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When ε-boundaries are manifolds

by

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Abstract. Let $A \subset R^n$. We define $\partial(\varepsilon, A)$ (the ε-boundary of A) to be the set of points in R^n whose distance from A is precisely ε. In this paper we study conditions which ensure that $\partial(\varepsilon, A)$ is an $(n-1)$-manifold.

We prove that for n equal to 2 or 3 $\partial(\varepsilon, A)$ is an $(n-1)$-manifold for almost all ε. We show that if A is a finite polyhedron in R^n then $\partial(\varepsilon, A)$ is an $(n-1)$-manifold for all sufficiently small values of ε. We prove a collaring theorem and exhibit counterexamples to several possible conjectures. The most interesting counterexample is a Cantor set K in R^4 such that $\partial(\varepsilon, K)$ is not a 3-manifold for any ε between 0 and 1. This example generalizes easily to all higher dimensions.

0. Introduction. Let $A \subset R^n$. We define $\partial(\varepsilon, A)$ (the ε-boundary of A) to be the set of points in R^n whose distance from A is precisely ε. In this paper we will study conditions which ensure that $\partial(\varepsilon, A)$ is an $(n-1)$-manifold.

This problem was stated by M. Brown in [1]. He proved that when $n = 2$ the components of $\partial(\varepsilon, A)$ are 1-manifolds for all but countably many ε. R. Gariepy and D. Pepe [3] have shown that for $n = 2$ $\partial(\varepsilon, A)$ is a 1-manifold for almost all ε. For arbitrary n they have shown that for almost all ε $\partial(\varepsilon, A)$ contains an open subset which is an $(n-1)$-manifold whose complement in $\partial(\varepsilon, A)$ has $(n-1)$-dimensional Hausdorff measure zero. See [2] and [4] for other related results.

In this paper we prove that for n equal to 2 or 3 $\partial(\varepsilon, A)$ is an $(n-1)$-manifold for almost all ε. We also show that if A is a finite polyhedron in R^n then $\partial(\varepsilon, A)$ is an $(n-1)$-manifold for all sufficiently small values of ε. This result is true for all n. Finally, we prove a collaring theorem and exhibit counterexamples to several possible conjectures. The most interesting example is a Cantor set K in R^4 such that $\partial(\varepsilon, K)$ is not a 3-manifold for any ε between 0 and 1. This example generalizes easily to all higher dimensions.

1. In this section we establish a sufficient condition for $\partial(\varepsilon, A)$ to be an $(n-1)$-manifold. As a direct consequence, we obtain our result concerning polyhedra.

First, we establish some notation. By $\delta(x)$ we will mean the distance from x to A in the Euclidean norm. By $N(x)$ we will mean the set of points in $\bar{A}$ whose distance from x is precisely $\delta(x)$. $\mathcal{H}(N(x))$ will denote the closed convex hull of $N(x)$.

PROPOSITION 1.1. *Let $A\subset R^n$, $x\in R^n-A$. If $x\notin\mathcal{H}(N(x))$ there exists a neighborhood U of x such that $\partial(\delta(x),A)\cap U$ is homeomorphic to R^{n-1}.*

Proof. We can assume that $x=0$ (the origin in R^n). If $0\notin\mathcal{H}(N(0))$ there is a hyperplane separating $N(0)$ from 0. More precisely, there is a unit vector v and a real number $s>0$ such that $s<a\cdot v$ for each $a\in N(0)$. It follows that there is a positive real number ε_0 such that $\|a\|>\delta(0)+\varepsilon_0$ for all $a\in A$ such that $s\geqslant a\cdot v$. For convenience, we choose $\varepsilon_0<\min(\delta(0),4s)$.

Choose an integer n large enough that the expressions

$$(s-\tfrac{1}{4}\varepsilon_0)n^2-2n\delta(0)-\varepsilon_0 \tag{1}$$

and

$$\varepsilon_0 s-(\tfrac{1}{2}\varepsilon_0)^2-2\delta(0)\varepsilon_0/n \tag{2}$$

are both positive. Let $p\in R^n$ be a vector with $\|p\|<\varepsilon_0/n$ and such that $p\cdot v=0$. We will show that there is precisely one δ with $|\delta|<\frac{1}{2}\varepsilon_0$ such that $(p+\delta v)\in\partial(\delta(0),A)$. In other words, there is precisely one point of $\partial(\delta(0),A)$ on each line parallel to v.

For each $a_0\in N(x)$ we have the four relations:

$$\|p+\tfrac{1}{2}\varepsilon_0 v-a_0\|^2=\|p\|^2+\tfrac{1}{4}\varepsilon_0^2+\delta^2(0)-2p\cdot a_0-\varepsilon_0 v\cdot a_0\,,$$

$$\|p\|^2\leqslant(\varepsilon_0/n)^2\,,$$

$$\varepsilon_0 s\leqslant\varepsilon_0 v\cdot a_0\,,$$

$$-2\delta(0)\varepsilon_0/n<2p\cdot a_0\,.$$

Adding these together and applying (1) we get

$$\|p+\tfrac{1}{2}\varepsilon_0 v-a\|^2<\delta^2(0)\,.$$

A similar calculation involving (2) shows that

$$\|p-\tfrac{1}{2}\varepsilon_0 v-a\|>\delta(0)\,.$$

Thus there is at least one δ with $|\delta|<\varepsilon_0$ such that $p+\delta v$ is in $\partial(\delta(0),A)$. To see that there is only one such δ, observe that if $\|p+\delta v-a\|=\delta(0)$ then $\|a\|<\delta(0)+\varepsilon_0$. Therefore $a\cdot v>s$. It follows that if $\|p+\delta_1 v-a\|^2-\|p+\delta_2 v-a\|^2\leqslant 0$ with $\delta_1\leqslant\delta_2$ then

$$(2p+(\delta_1+\delta_2)\cdot v-2a)(\delta_1-\delta_2)v\leqslant 0$$

and

$$(\delta_1+\delta_2)(\delta_1-\delta_2)\leqslant 2a\cdot v(\delta_1-\delta_2)\,.$$

Since $a\cdot v>\varepsilon_0$ and $\delta_1+\delta_2<\varepsilon_0$, we have $\delta_1=\delta_2$.

This establishes a 1-1 correspondence between the points of $\partial(\delta(0),A)$ in a certain neighborhood of 0 and vectors p with $\|p\|<\frac{1}{2}\varepsilon_0/n$ and $p\cdot v=0$. It is not hard to show (using the fact that $\partial(\delta(0),A)$ is closed) that this is a homeomorphism. ■

COROLLARY 1.2. *If A is bounded, then $\partial(\varepsilon,A)$ is an $(n-1)$-manifold for all sufficiently large ε. (This is also a result in* [1].) ■

DEFINITION 1.3. A point $x\in R^n$ is a *critical point* if $x\in\mathcal{H}(N(x))$. ε is a *critical value* if there is a critical point x with $\delta(x)=\varepsilon$. From now on, we will use C to denote the set of critical points.

THEOREM 1.4. *If $P\subset R^n$ is a finite polyhedron, there is a real number $\varepsilon_0>0$ such that for each $\varepsilon<\varepsilon_0$ $\partial(\varepsilon,P)$ is an $(n-1)$-manifold.*

Proof. Let the vertices of P be $v_0,\dots,v_k$. By thickening the open star of each v_i we can construct an open cover of P consisting of open sets $O_0,\dots,O_k$ such that each O_i is an open subset of R^n and such that each simplex of P which intersects O_i contains v_i. Let K be a compact neighborhood of P such that $P\subset\mathring{K}\subset K\subset\bigcup_i O_i$. Let λ be a Lebesgue number for the covering $\{O_i\cap K\}$ of K. Choose $\varepsilon_0<\min(\lambda,\frac{1}{2}d(P,R^n-K))$, "$d$" being the distance function.

If $x\in\partial(\varepsilon,P)$, then $N(x)\subset O_i$ for some i. Therefore, $N(x)$ is contained in a union of $(n-1)$-simplices which contain a single vertex. Thus $N(x)$ lies in a proper hemisphere of the sphere of radius $\delta(x)$ about x. By Proposition 1.1, x cannot be a critical point. ■

PROPOSITION 1.5. *Let x and y be elements of C. Then*

$$|\delta^2(x)-\delta^2(y)|\leqslant\|x-y\|^2\,.$$

Proof. If x is a critical point there exist $x_i\in N(x)$ and $t_i>0$ with $\sum t_i=1$ such that $x=\sum t_i x_i$. We also have

(i) $\delta^2(x)=\|x-x_i\|^2=\|x\|^2-2x\cdot x_i+\|x_i\|^2$ and

(ii) $\|y-x_i\|^2=\|x_i\|^2-2x_i\cdot y+\|y\|^2\geqslant\delta^2(y)$.

Subtracting (i) from (ii), multiplying by t_i and summing, we get

$$\|y\|^2-2x_i\cdot(y-x)-\|x\|^2\geqslant\delta^2(y)-\delta^2(x)\,.$$

Reversing the roles of x and y proves the other half of the inequality. ■

2. In this section we will study functions f having the property that $|f(x)-f(x)|\leqslant M\|x-y\|^2$. Our main result is the following theorem.

THEOREM 2.1. *Let $A\subset R^n$ and let $f\colon A\to R^1$ be a real-valued function such that $|f(x)-f(y)|\leqslant M\|x-y\|^n$ for some $M>0$. Then $f(A)$ has measure zero.*

We will prove this for the case $n = 2$, the only case we use. The generalization to higher dimensions is not difficult. We will need the following standard result from measure theory.

LEMMA 2.2 (Lebesgue density). *If S is a measurable subset of R^n, β is a real number with $0 < \beta < 1$, ε is a real number greater than zero, and μ is Lebesgue measure, then there exist disjoint balls D_i, $i = 1, 2, \ldots$ centered at points of S such that*

(i) $\mu(\bigcup_i D_i - S) \leqslant \varepsilon$,

(ii) $\mu(S - \bigcup_i D_i) = 0$,

(iii) $\mu(D_i \cap S)/\mu(D_i) > \beta$.

Proof. See Saks [3].

Proof (of Theorem 2.1). Without loss of generality, we may assume that A is closed and has finite measure. Choose a positive integer K and let $\beta > 1 - 1/(2\pi K^3)$. Let $\varepsilon = \mu(A) > 0$. The case $\mu(A) = 0$ will be taken care of in Step 1. Applying Lemma 2.2, we obtain a set of balls D_i having properties (i)-(iii). Let d_i be the radius of D_i. Let $A_0 = A - \bigcup_i D_i$ and let $A_1 = A - A_0$. We will be through if we can show that the measure of $f(A_i)$ is zero for $i = 0, 1$.

Step 1. Since A_0 has measure zero, for any given $\varepsilon' > 0$ we can find a set of balls Q_i of radii q_i covering A_0 such that $\sum q_i^2 < \varepsilon'$. For each i, $f(Q_i)$ is contained in an interval of length $< 8q_i^2$. Therefore $f(A_0)$ is contained in a set of measure less than $8\varepsilon'$. Since ε' is arbitrary, the measure of $f(A_0)$ must be zero.

Step 2. We now show that $f(A_1)$ has measure zero. Let a_i be the center of D_i. Let a be any other point of $A_1 \cap D_i$. Consider a wedge of D_i with central angle $1/K$ containing a. Cut this wedge into K segments whose distance from a_i is between $(k-1)/K$ and k/K units. Each such segment has area at least $(d_i^2/2K^3)$. Our choice of β guarantees that each segment contains at least one point of A_1 (see Fig. 1). Let $b_1 = a_i, b_2, \ldots, b_r = a$ be points of A_1 with b_i in the ith segment. Then

$$|f(a) - f(a_i)| = \Big|\sum_{l=2}^{r} f(b_l) - f(b_{l-1})\Big|$$
$$\leqslant \sum_{l=2}^{r} M\|b_l - b_{l-1}\|^2$$
$$\leqslant \sum_{l=2}^{r} 5Md_i^2/K^2$$
$$\leqslant 5Md_i^2/K.$$

Thus, $f(A_i)$ is contained in a set of measure $(10M/K)(\Sigma d_i^2)$. K is arbitrary and condition (i) of Lemma 2.2 guarantees that $\Sigma d_i^2 \leqslant 2\mu(A)$. Therefore $f(A_1)$ also has measure zero. ■

COROLLARY 2.3. *If $A \subset R^2$, then $\partial(\varepsilon, A)$ is a 1-manifold for almost all ε.*

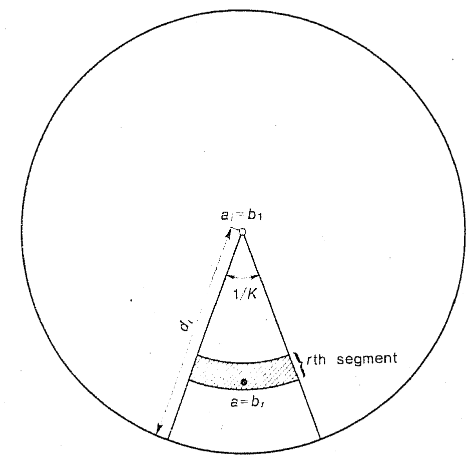

Fig. 1

Proof. By Proposition 1.1, we need to show that $\delta(C)$ has measure zero. By Proposition 1.5 and Theorem 2.1, $\delta^2(C)$ has measure zero. Since the square root function is absolutely continuous $\delta(C)$ also has measure zero. ■

3. In this section we will prove the analog of Corollary 2.3 for subsets of R^3. The main observation is that the critical points are constrained to lie near a countable set of planes. We can project onto these planes and obtain a countable number of functions like those studied in the last section.

THEOREM 3.1. *If $A \subset R^3$, then $\partial(\varepsilon, A)$ is a 2-manifold for almost all ε.*

Proof. The proof is divided into two steps.

Step 1. We will show that for each $x \in C$ there is a vector v such that

(i) $\|(x+v) - y\| \geqslant \|v\|$ and

(ii) $\|(x-v) - y\| \geqslant \|v\|$

for every $y \in C$. These inequalities say that the critical set lies outside of two tangent balls.

Choose v so that $x + 2v$ and $x - 2v$ are both in $\mathcal{H}(N(x))$. This is possible since x is not an extreme point of $\mathcal{H}(N(x))$. Let y be an arbitrary critical point. By definition, $y \in \mathcal{H}(N(x))$ and there are sequences $\{x_i\}$ and $\{y_i\}$ of points and $\{s_i\}$ and $\{t_i\}$ of scalars such that

(iii) $\Sigma s_i x_i = 2v$ and $\Sigma t_i y_i = 0$,

(iv) $\Sigma s_i = \Sigma t_i = 1,\ s_i, t_i > 0$,

(v) $x + x_i \in N(x)$ and $y + y_i \in N(y)$ for all i.

For each i,

$$\|x+x_i-y\|^2 \geqslant \delta^2(y) \quad \text{and} \quad \|y+y_i-x\|^2 \geqslant \delta^2(x).$$

Therefore,

(vi) $\sum_i (\|x\|^2+\|x_i\|^2+\|y\|^2+2x\cdot x_i-2x\cdot y-2x_i\cdot y)s_i = \|x\|^2+\delta^2(x)+\|y\|^2+ +4x\cdot v+2x\cdot y-4v\cdot y \geqslant \delta^2(y)$.

Similarly,

(vii) $\|y\|^2+\delta^2(y)+\|x\|^2-2x\cdot y \geqslant \delta^2(x)$.

Adding (vi) and (vii), we get

$$2\|x\|^2+2\|y\|^2+4x\cdot v-4v\cdot y-4x\cdot y \geqslant 0.$$

This is equivalent to $\|(x+v)-y\| \geqslant \|v\|$. The inequality (ii) follows by an entirely analogous argument. This completes step 1.

For each $x \in C$, choose $v(x)$ such that inequalities (i) and (ii) hold. Using the fact that a countable union of sets of measure zero has measure zero, we can make the following assumptions about C.

(viii) There is an integer k, independent of x, such that $\|v(x)\| \geqslant 1/k$ for each $x \in C$.

(ix) $\|x-y\| < 1/k$ for all $x, y \in C$.

Choose $x_0 \in C$ and let $P\colon R^3 \to R^3$ be the projection of R^3 onto the plane normal to $v(x_0)$.

Step 2. We will show that $\|P(x)-P(y)\| \geqslant \frac{1}{4}\|x-y\|$ for all x, y in C.

$$\|P(x)-P(y)\| = \|P(x-y)\| = \left\|(x-y)-\frac{(x-y)\cdot v(x_0)}{\|v(x_0)\|}\right\|$$

$$\geqslant \|x-y\|\left\|1-\frac{(x-y)\cdot v(x_0)}{\|x-y\|\|v(x_0)\|}\right\|.$$

We estimate the expression on the right.

$$\left\|\frac{(x-y)\cdot v(x_0)}{\|x-y\|\|v(x_0)\|}\right\| = \left\|\frac{(x-y)\cdot v(x)}{\|x-y\|\|v(x)\|}-\frac{(x-y)}{\|x-y\|}\cdot\left(\frac{v(x)}{\|v(x)\|}-\frac{v(x_0)}{\|v(x_0)\|}\right)\right\|$$

$$\leqslant \left|\frac{(x-y)\cdot v(x)}{\|x-y\|\|v(x)\|}\right| + \frac{1}{4}.$$

Therefore,

$$\|P(x)-P(y)\| \geqslant \|x-y\|\left\|\left(\frac{3}{4}-\left|\frac{(x-y)\cdot v(x)}{\|x-y\|\|v(x)\|}\right|\right)\right\|.$$

Inequalities (i) and (ii) guarantee that

$$\tfrac{1}{2}\|x-y\|^2 \geqslant |(x-y)v(x)|.$$

Therefore,

$$\|P(x)-P(y)\| \geqslant \|x-y\|\left(\tfrac{3}{4}-\tfrac{1}{2}(\|x-y\|/\|v(x)\|)\right).$$

Inequalities (viii) and (ix) insure that $\|x-y\|/\|v(x)\| \leqslant 1$. This completes step 2.

The rest is easy. $P|C$ has an inverse function, so we can define $\delta^2 P^{-1}\colon P(C) \to R^1$. From the above it is clear that

$$\|\delta^2 P^{-1}(x)-\delta^2 P^{-1}(y)\| \leqslant 16\|x-y\|^2$$

for all x, y in C. By Theorem 2.1, the image of $\delta^2 P^{-1}$ has measure zero. Since the image of $\delta^2 P^{-1}|P(C)$ is precisely $\delta^2(C)$, we are done. ■

4. Before proving the collaring theorem, we note that Theorem 3.1 can be improved if A is bounded.

THEOREM 3.1*. *Let $A \subset R^n$ be a bounded set, $n \leqslant 3$. The set of ε for which $\partial(\varepsilon, A)$ is an $(n-1)$-manifold contains an open dense set.*

Proof. The critical set C is bounded when A is bounded. It is always closed. Therefore, $\delta(C)$ is compact. The result now follows from Theorem 3.1. ■

THEOREM 4.1. *Let $A \subset R^n$. If $0 < a < b$ and $[a, b]$ contains no critical values, then $\delta^{-1}[a, b]$ is homeomorphic to $[0, 1]\times\delta^{-1}(a)$.*

Proof. Assume that A is compact. Let a and b be given as above. As usual, the proof has two steps.

Step 1. We show there is a fixed $\varepsilon > 0$ such that for each x in $\delta^{-1}[a, b]$ the ball of radius ε about x does not intersect $\mathcal{H}\{a \in A \mid \|x-a\| \leqslant \delta(x)+\varepsilon\}$. The proof is by contradiction. If the assertion is false, there are sequences $\{\varepsilon_i\}_1^\infty$, $\{x_i\}_1^\infty$, $\{t_{ij}\}_{i=1,\, j=0}^{\infty,\ n}$ and $\{a_{ij}\}$ such that

(i) $\delta(x_i) \in [a-\varepsilon_i, b+\varepsilon_i]$,

(ii) $x_i = \sum_{j=0}^{n} t_{ij} a_{ij},\ a_{ij} \in A,\ t_{ij} \geqslant 0,\ \sum_{j=0}^{n} t_{ij} = 1$,

(iii) $\|x_i - a_{ij}\| < 2\varepsilon_i + \delta(x_i)$,

(iv) $\lim \varepsilon_i = 0$.

Here we are using the fact that if B is a compact subset of R^n and $x \in \mathcal{H}(B)$, then there are elements $b_0, \ldots, b_n$ of B such that $x \in \mathcal{H}(\{b_i\})$.

$\{x_i\}$ has a convergent subsequence. By taking subsequences of subsequences, we can find x^*, $\{a_i\}_0^n$, $\{t_i\}_0^n$, such that $\sum_{i=0}^{n} t_i a_i = x^*$ and $\|x^* - a_i\| = \delta(x^*)$, $a_i \in A$, $t_i \geqslant 0$, $\Sigma t_i = 1$. This makes x^* a critical point, a contradiction with completes the proof.

Step 2. We wish to define a vector field $w(x)$ in such a way that trajectories of the vector field intersect each surface $\delta^{-1}(c)$ in a single point. Cover $\delta^{-1}[a, b]$ by a locally finite collection of balls of radius $\frac{1}{2}\varepsilon$. Let x_i be the center of the ball B_i, $i = 1, 2, \ldots$ Choose a vector $v(x_i)$ such that for each y such that $\|x_i - y\| < \varepsilon$ and $a \in A$ such that $\|a - x_i\| < \delta(x_i) + \varepsilon$ we have $v(x_i)\cdot(y-a) > 0$. That there is such a vector follows from our choice of ε. Let $\{\varphi_i\}$ be a smooth partition of unity subordinate to the cover $\{B_i\}$ of $\delta^{-1}[a, b]$. Define a vector field by $w(z) = \Sigma\varphi_i(z)\cdot v(x_i)$. Let $\psi(t)$ be an integral curve of this vector field. This means that ψ: $R^1 \to R^n$ and $\psi'(t) = w(\psi(t))$. Let t and s be real numbers, $t > s$

$$\begin{aligned}\psi(t) - \psi(s) &= \int_s^t \psi'(x)\,dx \\ &= \int_s^t \sum_{i=1}^{m} \varphi_i(\psi(x))\cdot v(x_i)\,dx \\ &= \sum_{i=1}^{m} c_i v(x_i), \qquad c_i \geqslant 0,\ c_i = t - s\,.\end{aligned}$$

We may assume that t and s are chosen so that for some fixed i, $\psi(r) \in B_i$ for all r between s and t. Let $A_i = \{a \in A \mid \|x_i - a\| \leqslant \delta(x_i) + \varepsilon\}$. A_i is compact. Let d be the Euclidean distance function. It is clear that $d(\psi(r), A) = d(\psi(r), A_i)$ for all r between s and t. It will suffice to show that for each $a \in A_i$ we have $d(\psi(t), a) > d(\psi(s), a)$. If $a \in A_i$,

$$\|\psi(t) - a\|^2 - \|\psi(s) - a\|^2 = \|\psi(s) - \psi(t)\|^2 + 2(\psi(t) - \psi(s))\cdot(\psi(s) - a)\,.$$

But,

$$(\psi(t) - \psi(s))\cdot(\psi(s) - a) = \Sigma c_i v(x_i)\cdot(\psi(s) - a)\,.$$

Since $\|x_i - \psi(s)\| < \varepsilon$ for each i with $\varphi_i \neq 0$ we have $v(x_i)\cdot(\psi(s) - a) > 0$. Therefore, $\delta(\psi(t)) > \delta(\psi(s))$.

The proof in the noncompact case is somewhat more complicated. Instead of ε we must use $\varepsilon(x)$. We omit the details. ∎

5. Counterexamples. The examples in this paper are variations of a simple construction which we now outline. Let $A \subset R^n$ be a closed set and let $* \in A$ be a point. Define a function f: $A \to R^1$ by the formula

$$f(a) = \inf\Big\{\sum_{i=1}^{k} \|a_i - a_{i-1}\|^2 \,\Big|\, a_i \in A,\ a_0 = *,\ a_k = a, \text{ and } k \text{ an integer}\Big\}\,.$$

It is clear that $|f(x) - f(y)| \leqslant \|x - y\|^2$ for each $x, y \in A$. Now, form $A^* = \{(a, \pm\sqrt{f(a)}) \in R^n \times R^1 = R^{n+1}\}$. It is not hard to see (and will be shown in the examples) that $A \times 0 \subset R^{n+1}$ is contained in the critical set of A^*.

Example 1. We construct a set $A \subset R^2$ such that $\partial(\varepsilon, A)$ fails to be a 1-manifold for uncountably many ε.

Proof. Let C be the Cantor set obtained by starting with $[0, 1]$ and removing the middle *two* thirds at each stage. The complement of C in $[0, 1]$ is a countable union of disjoint open intervals. We write $[0, 1] - C = \bigcup_n (a_n, b_n)$. Let $f(a) = \frac{1}{2}\sum_{b_n \leqslant a} \|b_n - a_n\|^2$ for each $a \in C$. We define

$$A = \{(a, \pm\sqrt{f(a)}):\ a \in C\}\,.$$

We will identify C with $\{(a, 0):\ a \in C\} \subset R^2$. For convenience, we will refer to the point $(a, 0)$ simply as a. We will refer to the points $(a, \pm\sqrt{f(a)})$ as a^+ and a^-, respectively.

Step 1. We show that $\delta(a) = \sqrt{f(a)}$.

Proof. Since $\|a - a^+\| = \sqrt{f(a)}$, we have $\delta(a) \leqslant \sqrt{f(a)}$. On the other hand, let $\beta \in C$. Then $\|a - \beta\|^2 = |a - \beta|^2 + f(\beta)$. If $\beta > a$, we are in fine shape, since f is increasing. If $\beta < a$

$$f(a) - f(\beta) = \tfrac{1}{2}\sum_{\beta < b_n \leqslant a_n} |b_n - a_n|^2 \leqslant \tfrac{1}{2}|a - \beta|^2\,.$$

Therefore

$$\|a - \beta^+\|^2 > |a - \beta|^2 + f(a) - \tfrac{1}{2}|a - \beta|^2 > f(a)\,.$$

This completes the proof of step 1.

Step 2. We wish to show that if $b_m < a$ is sufficiently close to a, there is a point of $\partial(\delta(a), A)$ in the interval (a_m, b_m).

First, we notice that if a is not one of the b_m's, then for large n there is an interval (a_m, b_m) of length $\frac{4}{6}(\frac{1}{6})^n$ with $\|b_m - a\| < (\frac{1}{6})^{n+1}$ and $b_m < a$. This is clear from the construction of C.

Second, we shall show that the point $t_m = \frac{1}{4}(a_m + 3b_m)$ has the property that $d(t_m, A) > \delta(a)$. Since $d(a_m, A) < d(b_m, A) = \delta(b_m) < \delta(a)$, this will complete step 2. We can compute that

$$\|t_m - b_m^+\|^2 = \|t_m - a_m^+\|^2 = f(b_m) + (\tfrac{1}{6})^{2n+2}\,.$$

Since we know that $\|b_m - a\| < (\frac{1}{6})^{n+1}$, we have $f(a) - f(b_m) \leqslant \frac{1}{2}(\frac{1}{6})^{2n+2}$. Therefore $\|t_m - b_m^+\|^2 \geqslant f(a)$. Now we need to show that b_m^+ and a_m^+ are the closest points of A to t_m. If $\beta > b_m$, $\|\beta^+ - t_m\| > \|b_m^+ - t_m\|$. If $a < a_m$, we can write

$$\|t_m - a^+\|^2 = f(a) + \|t_m - a\|^2 \geqslant f(a) + \|a_m - a\|^2 + \|t_m - a_m\|^2$$
$$> f(a_m) + \|t_m - a_m\|^2 = \|t_m - a_m^+\|^2 .$$

Thus, the interval (a_m, b_m) contains points of $\partial(\delta(a), A)$. It is clear that the line from b_m^+ to b_m^- cannot contain any points of $\partial(\delta(a), A)$. Therefore, if a is not one of the b_m's, then $\partial(\delta(a), A)$ is not locally connected and cannot be a 1-manifold. This completes Example 1. ∎

EXAMPLE 2. We construct a set $B \subset R^3$ such that $\partial(\varepsilon, B)$ has components which are not 2-manifolds for uncountably many ε. Thus, Brown's theorem [1] does not generalize to R^3.

Proof. Let B be $A \times 0 \subset R^2 \times R^1 = R^3$, where A is the set from Example 1. The proof in Example 1 shows that the component of $\partial(\delta(a), A)$ between the lines $x = a_m$ and $x = b_m$ consists of a simple closed curve which is the union of segments of circles of radius $\delta(a)$ centered at $a_m^{\pm}$ and $b_m^{\pm}$.

There are no points of $\partial(\delta(a), B)$ on the line $l_m = \{(t_m, 0, z) \in R^3\}$. Therefore, this simple closed curve is not contractible in $R^3 - l_m \supset \partial(\delta(a), B)$. It is easy, on the other hand, to see that the part of $\partial(\delta(a), B)$ in the region $x \leqslant a$ of R^3 is connected. ∎

EXAMPLE 3. We construct a set $A \subset R^4$ such that $\partial(\varepsilon, A)$ is not a 3-manifold for any ε such that $0 < \varepsilon < \frac{1}{100}$.

Proof. This example is basically due to Whitney [6]. Omitted details, including a picture, may be found there. Let Q be the unit cube in R^3 and let $Q(0), \ldots, Q(7)$ be subcubes of side $\frac{2}{5}$ placed so that each is $\frac{1}{20}$ unit from the boundary of Q and $\frac{1}{10}$ unit from the boundary of adjacent subcubes. Choose the enumeration of the cubes in such a way that $Q(i)$ and $Q(i+1)$ are adjacent. Let $Q(0,1), \ldots, Q(0,7)$ be cubes inside of $Q(0)$ arranged in the analogous fashion. The intersection of these sets of cubes is a Cantor set, K.

We can write each point of K as $Q(i_1, \ldots, i_k, \ldots)$ with $0 \leqslant i_j \leqslant 7$ for each j. Define $f\colon K \to R^1$ by $f(Q(i_1, \ldots, i_k, \ldots)) = \frac{1}{100}\sum_{k=1}^{\infty} i_k/8^k$. Let

$$A = \{(a, \pm\sqrt{f(a)}) \in R^3 \times R^1 |\ a \in K\} .$$

We will use the same notation, a, a^+, a^- as before.

Step 1. We wish to show that if $a \in Q(i_1, \ldots, i_k)$ for k large and $i_k \neq 0$, there is a component of $\partial(\delta(a), A)$ in the area near $Q(i_1, \ldots, i_{k-1})$.

Proof. Let k be large enough that $(\frac{1}{8})^{k-1} < \frac{1}{400}(\frac{2}{5})^{k-1}$. Let $a \in Q(i_1, \ldots, i_k)$. Let Q^* be the boundary of a cube with sides of length $\frac{1}{2}(\frac{2}{5})^{k-2}$ parallel to $Q(i_1, \ldots, i_{k-1})$. We claim that if $x \in Q^*$, then $\delta(x) > \sqrt{f(a)}$.

Let $\beta \in K$. We will estimate $\|x - \beta^+\|^2$.

Case 1. $\beta \notin Q(i_1, \ldots, i_{k-1})$.

In this case, $\beta \in Q(i_1, \ldots, i_l)$ and $\beta \notin Q(i_1, \ldots, i_{l+1})$ for some l with $0 \leqslant l \leqslant k-2$. Then

$$|f(a) - f(\beta)| < \frac{1}{100}\sum_{r=l+1}^{\infty} (\tfrac{7}{8})^r = \frac{1}{100}(\tfrac{1}{8})^l ,$$

and

$$\|x - \beta^+\|^2 = \|x - \beta^+\|^2 + f(\beta) > \frac{1}{100}(\tfrac{4}{25})^l + f(\beta) > f(a) .$$

Case 2. If $\beta \in Q(i_1, \ldots, i_{k-1})$, then

$$\|x - \beta\|^2 > [\tfrac{1}{20}(\tfrac{2}{5})^{k-1}]^2 \quad \text{and} \quad f(a) - f(\beta) < \tfrac{1}{400}(\tfrac{1}{8})^{k-1} .$$

In this case we also have $\|x - \beta^+\|^2 \geqslant f(a)$. A similar calculation shows that $\delta(a) = \sqrt{f(a)}$. This completes step 1.

Step 2. We show that if $a = Q(i_1, \ldots, i_k, \ldots)$ with infinitely many $i_k \neq 0$ there is no neighborhood U of a with diameter $< \frac{1}{2}\delta(a)$ such that $U \cap \partial(\delta(a), A)$ is homeomorphic with R^3.

Let $\beta \in Q(i_1, \ldots, i_{k-1}) \cap K$. There are no elements of $\partial(\delta(a), A)$ on the line l_β from β^- to β^+. On the other hand, we have shown in Step 1 that $\delta^{-1}[\delta(a)]$ separates β from Q^*. Thus, by Alexander duality, there is a two-dimensional Čech cocycle in $\partial(\delta(a), A) \cap R^3$ which does not bound in $R^3 - \beta$. Therefore, it cannot bound in $U - l_\beta$, which contains $\partial(\delta(a), A) \cap U$.

This leaves us with the case where $a = Q(i_1, i_2, \ldots)$ and $i_k = 0$ for all $k \geqslant K$. In this case, there is a neighborhood U of a such that $f(\beta) > f(a)$ for all $\beta \in K \cap U$. Let $u_0 \in U \cap R^3$. Then the vertical ray through u_0 in the upper half space of R^4 will contain precisely one point of $\partial(\delta(a), A)$ which is within $\frac{1}{2}\delta(a)$ of R^3. The same is true of the ray in the negative direction. Therefore, a has a neighborhood in $\partial(\delta(a), A)$ which is homeomorphic to a double cone with a as the cone point. This completes the last example. ∎

6. Addendum.

There is a related theorem of independent interest. Let A be a closed subset of R^n.

THEOREM 6.1. *Let* $K = \{x \in R^n |\ \operatorname{Int}\mathcal{H}(N(x)) \neq \emptyset\}$. *$K$ is countable.*

Proof. Let $x, y \in K$ with $x \neq y$. Let $S(x)$ and $S(y)$ be the spheres of radius $\delta(x)$ and $\delta(y)$ about x and y, respectively. Then $N(x)$ and $N(y)$ lie on opposite sides of the plane through the intersection of $S(x)$ and $S(y)$. Therefore, $[\operatorname{Int}\mathcal{H}(N(x))] \cap [\operatorname{Int}\mathcal{H}(N(y))] = \emptyset$. A collection of disjoint open subsets of R^n is countable. ∎

COROLLARY. *For* $A \subset R^2$, *let* $K = \{x \in R^2 |\ N(x)$ *has at least 3 points*$\}$. *K is countable.*

Proof. The proof is immediate. ∎

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Superpositions of transformations of bounded variation

by

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Abstract. The work deals with some classification of continuous functions transforming plane into plane. For every finite or countable ordinal number was defined a class of superposition of functions of bounded variation (in the sense of Rado, see [3]). The main results of the work are following theorems: every class of superpositions is non-empty and there exists a continuous function which does not belong to the sum of all classes. Similar results for real functions of real variable are included in the classical work [1] by Nina Bary.

Nina Bary in [1] has studied the possibility of representing arbitrary real continuous functions of a real variable only by superpositions of continuous functions of bounded variation. She has introduced the notion of superposition of class a for every finite or countable ordinal a and she has proved that all classes of superpositions are non-empty and that their sum is not equal to the class of all continuous functions. This work contains similar results for plane transformations defined on the unit square (open or closed). The notion of transformation of bounded variation is taken from [3] and [4]. The definition of superposition of class a is similar to that in [1] if a is a countable ordinal of the first kind (i.e. having a predecessor) and differs from the definition in [1] by using uniform convergence instead of ordinary convergence if a is a countable ordinal of the second kind.

The work consists of two parts. The first part contains the proof of an auxiliary theorem which explains the structure of plane transformations F_1, F_2 such that their superposition $F = F_2 \circ F_1$ is of the form $F(x^1, x^2) = \big(f(x^1), x^2\big)$ for $(x^1, x^2) \in [0, 1] \times [0, 1]$. The second part contains several theorems concerning superpositions of transformations of bounded variation. The main results of the work are: Theorem 13, which states that every class of superpositions is non-empty, and Theorem 14, which gives the construction of continuous plane transformation which does not belong to the sum of all classes (both these theorems deal with transformations defined on the open unit square) and the corollary (after Theorem 14), which includes the same results for transformations defined on the closed unit square.