Semi-local lattices

by

Johnny A. Johnson (Houston, Tex.)

Abstract. Let $\bar{R}$ be the completion of a semi-local ring R. In this note it is shown that the lattice of ideals of $\bar{R}$ can be obtained by purely lattice-theoretical methods from the lattice of ideals of R.

1. Introduction. In this note we apply the theory of Noether lattices to the special case of the ideal-lattice of a semi-local ring R and its completion $\bar{R}$. We show that the lattice of ideals of $\bar{R}$ can be obtained by purely lattice-theoretical methods from the lattice of ideals of R. This is accomplished by using the concept of the completion of a Noether lattice introduced previously in [3].

2. Completions. Before proceeding we will require some terminology. For a Noether lattice $\mathfrak{L}$, let $\mathfrak{J}(\mathfrak{L})$ denote the greatest lower bound of the collection of maximal elements of $\mathfrak{L}$ and set

$$\partial\mathfrak{J}(\mathfrak{L}) = \{A \text{ in } \mathfrak{L} | A \geqslant \mathfrak{J}(\mathfrak{L})^n \text{ for some natural number } n\}$$

so that $\partial\mathfrak{J}(\mathfrak{L})$ is a sub-multiplicative lattice of $\mathfrak{L}$. A metric (called the $\mathfrak{J}(\mathfrak{L})$-adic metric) can be defined on $\mathfrak{L}$ as follows (cf. [3], Theorem 3.10, p. 352): for A and B in $\mathfrak{L}$, set $d(A, B) = 2^{-S(A,B)}$, where $S(A, B) = \bigvee \{n\colon A \vee \mathfrak{J}(\mathfrak{L})^n = B \vee \mathfrak{J}(\mathfrak{L})^n\}$. This metric gives rise to the $\mathfrak{J}(\mathfrak{L})$-adic completion $\mathfrak{L}^*$ of $\mathfrak{L}$ which is itself a Noether lattice (cf. [2], Theorem 5.9, p. 198). A Cauchy sequence $\langle B_i \rangle$ of elements $\mathfrak{L}$ is called completely regular in case $B_i = B_{i+1} \vee \mathfrak{J}(\mathfrak{L})^i$, for all integer $i \geqslant 1$. We adopt the lattice terminology of [3] and the ring terminology of [4]. In particular, all rings are commutative with identity.

In order to avoid repetition we fix our basic notation as follows. $(R, m_1, \ldots, m_n)$ is a semi-local ring, $\mathfrak{L}$ denotes the lattice of ideals of R, $m = \bigcap_{i=1}^{n} m_i$, the m-adic (ring) completion of R will be denoted by $\bar{R}$ and $\bar{\mathfrak{L}}$ denotes the lattice of ideals of $\bar{R}$. Then $\mathfrak{L}$ and $\bar{\mathfrak{L}}$ are semi-local Noether lattices. We will denote the $\mathfrak{J}(\mathfrak{L})$-adic (Noether lattice) completion of $\mathfrak{L}$

by $\mathfrak{L}^*$. We shall show that $\bar{\mathfrak{L}}$ and $\mathfrak{L}^*$ are isomorphic as multiplicative lattices. *It follows that the lattice of ideals of the completion $\bar{R}$ of a semi-local ring R can be obtained lattice theoretically from the lattice of ideals of R.*

We will require the following lemma in the sequel.

LEMMA 1. *Let a be an element of $\partial\mathfrak{J}(\mathfrak{L}^*)$ and let $\langle a_i\rangle$ be the completely regular representative of a (considered as an element of $\mathfrak{L}^*$). Then there exists a natural number k such that*

(i) $a_i\bar{R} \geqslant (m\bar{R})^k$ *for all* i.

(ii) $\bigcap_{j=1}^{\infty} a_j\bar{R} = a_i\bar{R}$ *for all* $i \geqslant k$.

Proof. If a is an element of $\partial\mathfrak{J}(\mathfrak{L}^*)$, then there exists a natural number n such that $a \geqslant (m\mathfrak{L}^*)^n = m^n\mathfrak{L}^*$. It follows that $a_i \geqslant m^n + m^i \geqslant m^n$, for all integers $i \geqslant 1$ (cf. [3], Remark 5.2, p. 356, and Proposition 5.9, p. 358), so that $a_i\bar{R} \geqslant m^n\bar{R} = (m\bar{R})^n$, for all i. Also, since $\bar{R}/(m\bar{R})^n$ is Artinian and $\langle a_i\bar{R}\rangle$ is decreasing, there exists a natural number w such that $\bigcap_{i=1}^{\infty} a_i\bar{R} = a_i\bar{R} = a_w\bar{R}$, for all $i \geqslant w$. Set $k = \max\{n, w\}$ which completes the proof.

For an element b in $\partial\mathfrak{J}(\mathfrak{L}^*)$, the completely regular representative $\langle b_i\rangle$ of b (in $\mathfrak{L}^*$) is uniquely determined ([3], Theorem 4.14, p. 356) and we set $\partial\varphi(b) = \bigcap_{i=1}^{\infty} b_i\bar{R}$, so that $\partial\varphi\colon \partial\mathfrak{J}(\mathfrak{L}^*)\to\partial\mathfrak{J}(\bar{\mathfrak{L}})$ by Lemma 1.

THEOREM 2. *The map $\partial\varphi\colon \partial\mathfrak{J}(\mathfrak{L}^*)\to\partial\mathfrak{J}(\bar{\mathfrak{L}})$ defined above is a multiplicative lattice homomorphism from $\partial\mathfrak{J}(\mathfrak{L}^*)$ into $\partial\mathfrak{J}(\bar{\mathfrak{L}})$.*

Proof. Let a and b be elements of $\mathfrak{L}^*$ with completely regular representatives $\langle a_i\rangle$ and $\langle b_i\rangle$, respectively.

By Lemma 1, there exist integers k_1, k_2, and k_3 such that

$$\bigcap_{j=1}^{\infty} a_j\bar{R} = a_i\bar{R} = a_{k_1}\bar{R} \quad \text{for all } i \geqslant k_1,$$

$$\bigcap_{j=1}^{\infty} b_j\bar{R} = b_i\bar{R} = b_{k_2}\bar{R} \quad \text{for all } i \geqslant k_2,$$

$$\bigcap_{j=1}^{\infty} (a_j+b_j)\bar{R} = (a_i+b_i)\bar{R} = (a_{k_3}+b_{k_3})\bar{R} \quad \text{for all } i \geqslant k_3.$$

If $w = \max\{k_1, k_2, k_3\}$, we have

$$\partial\varphi(a)+\partial\varphi(b) = (a_w+b_w)\bar{R} = \partial\varphi(a\vee b),$$

since the completely regular representative of $a\vee b$ is $\langle a_i\vee b_i\rangle$ ([3], Proposition 5.7, p. 358), and thus $\partial\varphi$ is a join-homomorphism.

Again, by Lemma 1, there exists an integer k_4 such that, for all integers $i \geqslant k_4$,

$$\bigcap_{j=1}^{\infty} [(a_jb_j+m^j)\bar{R}] = (a_ib_i+m^i)\bar{R} = (a_i\bar{R})(b_i\bar{R})+(m\bar{R})^i.$$

Thus, since the completely regular representative of ab is $\langle a_ib_i+m^i\rangle$ ([3], Corollary 5.15, p. 360), for $i = \max\{k_1, k_2, k_4\}$ we obtain

$$\partial\varphi(a)\partial\varphi(b) = (a_i\bar{R})(b_i\bar{R})+(m\bar{R})^i = \partial\varphi(ab)$$

and so $\partial\varphi$ preserves multiplication.

To see that $\partial\varphi$ is a meet-homomorphism choose k_5 (Lemma 1) such that, for all $i \geqslant k_5$,

$$\bigcap_{i=1}^{\infty}\Big(\big(\bigcap_{j=1}^{\infty}[(a_j\cap b_j)+m^i]\big)\bar{R}\Big) = \big(\bigcap_{j=1}^{\infty}[(a_j\cap b_j)+m^{k_5}]\big)\bar{R} = \big(\bigcap_{j=1}^{\infty}[(a_j\cap b_j)+m^i]\big)\bar{R}.$$

Set $w = \max\{k_1, k_2, k_5\}$ (so that in particular $a_i\bar{R}\cap b_i\bar{R} \geqslant (m\bar{R})^w$). Since R/m^w is artinian and $\langle(a_i\cap b_i)+m^w\rangle$, $i = 1, 2, \ldots$, is decreasing in R/m^w, there exists a natural number k_6 such that, for all integers $i \geqslant k_6$,

$$\bigcap_{j=1}^{\infty}[(a_j\cap b_j)+m^w] = (a_i\cap b_i)+m^w.$$

Setting $n = \max\{k_1, k_2, k_6\}$ we obtain

$$\bigcap_{j=1}^{\infty}[(a_j\cap b_j)+m^w]) = [(a_n\cap b_n)+m^w]\bar{R} = (a_n\bar{R}\cap b_n\bar{R})+(m^w\bar{R}).$$

Since the completely regular representative of $a\wedge b$ is

$$\Big\langle\bigcap_{j=1}^{\infty}\big((a_j\cap b_j)+m^i\big)\Big\rangle, \quad i = 1, 2, \ldots,$$

by combining the above we obtain

$$\partial\varphi(a)\cap\partial\varphi(b) = (a_n\bar{R})\cap(b_n\bar{R}) = (a_n\bar{R}\cap b_n\bar{R})+(m^w)\bar{R} = \partial\varphi(a\wedge b)$$

which completes the proof.

LEMMA 3. *If a is an element of $\partial\mathfrak{J}(\bar{\mathfrak{L}})$, then $(a\cap R)\bar{R} = a$.*

Proof. If a is in $\partial\mathfrak{J}(\bar{\mathfrak{L}})$, then there is an integer n such that $(m\bar{R})^n \leqslant a = \sum_{i=1}^{k} \bar{r}_i \bar{R}$, where $\bar{r}_i$ are in $\bar{R}$, and $a \cap R$ is dense in a. Thus, there exist r_i in $a \cap R$ such that $r_i \equiv \bar{r}_i \pmod{(m\bar{R})^{n+1}}$. It follows that

$$a \leqslant (a \cap R)\bar{R} + (m\bar{R})^{n+1} \leqslant (a \cap R)\bar{R} + (m\bar{R})a$$

and so $a = (a \cap R)\bar{R}$ by the Krull-Azumaya theorem.

THEOREM 4. *The map* $\partial\varphi\colon \partial\mathfrak{J}(\mathfrak{L}^*) \to \partial\mathfrak{J}(\bar{\mathfrak{L}})$ *is a bijection.*

Proof. Let a be an element of $\partial\mathfrak{J}(\bar{\mathfrak{L}})$. Then there exists an integer n such that $(m\bar{R})^n \leqslant a$. Thus

$$m^n = (m\bar{R})^n \cap R \leqslant a \cap R$$

and so

$$(m^n)\mathfrak{L}^* = (m\mathfrak{L}^*)^n \leqslant (a \cap R)\mathfrak{L}^*$$

which shows that $(a \cap R)\mathfrak{L}^*$ is in $\partial\mathfrak{J}(\mathfrak{L}^*)$. For each i, $1 \leqslant i < \infty$, set

$$b_i = (a \cap R) + m^i$$

so that $\langle b_i \rangle$ is the completely regular representative of $(a \cap R)\mathfrak{L}^*$ ([3], Remark 5.2, p. 356). Then by Lemma 3

$$\partial\varphi\big((a \cap R)\mathfrak{L}^*\big) = \bigcap_i \big((a \cap R) + m^i\big)\bar{R}$$

$$= (a \cap R)\bar{R} = a$$

and hence $\partial\varphi$ surjective.

Let a and b be elements of $\partial\mathfrak{J}(\mathfrak{L}^*)$, with completely regular representatives $\langle a_i \rangle$ and $\langle b_i \rangle$ respectively, and suppose that $\partial\varphi(a) = \partial\varphi(b)$. Then (Lemma 1) there exists an integer k such that, for all $i \geqslant k$,

$$a_i\bar{R} = \bigcap_j a_j\bar{R} = \bigcap_j b_j\bar{R} = b_i\bar{R},$$

and thus, for all $i \geqslant k$, we have

$$a_i = (a_i\bar{R}) \cap R = (b_i\bar{R}) \cap R = b_i$$

which implies $a = b$ ([3], Proposition 5.10, p. 359). Hence $\partial\varphi$ is injective.

We can now prove our main result.

THEOREM 5. *Let* $(R, m_1, \ldots, m_n)$ *be a semi-local ring, let* $m = \bigcap_{i=1}^{n} m_i$, *let* $\bar{R}$ *be the* m*-adic ring completion of* R, *let* $\mathfrak{L}$ *and* $\bar{\mathfrak{L}}$ *be the lattice of ideals of* R *and* $\bar{R}$, *respectively, and let* $\mathfrak{L}^*$ *be the* $\mathfrak{J}(\mathfrak{L})$*-adic Noether lattice completion of* $\mathfrak{L}$. *Then* $\mathfrak{L}^*$ *and* $\bar{\mathfrak{L}}$ *are isomorphic as multiplicative lattices.*

Proof. Use Theorems 2 and 4 together with Theorem 2.4, p. 662, of [1] to extend $\partial\varphi\colon \partial\mathfrak{J}(\mathfrak{L}^*) \to \partial\mathfrak{J}(\bar{\mathfrak{L}})$ to a multiplicative lattice isomorphism $\varphi\colon \mathfrak{L}^* \to \bar{\mathfrak{L}}$.

If $\mathfrak{L}$ is complete in the $\mathfrak{J}(\mathfrak{L})$-adic metric, then $\mathfrak{L} = \mathfrak{L}^*$, and in the terminology of Theorem 5 we obtain the following

COROLLARY 6. *If* $\mathfrak{L}$ *is complete in the* $\mathfrak{J}(\mathfrak{L})$*-adic metric, then* $\mathfrak{L}$ *and* $\bar{\mathfrak{L}}$ *are isomorphic as multiplicative lattices.*

A semi-local ring $(R, m_1, \ldots, m_n)$ is called *quasi-complete* if, whenever given a decreasing sequence $\langle a_i \rangle$ of ideals of R and a natural number n, then there exists a natural number $S(n)$ such that $a_i \leqslant (\bigcap_{j=1}^{\infty} a_j) + m^n$, for all integers $i \geqslant S(n)$.

It is easy to see by example that there exist semi-local rings which are quasi-complete but not topologically complete. The following corollary shows that for ideal theoretical considerations, a quasi-complete semi-local ring may be assumed to be complete in its natural topology.

COROLLARY 7. *If* R *is quasi-complete, then* $\mathfrak{L}$ *and* $\bar{\mathfrak{L}}$ *are isomorphic as multiplicative lattices.*

Proof. It is easily verified that the lattices of ideals of a quasi-complete semi-local ring is complete in the $\mathfrak{J}(\mathfrak{L})$-adic metric and we omit the details.

COROLLARY 8. *A quasi-complete semi-local domain is analytically irreducible.*

References

[1] E. W. Johnson and J. A. Johnson, *A structural approach to Noether lattices*, Canad. J. Math. 22 (1970), pp. 657–665.

[2] — — *Lattice modules over semi-local Noether lattices*, Fund. Math. 68 (1970), pp. 187–201.

[3] J. A. Johnson, *A-adic completions of Noetherian lattice modules*, Fund. Math. 66 (1970), pp. 347–373.

[4] M. Nagata, *Local Rings*, New York 1962.

UNIVERSITY OF HOUSTON, DEPARTMENT OF MATHEMATICS
Houston, Texas

Accepté par la Rédaction le 5. 1. 1974