline{F}_n \in U_{\beta_n}(\mathrm{Int}K)$. Simultaneously $\lim\limits_{n\to\infty} \overline{F}_n = F$ and the convergence is uniform; so $F \in S_\gamma(\mathrm{Int}K)$, where $\gamma \leqslant \beta$. In virtue of Theorem 6, we have $F|(a_{n-1}, a_n) \times (0, 1) \in S_{\beta_n}((a_{n-1}, a_n) \times (0, 1))$. Hence $\gamma \geqslant \beta_n$ for every natural number n, and so $\gamma \geqslant \beta$. Finally $F \in S_\beta(\mathrm{Int}K)$. Since $F^1 \in s_\beta([0, 1])$ (after extension, see [1], pp. 356–358), then $F \in U_\beta(\mathrm{Int}K)$.

THEOREM 12. *Let β be a countable ordinal of the second kind. If there exists a function $F \in U_\beta(\mathrm{Int}K)$, $F(x^1, x^2) = (F^1(x^1), x^2)$, where F^1 fulfils condition* (m), *then there exists a function $\overline{F} \in U_{\beta+1}(\mathrm{Int}K)$, $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$, where $\overline{F}^1$ fulfils a condition* (m).

Proof. Let us put $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$ for $(x^1, x^2) \in \mathrm{Int}K$, where $\overline{F}^1(x^1) = 3^{-1} + 3^{-1}\hat{F}^1(5x^1 - 1)$ for $x^1 \in (0, 1)$ and the function $\hat{F}^1$: $(-1, 4) \to R$ is defined in the following way:

$$\hat{F}^1(x^1) = \begin{cases} x^1 & \text{for} \quad x^1 \in (-1, 0], \\ F^1(x^1) & \text{for} \quad x^1 \in (0, 1), \\ 2 - x^1 & \text{for} \quad x^1 \in [1, 2), \\ x^1 - 2 & \text{for} \quad x^1 \in [2, 4). \end{cases}$$

From Theorem 11 from [1], p. 359, it follows that $\hat{F}^1 \in s_{\beta+1}([-1, 4])$ (after extension); hence, in virtue of Theorem, 6 $\overline{F} \in S_\gamma(\mathrm{Int}K)$, where $\gamma \leqslant \beta+1$. Simultaneously, also in virtue of Theorem 6, $\overline{F}|(5^{-1}, 2\cdot 5^{-1}) \times (0, 1) \in S_\beta((5^{-1}, 2\cdot 5^{-1}) \times (0, 1))$ and so $\gamma \geqslant \beta$. We shall prove that $\gamma \neq \beta$.

Suppose that $\gamma = \beta$. Then there exists a sequence $\{g_n\}$ of functions such that g_1: $\mathrm{Int}K \to R^2$, g_{n+1}: $g_n(g_{n-1}(\ldots(g_1(\mathrm{Int}K))\ldots)) \to R^2$ for $n = 1, 2, \ldots$ and $\overline{F}(x^1, x^2) = \lim\limits_{n\to\infty} F_n(x^1, x^2)$ uniformly on $\mathrm{Int}K$, where $F_1 = g_1$, $F_{n+1} = g_{n+1} \circ F_n$ for $n = 1, 2, \ldots$ and $F_n \in S_{\beta_n}(\mathrm{Int}K)$, where $\beta_n < \beta$. If we put $r_n = \lim\limits_{k\to\infty} (g_{n+k} \circ g_{n+k-1} \circ \ldots \circ g_{n+1})$ (such a sequence is uniformly convergent on $g_n(g_{n-1}(\ldots(g_1(\mathrm{Int}K))\ldots))$, then $\overline{F} = r_n \circ F_n$ for $n = 1, 2, \ldots$ The function $\overline{F}$ fulfils all the assumptions of Lemma 11 for $d_1 = (5^{-1}, 2\cdot 5^{-1})$ and $d_2 = (2\cdot 5^{-1}, 3\cdot 5^{-1})$, and so from Lemma 11 it follows that $F_n((5^{-1}, 2\cdot 5^{-1}) \times (0, 1)) = F_n((2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1))$ or $F_n((5^{-1}, 2\cdot 5^{-1}) \times (0, 1)) \cap F_n((2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1)) = \emptyset$ for every natural number n.

If for some n the equality holds, then r_n is one-to-one on $F_n((2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1))$, because $\overline{F}$ is one-to-one on $(2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1)$. From this it follows that $F \in S_{\beta_n+1}(\mathrm{Int}K)$ in virtue of Theorem 6. This is a contradiction, for $\beta_n + 1 < \beta$. Hence, for every natural number n, $F_n((5^{-1}, 2\cdot 5^{-1}) \times (0, 1)) \cap F_n((2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1)) = \emptyset$. This also leads to a contradiction, because from Theorem 1 it follows that the set $F_n((5^{-1}, 2\cdot 5^{-1}) \times (0, 1))$ is a region and $\lim\limits_{n\to\infty} F_n(x^1, x^2) = \overline{F}(x^1, x^2)$ for $(x^1, x^2) \in \mathrm{Int}K$ and $\overline{F}((5^{-1}, 2\cdot 5^{-1}) \times (0, 1)) = F((2\cdot 5^{-1}, 3\cdot 5^{-1}) \times (0, 1))$. Hence $\gamma \neq \beta$ so $\gamma = \beta+1$ and $\overline{F} \in U_{\beta+1}(\mathrm{Int}K)$.

THEOREM 13. *For every ordinal $\beta < \Omega$ the class $S_\beta(\mathrm{Int}K)$ is non-empty.*

Proof. We shall prove that for every ordinal $\beta < \Omega$ the smaller class $U_\beta(\mathrm{Int}K)$ is non-empty and if β is a natural number or a countable ordinal of the first kind, then there exists an irreducible superposition of the class β. For $\beta = 1$ this is obvious. If for every ordinal $\alpha < \beta$ the class $U_\alpha(\mathrm{Int}K)$ is non-empty, then the class $U_\beta(\mathrm{Int}K)$ is non-empty:

a) In virtue of Theorem 10, if β is a natural number or a countable ordinal of the first kind such that $\beta - 1$ is a countable ordinal of the first kind.

b) In virtue of Theorem 11, if β is a countable ordinal of the second kind.

c) In virtue of Theorem 12, if β is a countable ordinal of the first kind and $\beta - 1$ is a countable ordinal of the second kind. The fact that the function $\overline{F}$ constructed in the proof of Theorem 12 is an irreducible superposition follows immediately from Theorem 5.

Hence, in virtue of transfinite induction, $U_\beta(\mathrm{Int}K) \neq \emptyset$ for every ordinal $\beta < \Omega$.

DEFINITION 7. The function F: $\mathrm{Int}K \to R^2$ is called *monotone* if and only if for every $(y^1, y^2) \in F(\mathrm{Int}K)$ the set $F^{-1}(\{(y^1, y^2)\})$ is a connected set (see [3], II. 1.1, p. 45).

THEOREM 14. *Let $0 < \beta < \Omega$ be an arbitrary ordinal. If $F \in S_\beta(\mathrm{Int}K)$ and there exists a homeomorphism H: $F(\mathrm{Int}K) \xrightarrow[\text{onto}]{} \mathrm{Int}K$ such that $(H \circ F)(x^1, x^2) = (F^1(x^1), x^2)$, where F^1 fulfils condition* (m), *F^1 is not monotone and F^1 has no interval of constancy, then there exist functions g: $\mathrm{Int}K \to R^2$, f: $g(\mathrm{Int}K) \to R^2$, such that $F = f \circ g$, g is BVB in $\mathrm{Int}K$ and g is not a monotone function and f is a continuous function on $g(\mathrm{Int}K)$.*

Proof. If $\beta = 1$, then it suffices to put $g = F$ and $f(x^1, x^2) = (x^1, x^2)$.

Suppose now that the theorem holds for every ordinal number $\alpha < \beta$. We shall prove that it holds for β. Consider two cases:

If $\beta > 1$ is a natural number or a countable ordinal of the first kind, then there exist functions F_0, f_1 such that $F_0 \in S_{\beta-1}(\mathrm{Int}K)$, $f_1 \in S_1(F_0(\mathrm{Int}K))$ and $F = f_1 \circ F_0$. If H: $F(\mathrm{Int}K) \xrightarrow[\text{onto}]{} \mathrm{Int}K$ is a homeomorphism such that

$(H\circ F)(x^1,x^2)=(F^1(x^1),x^2)$, then $H\circ F=(H\circ f_1)\circ F_0$ and from Theorem 1 it follows that there exists a homeomorphism H_0: $F_0(\mathrm{Int}K)\xrightarrow[\text{onto}]{}\mathrm{Int}K$ such that $(H_0\circ F_0)(x^1,x^2)=(F_0^1(x^1),x^2)$. If F_0^1 is a monotone function, then F_0 is one-to-one and, in virtue of Theorem 4, F is a superposition of the class 0 or 1 — a contradiction. Hence F_0^1 is not a monotone function. It is not difficult to see that F_0^1 has no interval of constancy and F_0^1 fulfils condition (m), and so F_0 fulfils all the assumptions of the theorem. By hypothesis $F_0=f_0\circ g$, where g: $\mathrm{Int}K\to R^2$ is BVB in $\mathrm{Int}K$ and g is not a monotone function and f_0: $g(\mathrm{Int}K)\to R^2$ is a continuous function. Hence $F=(f_1\circ f_0)\circ g$, and so F has a representation of the required form.

If β is a countable ordinal of the second kind, then $F=\lim_{n\to\infty}F_n$, where $F_n\in S_{\beta_n}(\mathrm{Int}K)$, $\beta_n<\beta$ for every natural number n, and the convergence is uniform. Simultaneously $F=r_n\circ F_n$ (cf. the proof of Theorem 11) and, as in the previous part of the proof, one can prove that for every natural number n there exists a homeomorphism H_n: $F_n(\mathrm{Int}K)\xrightarrow[\text{onto}]{}\mathrm{Int}K$ such that $(H_n\circ F_n)(x^1,x^2)=(F_n^1(x^1),x^2)$ and F_n^1 fulfils condition (m) and has no interval of constancy. If for every n the function F_n^1 is one-to-one, then it is not difficult to see that F^1 is non-decreasing. Hence there exists a number n_0 such that F_{n_0} fulfils all the assumptions of the theorem. We have $F_{n_0}\in S_{\beta_{n_0}}(\mathrm{Int}K)$ and $\beta_{n_0}<\beta$, and so the existence of a representation of the required form follows as in the first part of the proof.

THEOREM 15. *There exists a continuous function* F: $\mathrm{Int}K\to R^2$, $F(x^1,x^2)=(F^1(x^1),x^2)$ *for* $(x^1,x^2)\in\mathrm{Int}K$ *such that* $F\notin\bigcup_{\beta<\Omega}S_\beta(\mathrm{Int}K)$.

Proof. In the first place we shall prove that there exists a continuous function F^1: $(0,1)\to R$ having the following properties:

a) F^1 fulfils condition (m),

b) if F^1 has in $x_1^1, x_2^1\in(0,1)$, $x_1^1\neq x_2^1$, a maximum (a minimum), then $F^1(x_1^1)\neq F^1(x_2^1)$,

c) for every interval $(c,d)\subset(0,1)$ there exists an interval $[a,b]\subset(c,d)$ such that $\min(F^1(a),F^1(b))<F^1(x^1)<\max(F^1(a),F^1(b))$ for every $x^1\in(a,b)$ and for every $y^1\in F^1((a,b))$ the set $(F^1)^{-1}(\{y^1\})\cap(a,b)$ is infinite.

Let h: $(a,b)\to(h(a),h(b))$ be a linear function and let $\{a_n\}$, $n=0,\pm1,\pm2,\ldots$ be a sequence of numbers fulfilling the following conditions: $0<a_{n+1}-a_n<2^{-1}(b-a)$ for every integer n, $\lim_{n\to-\infty}a_n=a$, $\lim_{n\to\infty}a_n=b$. Let us put

$$h^*(t)=\begin{cases}3h(t)-2h(a_n) & \text{for } t\in[a_n,\,2\cdot3^{-1}a_n+3^{-1}a_{n+1}),\\ -3h(t)+2\big(h(a_n)+h(a_{n+1})\big) & \text{for } t\in[2\cdot3^{-1}a_n+3^{-1}a_{n+1},\,3^{-1}a_n+2\cdot3^{-1}a_{n+1}),\\ 3h(t)-2h(a_{n+1}) & \text{for } t\in[3^{-1}a_n+2\cdot3^{-1}a_{n+1},\,a_{n+1}),\end{cases}$$

and for every integer n. We shall say that the function h on the interval (a,b) has been replaced by a polygonal line of type P. It is not difficult to verify that h^* is a continuous function on (a,b) and h^*: $(a,b)\xrightarrow[\text{onto}]{}(h(a),h(b))$. If we write $b_{2n}=2\cdot3^{-1}a_n+3^{-1}a_{n+1}$, $b_{2n+1}=3^{-1}a_n+2\cdot3^{-1}a_{n+1}$, then h^* is a linear function on every interval (b_n,b_{n+1}) and it is increasing on (b_{2n-1},b_{2n}) and decreasing on (b_{2n},b_{2n+1}). At every b_{2n} h^* has a local maximum and at every b_{2n+1} — a local minimum. It is easy to see that h^* fulfils condition b).

In the case where h is a decreasing linear function the function h^* is defined similarly.

Let us observe that from the condition $a_{n+1}-a_n<2^{-1}(b-a)$ it follows that, for every $t\in(a,b)$, $|h^*(t)-h(t)|<2^{-1}|h(b)-h(a)|$.

Now we shall construct the function F^1. Let $f_0(x^1)=x^1$ for $x^1\in(0,1)$. To construct f_1 we replace f_0 on $(0,1)$ by a polygonal line of type P constructed for the sequence $\{a_n\}$ fulfilling all the conditions mentioned. Suppose that we have already constructed functions $f_0,f_1,\ldots,f_{k-1}$ such that $|f_{k-1}(x^1)-f_{k-2}(x^1)|<2^{-k+1}$ for every $x^1\in(0,1)$, the sum of intervals on which f_{k-1} is a linear function is dense in $(0,1)$ and the oscillation of f_{k-1} on every such interval is not greater than 2^{-k+1}, and f_{k-1} fulfils condition b). To construct f_k we replace f_{k-1} on every interval of linearity (a maximal interval, of course) by a polygonal line of type P constructed for the sequence $\{a_n\}$ fulfilling all the conditions mentioned and the following additional one: for every natural number n $f_{k-1}(a_n)$ is different from all the values taken by f_{k-1} at those points where f_{k-1} has an extremum. If (a',b') and (a'',b'') are two different intervals of linearity of f_{k-1}, then we choose sequences $\{a_n'\}$, $\{a_n''\}$ in these intervals in such a way that $f_{k-1}(a_m')\neq f_{k-1}(a_p'')$ for every integers m,p. The function f_k constructed in this manner is a continuous function, it has a dense set of intervals of linearity, its oscillation on every such interval is not greater than 2^{-k}, f fulfils condition b) and $|f_k(x^1)-f_{k-1}(x^1)|<2^{-k}$ for every $x^1\in(0,1)$. From the construction it follows also that the length of every interval of linearity of f_k is less than 2^{-k}.

Hence, by induction, we have defined a sequence of continuous functions $\{f_k\}$, $k=0,1,\ldots$ This sequence is uniformly convergent. Let us put $F^1(x^1)=\lim_{k\to\infty}f_k(x^1)$ for $x^1\in(0,1)$. In [1], pp. 367–368 it was proved that F^1 fulfils conditions a) and b). We shall prove that F^1 fulfils also condition c). Let $(c,d)\subset(0,1)$ be an arbitrary interval. From the construction it follows that there exist a natural number k_0 and the interval $[a,b]\subset(c,d)$ such that f_{k_0} is linear on (a,b) and is not linear on any greater interval. We have $F^1((a,b))=f_{k_0}((a,b))$. Denote by B the set of values of F^1 taken by F^1 it those points where F^1 has an extremum. For $y^1\in(0,1)-B$ the set $(F^1)^{-1}(\{y^1\})$ is a perfect set (see [1], pp. 368–369).

Hence for $y^1 \in F^1((a,b)) - B$ we have $(F^1)^{-1}(\{y^1\}) \cap (a,b) \neq \emptyset$, and so the set $(F^1)^{-1}(\{y^1\}) \cap (a,b)$ is infinite. Suppose now that $y^1 \in F^1((a,b)) \cap B$. There exists a point $x^1 \in (0,1)$, at which F^1 has an extremum and $F^1(x^1) = y^1$. If neither of the points having these properties (in the first F^1 has a maximum and in the second — a minimum) is in (a,b), then, in the same way as in [1], pp. 368–369, one can prove that the set $(F^1)^{-1}(\{y^1\}) \cap \cap (a,b)$ is non-empty and perfect in (a,b), and hence infinite. If at least one of those points belongs to (a,b), then from the construction of f_{k_0+1} it follows that there exists an interval $[b_n, b_{n+1}] \subset (a,b)$ such that F^1 has no extremum at any point of the set $(F^1)^{-1}(\{y^1\}) \cap (b_n, b_{n+1})$. On the same way as in [1], one can prove that the set $(F^1)^{-1}(\{y^1\}) \cap (b_n, b_{n+1})$ is non-empty and perfect in (b_n, b_{n+1}), and so the set $(F^1)^{-1}(\{y^1\}) \cap (a,b)$ is infinite. Hence F^1 fulfils condition c).

Let $F(x^1, x^2) = (F^1(x^1), x^2)$ for $(x^1, x^2) \in \operatorname{Int} K$. Suppose that there exists an ordinal number $\beta < \Omega$ such that $F \in S_\beta(\operatorname{Int} K)$. It is easy to see that F fulfills all assumptions of Theorem 14. Hence there exist $g\colon \operatorname{Int} K \to R^2$, $f\colon g(\operatorname{Int} K) \to R^2$ such that $F = f \circ g$, g is BVB in $\operatorname{Int} K$ and is not monotone and f is continuous on $g(\operatorname{Int} K)$. In virtue of Theorem 1 there exists a homeomorphism $H\colon g(\operatorname{Int} K) \xrightarrow[\text{onto}]{} \operatorname{Int} K$ such that $(H \circ g)(x^1, x^2) = (f_1(x^1), x^2)$. We have also $(f \circ H^{-1})(y^1, y^2) = (f_2(y^1), y^2)$ for $(y^1, y^2) \in \operatorname{Int} K$, because $F = (f \circ H^{-1}) \circ (H \circ g)$. Hence $F^1 = f_2 \circ f_1$. From Lemma 3 of [1], p. 363 it follows that f_1 is one-to-one on $(0,1)$ or there exists an open interval $d_0 \subset f_1((0,1))$ on which f_2 is one-to-one. The supposition that f_1 is one-to-one leads to a contradiction, because g is not monotone on $\operatorname{Int} K$. Hence the second possibility ought to be fulfilled. Let $(c,d) \subset (0,1)$ be a component of the open set $f_1^{-1}(d_0)$ and $(a,b) \subset (c,d)$ — the interval chosen for F^1 in virtue of condition c). Let $d_1 = f_1((a,b))$. Obviously $d_1 \subset d_0$, the function f_2 is one-to-one on d_1 and it is not difficult to see that d_1 is an open interval. We have $F^1((a,b)) = f_2(d_1)$ and for every $y^1 \in F^1((a,b))$ the set $(F^1)^{-1}(\{y^1\}) \cap (a,b)$ is infinite, thus for every $z^1 \in d_1$ the set $f_1^{-1}(\{z^1\}) \cap (a,b)$ is infinite. Hence $N((z^1, z^2); H \circ g, (a,b) \times (0,1)) = +\infty$ for $(z^1, z^2) \in d_1 \times (0,1)$ and $N((t^1, t^2); g; (a,b) \times (0,1)) = +\infty$ for $(t^1, t^2) \in H^{-1}(d_1 \times (0,1))$. Since $H^{-1}(d_1 \times (0,1))$ is an open region, g is not BVB in $(a,b) \times (0,1)$ — a contradiction.

The supposition that $F \in \bigcup_{\beta < \Omega} S_\beta(\operatorname{Int} K)$ leads to a contradiction, and the theorem is proved.

COROLLARY. *For every ordinal number $\beta < \Omega$ the class $S_\beta(K)$ is non-empty. There exists a function $\overline{F}\colon K \to R^2$ such that $\overline{F} \notin \bigcup_{\beta < \Omega} S_\beta(K)$.*

Proof. Let $\beta < \Omega$ and let $F\colon \operatorname{Int} K \to R^2$ be a superposition of the class β constructed in the proof of Theorems 10, 11 or 12. The function F may be extended on K. Let $\overline{F}\colon K \to R^2$ denote this extension. We have $\overline{F}(x^1, x^2) = (\overline{F}^1(x^1), x^2)$ for $(x^1, x^2) \in K$. From Theorems 8, 10 or 11 of [1] (or from the proof of Theorems 10, 11 or 12 in this paper) it follows that $\overline{F}^1 \in s_\beta([0,1])$. Hence $\overline{F} \in S_\gamma(K)$, where $\gamma \leqslant \beta$. The opposite inequality follows from the fact that $F \in S_\beta(\operatorname{Int} K)$ and $\operatorname{Int} K \subset K$. Finally $\overline{F} \in S_\beta(K)$.

Let $F\colon \operatorname{Int} K \to R^2$ be the function constructed in the proof of Theorem 15 and let $\overline{F}$ denote a continuous extension of F on K. If $\overline{F} \in S_\beta(K)$ for some $\beta < \Omega$, then $F \in S_\gamma(\operatorname{Int} K)$, where $\gamma \leqslant \beta$ — a contradiction. Hence $\overline{F} \notin \bigcup_{\beta < \Omega} S_\beta(K)$.

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