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Function spaces with intervals as domain spaces

by

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Abstract. An example is given of a pseudo-complete, separable metric space Y such that the space of continuous functions from the closed unit interval into Y is of first category, where the topology on the function space may be taken to be any of the following: supremum metric, compact-open, pointwise convergence. Then conditions are given which guarantee that a function space with an interval as domain space and with compact-open topology be pseudo-complete, and hence of second category.

A well-known theorem in topology and analysis says that the supremum metric on a function space is complete whenever the metric on the range space is complete (the converse is also true). In this paper we take a particular space — the closed unit interval I — and consider the general question as to what "complete-type" properties can one obtain on a function space with domain space I when the property of completeness on the range space is relaxed. An example is given showing that even if the range space is a pseudo-complete, separable metric space, with no further conditions the function space with domain space I may be of first category — far from complete. However, we then give certain conditions on the range space (which do not imply completeness) insuring that the function space with I as domain space be pseudo-complete, and hence of second category.

1. Basic definitions. A subset of the topological space X is *of first category in X* provided that it can be written as the countable union of nowhere dense subsets of X (i.e., subsets of X whose closures have no interior points). If a subset of X is not of first category in X, then it is *of second category in X*. A space is *of first category* (*second category*, respectively) if it is of first category (second category, respectively) in itself. A space having the property that every open subspace is of second category is called a *Baire space*.

The Baire Category Theorem says that every complete metric space is a Baire space. In some cases one needs to have a complete space only to use such a theorem as the Baire Category Theorem, so that a natural question is whether one may weaken the completeness property on the range space and still retain some generalization of completeness, such as

Baire space, on the function space. A property which is very close to completeness is that of pseudo-completeness. A space is *pseudo-complete* provided that it is a quasi-regular space having a sequence $\{\mathcal{P}_n\}$ of pseudo-bases such that if $P_n \in \mathcal{P}_n$ and $\overline{P_{n+1}} \subset P_n$ for each n, then $\bigcap_{n=1}^{\infty} P_n \neq \emptyset$, where quasi-regular and pseudo-base are defined as follows. A space is *quasi-regular* if every nonempty open set contains the closure of some nonempty open set; and a collection of nonempty open sets is a *pseudo-base* for a space if each nonempty open set contains some member of this collection. Every pseudo-complete space is known to be a Baire space. Also it was shown in [1] that a metrizable space is pseudo-complete if and only if it contains a dense topologically complete subspace.

We shall be concerned with three different commonly used topologies on function spaces — the supremum metric topology (topology of uniform convergence), the compact-open topology, and the topology of pointwise convergence. If X and Y are topological spaces, the set of continuous functions from X into Y will be denoted by $C(X, Y)$. In the case that (Y, d) is a bounded metric space, a metric $\hat{d}$, called the supremum metric, can be defined on $C(X, Y)$ by $\hat{d}(f, g) = \sup\{d(f(x), g(x))|\ x \in X\}$. We shall use the notation $C_d(X, Y)$ for this metric space. Also $C(X, Y)$ with the compact-open topology will be denoted by $C_k(X, Y)$, where the compact-open topology is the topology on $C(X, Y)$ generated by the subbase of all sets $\langle K, U\rangle \equiv \{f \in C(X, Y)|\ f(K) \subset U\}$, where K is compact in X and U is open in Y. In the case that X is compact and (Y, d) is a bounded metric space, it is a standard theorem that $C_d(X, Y)$ and $C_k(X, Y)$ are identical spaces. Finally, the topology of pointwise convergence on $C(X, Y)$ is defined the same as the compact-open topology except that points are used instead of compact sets. This space will be denoted by $C_p(X, Y)$, and can be considered as a subspace of $\prod_{x \in X} Y_x$ with the product topology, where each Y_x is a copy of Y.

Throughout this paper, Y will be assumed to have metric d whenever the space $C_d(X, Y)$ is discussed, otherwise Y need not be a metrizable space unless explicitly stated. In certain cases, the domain space X will be taken to be the closed interval from 0 to 1 with the usual topology; this space will be denoted by I. The term J will be used to denote an arbitrary interval. Also N will denote the set of natural numbers.

2. A first category function space with pseudo-complete range space. The first theorem gives a condition on a subspace of a function space implying that it be of first category, and will be used to establish Theorems 2.2 and 2.5.

THEOREM 2.1. *Let F be a subspace of either $C_d(X, Y)$, $C_k(X, Y)$, or $C_p(X, Y)$ such that for some $x \in X$, $\{f(x)|\ f \in F\} = \bigcup_{n=1}^{\infty} Y_n$, where each Y_n is closed and nowhere dense in Y, and for every positive integer n, for every $f \in F$ with $f(x) \in Y_n$, and for every neighborhood W of f in F, there exists a $g \in W$ such that $g(x) \notin Y_n$. Then F is of first category.*

Proof. For each $n \in N$, let $F_n = \{f \in F|\ f(x) \in Y_n\}$.

To see that each F_n is closed in F, let $f \in F \setminus F_n$. Now let V be a neighborhood of $f(x)$ contained in $Y \setminus Y_n$. In the case that F is a subspace of $C_d(X, Y)$, such a neighborhood can be taken to be the ε-nieghborhood about $f(x)$ for some $\varepsilon > 0$. Then let W be the ε-neighborhood about f in F. In the case that F is a subspace of $C_k(X, Y)$ or $C_p(X, Y)$, define W to be $\langle\{x\}, V\rangle \cap F$. In any case, if $g \in W$, then $g(x) \in Y \setminus Y_n$, so that $g \in F \setminus F_n$. Therefore W is a neighborhood of f contained in $F \setminus F_n$, so that F_n is closed. Now by the hypotheses, each F_n has no interior point, so is nowhere dense. Since $F = \bigcup_{n=1}^{\infty} F_n$, then F is of first category.

THEOREM 2.2. *There exists a pseudo-complete, separable metric space (Y, d) such that $C_p(I, Y)$, $C_k(I, Y)$, and $C_d(I, Y)$ are all of first category.*

Proof. Let I_D be the dyadic rationals in I, and let I_p be the irrationals in I. Define Y to be the set $(I_D \times I) \cup (I_p \times I_p)$, and let it have the metric d which is inherited from the usual metric on the plane. Notice that Y is pseudo-complete since $I_p \times I_p$ is a topologically complete dense subspace of Y. We shall only consider the case of $C_k(K, Y)$ since the proof for the case of $C_p(I, Y)$ is similar and since $C_d(I, Y)$ and $C_k(I, Y)$ are identical (because I is compact).

If $I_D = \{r_n|\ n \in N\}$, define $F_n = \{f|\ \pi_1 f(I) = \{r_n\}\}$ and let $F = \bigcup_{n=1}^{\infty} F_n$. Also for each n, define $Y_n = \{r_n\} \times I$, which is closed and nowhere dense in Y. We wish to establish that F and $\{Y_n\}$ satisfy the hypotheses of Theorem 2.1. Let $n \in N$, let $f \in F_n$, and let W be a neighborhood of f in F. We may suppose that $W = \langle K_1, V_1\rangle \cap \ldots \cap \langle K_m, V_m\rangle \cap F$ where $K_1, \ldots, K_m$ are compact in I and $V_1, \ldots, V_m$ are open in Y. Let

$$\varepsilon = \min\{d(f(K_i), Y \setminus V_i)|\ i = 1, \ldots, m\},$$

which is positive since each K_i is compact. Now there exists an $m \in N$ such that $\max\{0, 1-\varepsilon\} < r_m < 1$. Define $\alpha\colon Y \to Y$ by $\alpha(s, t) = (r_m s, t)$. Let $g = \alpha \circ f$, which by construction of α is in $F \setminus F_n$. Also since α moves each point less that ε, $g \in W$. Therefore by Theorem 2.1, F is of first category.

Finally we wish to establish that $C_k(I, Y) \setminus F$, call it F_p, is nowhere dense in $C_k(I, Y)$. Because continuous functions preserve connectedness, F_p consists only of the constant maps from I into $I_p \times I_p$. Now suppose that $f \in C_k(I, Y)$ with f not a constant map. Then $\pi_2 f(I) = [a, b]$ for some $0 \leqslant a < b \leqslant 1$. Let $\varepsilon = (b-a)/2$, let $t_1, t_2 \in I$ such that $\pi_2 f(t_1) = a$ and $\pi_2 f(t_2) = b$, and let V_1 and V_2 be ε-neighborhoods in Y about $f(t_1)$ and $f(t_2)$, respectively. Then if $W = \langle\{t_1\}, V_1\rangle \cap \langle\{t_2\}, V_2\rangle$, $f \in W$. By the choice of ε, $W \subset C_k(I, Y)\setminus F_p$, so that $\overline{F}_p \subset \{f \in C_k(I, Y)|\ f$ is a constant map$\}$. (This last containment is actually an equality.) To see that $\overline{F}_p$ has no interior point, let $f \in \overline{F}_p$ and let $W = \langle K_1, V_1\rangle \cap \ldots \cap \langle K_m, V_m\rangle$ be a neighborhood of f. Since f is a constant map, say that constant is (s, t), there exists $\varepsilon > 0$ such that the ε-neighborhood of (s, t) is contained in $V_1 \cap \ldots \cap V_m$. Choose an $n \in N$ such that $|r_n - s| < \frac{1}{2}\varepsilon$, and let $a = \max\{0, t - \frac{1}{2}\varepsilon\}$ and $b = \min\{1, t + \frac{1}{2}\varepsilon\}$. Define $g \in C_k(I, Y)$ by $g(p) = (r_n, (b-a)p + a)$. It can be seen that $g \in W$ and g is not a constant map. Therefore $W \cap [C_k(I, Y)\setminus\overline{F}_p] \neq \emptyset$, so that F_p is nowhere dense. Since $C_k(I, Y) = F \cup F_p$, it follows that $C_k(I, Y)$ is of first category.

THEOREM 2.3. *If X is compact and (Y, d) has an open topologically complete subspace, then $C_d(X, Y)$ has an open topologically complete subspace, and hence is of second category.*

Proof. Let Z be an open topologically complete subspace of Y. Consider $C_d(X, Z)$ as a subspace, call it C_Z, of $C_d(X, Y)$. Let ϱ be a complete bounded metric on Z. Then since X is compact, $C_\varrho(X, Z)$ has the same topology as $C_d(X, Z)$. Since (Z, ϱ) is complete, then $C_\varrho(X, Z)$ is complete. Therefore C_Z is topologically complete. Now let $f \in C_Z$, so that $f(X) \subset Z$. Let ε be the distance between $f(X)$ and $Y\setminus Z$. Then the ε-neighborhood of f in $C_d(X, Y)$ is contained in C_Z, so that C_Z is open in $C_d(X, Y)$.

THEOREM 2.4. *If X is compact and Y has an open completely metrizable subspace, then $C_k(X, Y)$ has an open completely metrizable subspace, and hence is of second category.*

Proof. This is similar to the proof of Theorem 2.3.

We might add that the analog to Theorem 2.4 for $C_p(X, Y)$ instead of $C_k(X, Y)$ is false since $C_p(I, E^1)$ is not metrizable [3], where E^1 is the set of real numbers with the usual topology.

THEOREM 2.5. *There exists a separable metric space (Y, d) which has a dense, open, arcwise connected, topologically complete subspace such that $C_d(I, Y)$, $C_k(I, Y)$, and $C_p(I, Y)$ are all not Baire spaces. (However, $C_d(I, Y)$ and $C_k(I, Y)$ are of second category by Theorems 2.3 and 2.4).*

Proof. We shall modify the example in Theorem 2.2 and shall use the terminology defined in that proof. Also, as in Theorem 2.2, we shall consider only the case for $C_k(I, Y)$. Define Y to be the set $(I_D \times I \times \{0\}) \cup (I_p \times I_p \times I) \cup (I \times I \times \{1\})$, and let it have the metric d which is inherited from the usual metric on Euclidean 3-space. The desired dense, open, arcwise connected, topologically complete subspace of Y is $(I_p \times I_p \times I\setminus\{0\}) \cup (I \times I \times \{1\})$.

Let $Z = I_D \times I \times \{0\}$, and let $F = \{f \in C_k(I, Y)|\ f(I) \subset Z$ and f is not a constant map$\}$. By using Theorem 2.1, as was done in the proof of Theorem 2.2, it can be seen that F is of first category. Also F can be shown to be open in $C_k(I, Y)$ in a way very similar to the way in which F_p was shown to be nowhere dense in the proof of Theorem 2.1. Therefore $C_k(I, Y)$ is not a Baire space.

The underlying reason that $C_k(I, Y)$ is not a Baire space in Theorem 2.5 is that Y is not locally connected. This can be seen from Theorem 4.1, which will be proved in Section 4. However, we shall first need to discuss the topic of when a continuous function from a closed subspace of an interval J into some space has a continuous extension to all of J.

3. Absolute extensors of finite-dimensional metric spaces. Let Y be a space and let χ be a class of spaces. Then Y is called an *absolute extensor for χ* provided that for any closed subspace A of any $X \in \chi$, every continuous function $f\colon A \to Y$ has a continuous extension to all of X. In the case that $\chi = \{X\}$, we shall say that Y is an *absolute extensor for X*.

The concept of n-connectedness will appear in the next couple of theorems. If n is a nonnegative integer, a space Y is called *n-connected* provided that for every integer k with $0 \leqslant k \leqslant n$, every continuous function from the k-sphere, S^k, (lying in Euclidean $(k+1)$-space, E^{k+1}) into Y has a continuous extension to all of E^{k+1}. Also Y is called *locally n-connected* if for every integer k with $0 \leqslant k \leqslant n$, for every $y \in Y$, and for every neighborhood U of y in Y, there exists a neighborhood V of y contained in U such that every continuous function from S^k into V extends to a continuous function from E^{k+1} into U. Finally, the abbreviation $\dim X$ will be used to denote the covering dimension of X.

The following two theorems can be deduced from results of Dugundji in [2].

THEOREM 3.1. *Let n be a positive integer, let Y be a metric space, and let χ be any class of metric spaces such that:*

1) *for every $X \in \chi$, $\dim X \leqslant n$, and*

2) *there exists $X \in \chi$ such that X contains a copy of E^n embedded in it.*

Then Y is an absolute extensor of χ if and only if Y is $(n-1)$-connected and locally $(n-1)$-connected.

A space is *pathwise* connected if it is 0-connected, and it is *locally pathwise connected* if it is locally 0-connected. It is not difficult to see that a connected, locally pathwise connected space is pathwise connected. Now as a corollary to Theorem 3.1, we get the following result which will be used in the next section.

COROLLARY 3.3. *Let J be an interval and let Y be a metric space. Then Y is an absolute extensor of J if and only if Y is connected and locally pathwise connected.*

One half of this corollary can be generalized as follows.

THEOREM 3.4. *Let X be a Hausdorff space containing some nondegenerate path. Then if Y is a first countable absolute extensor of X, Y must be pathwise connected and locally pathwise connected.*

Proof. Let $g\colon I\to X$ be a nondegenerate path in X. Since X is a Hausdorff space, $g(I)$ is arcwise connected, so that there exists an embedding $h\colon I\to X$. Clearly Y must be pathwise connected. Now suppose that Y is not locally pathwise connected. Then there exist $y_0\in Y$ and neighborhood V of y_0 in Y such that for every neighborhood W of y_0 contained in V, there exists $w\in W$ such that there is no path from w to y_0 whose image lies entirely within V. Let $\{B_i|\ i\in N\}$ be a countable local base at y_0 with $B_{i+1}\subset B_i$ for every $i\in N$ and $B_1\subset V$. Then for every $i\in N$, let $y_i\in B_i$ such that there is no path from y_i to y_0 whose image lies entirely in V. Let $K=\{0\}\cup\left\{\frac{1}{i}\middle|\ i\in N\right\}$, which is closed in I and hence compact. Therefore $h(K)$ is compact and thus closed in X. Define the continuous function $f\colon h(K)\to Y$ by $f[h(0)]=y_0$ and for every $i\in N$, let $f\left[h\left(\frac{1}{2i}\right)\right]=y_0$ and $f\left[h\left(\frac{1}{2i-1}\right)\right]=y_i$. Now suppose $F\colon X\to Y$ is a continuous extension of f. Then for each $i\in N$, $\mathrm{Fh}\left(\left[\frac{1}{2i-1},\frac{1}{2i}\right]\right)$ is the image of a path from y_i to y_0, so there exists t_i such that $\frac{1}{2i-1}<t_i<\frac{1}{2i}$ and $\mathrm{Fh}(t_i)\notin V$. But $\{t_i|\ i\in N\}$ converges to 0, while $\{\mathrm{Fh}(t_i)|\ i\in N\}$ cannot converge to $y_0=\mathrm{Fh}(0)$. Hence Fh is not continuous — which is a contradiction. Therefore f has no continuous extension to Y, so that Y would not be an absolute extensor of I. Thus Y must be locally pathwise connected after all.

The first countability hypothesis in Theorem 3.4 cannot be omitted, as the following example shows. Let X be the real line with discrete topology except that the neighborhoods of 0 are precisely all subsets of X with countable complements. Let $Y=(X\times I)/(X\times\{0\})$ with the quotient topology. Now Y is neither first countable nor locally pathwise connected at the point $(0,1)$. To see that Y is an absolute extensor of I, let K be a closed subspace of I and let $f\colon K\to Y$ be continuous. Set $X_0=\{x\in X|$ there exists $0<t\leqslant 1$ with $(x,t)\in f(K)\}$. If X_0 were uncountable, then $\{f^{-1}(\{x\}\times(0,1])|\ x\in X_0\setminus\{0\}\}$ would be an uncountable disjoint collection of nonempty open subsets of K — which contradicts the fact that K is separable. Therefore X_0 is countable, so that X_0 as a subspace of X has the discrete topology. Let $Y_0=(X_0\times I)/(X_0\times\{0\})$, which is then a subspace of Y containing $f(K)$. Now Y_0 is a connected, locally pathwise connected, metrizable space, so that Y_0 is an absolute extensor of I by Corollary 3.3.

4. Pseudo-complete function spaces with intervals as domain spaces. Throughout this section the domain space of most of the function spaces will be an arbitrary interval J. Whenever Z is a subspace of Y, we shall consider $C_k(J,Z)$ as a subspace of $C_k(J,Y)$.

THEOREM 4.1. *Let Y be a locally connected space which contains a dense, locally pathwise connected, metrizable subspace Z with the property that $V\cap Z$ is connected whenever V is connected and open in Y. Then $C_k(J,Z)$ is dense in $C_k(J,Y)$.*

Proof. Let $f\in C_k(J,Y)$ and let $W=\langle K_1,V_1\rangle\cap\ldots\cap\langle K_m,V_m\rangle$ be a basic open set containing f, where $K_1,\ldots,K_m$ are compact in J; $V_1,\ldots,V_m$ are open in Y; and each $\langle K_i,V_i\rangle=\{g\in C_k(J,Y)|\ g(K_i)\subset V_i\}$. In order to complete the proof, we need to find a $g\in W$ such that $g(J)\subset Z$.

Let Y_f be the component of Y containing $f(J)$, and let $Z_f=Y_f\cap Z$, which is a nonempty connected open subspace of Z and is hence locally pathwise connected. For each $1\leqslant k\leqslant m$, let $S(k,1),\ldots,S(k,p(k))$ be all possible sets of precisely k distinct positive integers less than or equal to m such that $\bigcap\{K_n|\ n\in S(k,i)\}\neq\emptyset$ for every $1\leqslant i\leqslant p(k)$, if such sets exist. If for some $1\leqslant k\leqslant m$, no such sets exist, let $p(k)=0$. Let m_0 be the largest positive integer less than or equal to m such that $p(m_0)>0$. Now for each $1\leqslant k\leqslant m_0$ and $1\leqslant i\leqslant p(k)$, there exists a finite number of components $V(k,i,1),\ldots,V(k,i,q(k,i))$ of $\bigcap\{V_n\cap Y_f|\ n\in S(k,i)\}$ such that $f[\bigcap\{K_n|\ n\in S(k,i)\}]\subset V(k,i,1)\cup\ldots\cup V(k,i,q(k,i))$. For each $1\leqslant j\leqslant q(k,i)$, let $Z(k,i,j)=Z_f\cap V(k,i,j)$, which is connected and open in Z_f. Also let $K(k,i,j)=[\bigcap\{K_n|\ n\in S(k,i)\}]\cap f^{-1}[V(k,i,j)]$, which can be seen to be a compact subset of J. For each $1\leqslant k\leqslant m_0$, let $K(k)=\bigcup\{K(n,i,j)|\ k\leqslant n\leqslant m_0,\ 1\leqslant i\leqslant p(n)$, and $1\leqslant j\leqslant q(n,i)\}$.

Now for each $1\leqslant i\leqslant p(m_0)$ and $1\leqslant j\leqslant q(m_0,i)$, there exists a continuous function $g(m_0,i,j)\colon J\to Z(m_0,i,j)$. Define the continuous function $g(m_0)\colon K(m_0)\to Z_f$ by $g(m_0)(t)=g(m_0,i,j)(t)$ if $t\in K(m_0,i,j)$. With the intent of defining $g(1)\colon K(1)\to Z_f$ by induction, we suppose that for each $1\leqslant k\leqslant n$, where $n<m_0$, a continuous function

$$g(m_0-k+1)\colon K(m_0-k+1)\to Z_f$$

has been defined so that $g(m_0-k+1)[K(m_0-k+1,i,j)]\subset Z(m_0-k+1,i,j)$ for every $1\leqslant i\leqslant p(m_0-k+1)$ and $1\leqslant j\leqslant q(m_0-k+1,i)$. Then define $g(m_0-n)\colon K(m_0-n)\to Z_f$ as follows. First let $1\leqslant i\leqslant p(m_0-n)$ and $1\leqslant j\leqslant q(m_0-n,i)$. Suppose that $K(m_0-n,i,j)\cap K(m_0-n+1)=\emptyset$. Then there exists a continuous function $g(m_0-n,i,j)\colon J\to Z(m_0-n,i,j)$. On the other hand suppose that $K(m_0-n,i,j)\cap K(m_0-n+1)\neq\emptyset$. Then by Corollary 3.3, there exists a continuous function $g(m_0-n,i,j)$:

$J \to Z(m_0-n, i, j)$ which is an extension of $g(m_0-n+1)|_{K(m_0-n,i,j)\cap K(m_0-n+1)}$. Define the continuous function $g(m_0-n)\colon K(m_0-n) \to Z_f$ by $g(m_0-n)(t) = g(m_0-n+1)(t)$ if $t \in K(m_0-n+1)$ and $g(m_0-n)(t) = g(m_0-n, i, j)(t)$ if

$$t \in K(m_0-n, i, j) \setminus K(m_0-n+1).$$

Then by finite induction, the continuous function $g(1)\colon K(1) \to Z_f$ is defined so that $g(1)[K(1, i, j)] \subset Z(1, i, j)$ for every $1 \leqslant i \leqslant p(1)$ and $1 \leqslant j \leqslant q(1, i)$. But for each $1 \leqslant k \leqslant m$,

$$K_k = \bigcup \{K(1, i, j) |\ 1 \leqslant j \leqslant q(1, i)\}$$

for some $1 \leqslant i \leqslant p(1)$. Also for this i, $\bigcup \{Z(1, i, j) |\ 1 \leqslant j \leqslant q(1, i)\} \subset V_k$. Therefore, for each $1 \leqslant k \leqslant m$, $g(1)(K_k) \subset V_k$.

Finally, since $K(1) = \bigcup \{K_k |\ 1 \leqslant k \leqslant m\}$, it is a closed subset of J. Also Z_f is connected and locally pathwise connected, so that by Corollary 3.3 again, $g(1)$ has a continuous extension $g\colon J \to Z_f$, which is the desired element of W.

COROLLARY 4.2. *Let Y be a locally connected space which contains a dense completely metrizable subspace Z (so that Y is pseudo-complete) with the property that $V \cap Z$ is connected whenever V is connected and open in Y. Then $C_k(J, Z)$ is a dense completely metrizable subspace of $C_k(J, Y)$, so that $C_k(J, Y)$ is pseudo-complete.*

COROLLARY 4.3. *Let (Y, d) be a locally connected metric space which contains a dense, locally pathwise connected subspace Z with the property that $V \cap Z$ is connected whenever V is connected and open in Y. Then for every continuous function $f\colon I \to Y$ and every $\varepsilon > 0$, there exists a continuous function $g\colon I \to Z$ such that $d(f(x), g(x)) < \varepsilon$ for every $x \in I$.*

Corollary 4.2 follows from Theorem 4.1 since Z will be locally connected, and since a locally connected, complete metric space is locally pathwise connected. Also Corollary 4.3 follows from Theorem 4.1 and the fact that when the domain space compact, as is I, the supremum metric on the function space generates the compact-open topology.

We saw from Theorem 2.5 that the local connectedness condition on Y cannot be omitted from Theorem 4.1 or its corollaries, since the subspace Z of the space Y constructed in the proof of Theorem 2.5 has the property that $V \cap Z$ is connected whenever V is connected and open in Y.

Corollary 4.2 has the following partial converse.

THEOREM 4.4. *Let Y be a locally pathwise connected metric space. Then if $C_k(J, Y)$ is pseudo-complete, so is Y.*

Proof. Let t be an arbitrary element of J, and let $p_t\colon C_k(J, Y) \to Y$ be the projection of $C_k(J, Y)$ onto Y determined by t. That is, for each $f \in C_k(J, Y)$, $p_t(f) = f(t)$. It is clear that p_t is a continuous surjection. In order to see that p_t is also open, let $B = \langle K_1, V_1\rangle \cap \ldots \cap \langle K_n, V_n\rangle$ be a nonempty basic open subset of $C_k(J, Y)$. If t is contained in some K_i, let $V = \bigcap \{V_i |\ t \in K_i\}$. On the other hand, if $t \notin K_1 \cup \ldots \cup K_n$, let $V = Y$. Now let $f \in B$, and define V_f to be the component of V which contains $f(t)$. Let y be any element of V_f. We can find $a, b \in J$, with $a < t < b$, such that the interval $[a, b]$ intersects only those K_i which contain t and $f([a, b]) \subset V_f$. Define $g\colon \{a, b, t\} \to V_f$ by $g(a) = f(a)$, $g(b) = f(b)$, and $g(t) = y$. Now g has a continuous extension $\bar{g}\colon [a, b] \to V_f$. Define $\bar{f}\colon J \to Y$ by $\bar{f}(x) = \bar{g}(x)$ if $x \in [a, b]$, and $\bar{f}(x) = f(x)$ if $x \in J \setminus [a, b]$. It is easy to see that $\bar{f} \in B$ and $p_t(\bar{f}) = y$. Therefore $p_t(B) = \bigcup \{V_f |\ f \in B$ and V_f is the component of V containing $f(t)\}$, which is open in Y. Hence p_t is a continuous open function from the pseudo-complete space $C_k(J, Y)$ onto the metric space Y. Then by a theorem in [1], Y must be pseudo-complete.

If X is a rimcompact (has a base having members with compact boundaries) Hausdorff space, then γX will denote the Freudenthal compactification of X. Most of the properties of γX used in proving the following theorem can be found for example in [4].

THEOREM 4.5. *If X is a connected, locally pathwise connected, rimcompact metric space, then $C_k(J, X)$ is dense in $C_k(J, \gamma X)$.*

Proof. To begin with, γX has the following two properties: (1) for every $y \in \gamma X$ and neighborhood V of y in γX, there exists an open subset W of γX such that $y \in W \subset V$ and $\mathrm{Bd}\, W \subset X$, and (2) $V \cap X$ is connected whenever V is connected and open in γX and $\mathrm{Bd}\, V \subset X$. Also since X is connected and locally connected, γX will be locally connected. Therefore we simply need to modify the proof of Theorem 4.1 to prove the following: if Y is a locally connected space which contains a dense, connected, locally pathwise connected, metrizable subspace Z with the two properties (1) for every $y \in Y$ and neighborhood V of y in Y, there exists an open subset W of Y such that $y \in W \subset V$ and $\mathrm{Bd}\, W \subset Z$, and (2) $V \cap Z$ is connected whenever V is connected and open in Y and $\mathrm{Bd}\, V \subset Z$; then $C_k(J, Z)$ is dense in $C_k(J, Y)$.

This modification is done as follows. First, since Z is connected, take $Z_f = Z$ and $Y_f = Y$. Also for each $1 \leqslant k \leqslant m$, since $f(K_k)$ is a compact subset of Y contained in V_k, there exists an open subset W_k of Y such that $f(K_k) \subset W_k \subset V_k$ and $\mathrm{Bd}\, W_k \subset Z$. Now in constructing the $V(k, i, j)$ in the modification of the proof of Theorem 4.1 for each $1 \leqslant k \leqslant m_0$ and $1 \leqslant j \leqslant p(k)$, take the $V(k, i, 1), \ldots, V(k, i, q(k, i))$ to be components of $\bigcap \{W_n |\ n \in S(k, i)\}$ such that

$$f[\bigcap \{K_n |\ n \in S(k, i)\}] \subset V(k, i, 1) \cup \ldots \cup V(k, i, q(k, i)).$$

Each $V(k, i, j)$ is connected and $\mathrm{Bd}\,V(k, i, j) \subset Z$. Therefore each $Z(k, i, j) = Z \cap V(k, i, j)$ is connected and open in Z. The rest of the proof now needs no further modification.

COROLLARY 4.6. *If X is a Peano space, then $C_k(J, X)$ is dense in $C_k(J, \gamma X)$.*

We might note that if X is a Peano space, (i.e., a connected, locally connected, locally compact metric space), then γX is metrizable, say with metric d. Then in this case, the above corollary assures us that for each continuous function $f\colon I \to \gamma X$ and for each $\varepsilon > 0$, there exists a continuous function $g\colon I \to X$ such that $d(f(t), g(t)) < \varepsilon$ for every $t \in I$.

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