$\subseteq A \cap (S_\xi \cup C_\xi) \subseteq A \cap S_\xi$. But now either $|A \cap S_\xi| < p$, or else $A \in \mathfrak{D}_\xi$. In either case, $|A \cap B_{\xi\alpha}| < p$. Since $A \cap A_{\xi\alpha} \subseteq A \cap (B_{\xi\alpha} \cup C_\xi) \subseteq A \cap B_{\xi\alpha}$, we have $|A \cap A_{\xi\alpha}| < p$. Thus (9) holds when $\mu = \xi$. This completes the construction.

Now put $\mathcal{A} = \bigcup \{\mathcal{A}_\mu;\ \mu < q^+\}$, so $\mathcal{A}$ is a (q^+, n, p)-family, by (9). We show (7) holds for $\mathcal{A}$. Take S in $[\bigcup \mathcal{A}]^q$. Then $S = S_\mu$ for some $\mu < q^+$; yet $\operatorname{card}\{A \in \mathcal{A}_\mu;\ |A \cap S_\mu| \geqslant p\} = q^+$, and $\mathcal{A}_\mu \subseteq \mathcal{A}$. Thus (7) holds, and the proof is complete.

Together, Theorems 2.1 and 2.4 establish the sufficiency of (1) and (2).

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UNIVERSITY OF QUEENSLAND
Brisbane, Australia

Accepté par la Rédaction le 18. 2. 1974

A note on Lusin's condition (N)

by

James Foran (Milwaukee, Wis.)

Abstract. A function is said to satisfy condition (N′) provided the image of closed sets of measure 0 is of measure 0. In this paper it is shown that for several classes of functions (N′) implies Lusin's condition (N). The Baire functions (in a general setting) are one such class. Using the continuum hypothesis, a real valued function is constructed which satisfies (N′) but does not satisfy Lusin's condition (N).

A function $f\colon X \to Y$, where X and Y are measure spaces, is said to *satisfy condition* (N) if the image under f of each set of measure 0 in X is of measure 0 in Y. Condition (N) arises quite naturally in the study of integrals. (See, e.g., [4], p. 224ff.) A function $f\colon X \to Y$ will be said to *satisfy condition* (N′) if X is a topological space and the image under f of compact sets of measure 0 is of measure 0. The purpose of this note is to show that for several classes of functions condition (N′) implies condition (N); that is, the compact sets of measure 0 are a determining class for condition (N).

Although greater generality is attainable, the spaces X and Y will always be σ-compact metric spaces. The following notation and definitions will be used:

1) $B(f; A)$ will denote the *graph of f on the set A*, i.e., if $f\colon X \to Y$, $B(f; A) = \{(x, y) \in X \times Y \mid x \in A,\ y = f(x)\}$.

2) $m(E)$ will denote the *measure of E* when it is clear which measure this is.

3) $n{-}m(E)$ will denote *Hausdorff n-measure*. Briefly, a set B has n-measure less than or equal to b if for any given $\varepsilon > 0$ there is a cover U_ε of B with each set $I \in U_\varepsilon$ having diameter less than ε and $\sum_{I \in U_\varepsilon} (\operatorname{diam} I)^n < b + \varepsilon$.

4) proj_Y will denote the *projection map* from $X \times Y$ to Y; similarly, proj_X denotes the projection map from $X \times Y$ to X.

5) A measure space X is of *σ-finite measure* if X is the countable union of sets of finite measure.

6) A measure on a topological space is *regular* if each measurable set A is contained in a G_δ set B with $m(B-A)=0$. Here, as usual, a set A is measurable if $m(T)=m(T\cap A)+m(T-A)$ for every set T.

The following three classes of functions will be shown to satisfy condition (N) provided they satisfy condition (N′):

(1) Functions $f\colon X\to Y$ where X and Y have regular σ-finite measures and f satisfies

i) $f(C)$ is measurable for every compact set C,

ii) $f^{-1}(y)$ is closed for almost every y in Y.

(2) Functions $f\colon X\to Y$ where X and Y have regular σ-finite measures and $B(f;X)$ is an analytic set in $X\times Y$.

(3) Functions $f\colon E_n\to E_n$, with Lebesgue n-measure on the range and domain, with f satisfying: $B(f;E_n)$ is of σ-finite Hausdorff n-measure and is n-measurable.

Continuous functions are a special case of (1) and Baire functions of (2). (A proof that Baire functions have analytic graphs is given in [2] p. 384 and, for the real valued case, in [1] p. 300.) Furthermore, given a measurable function f satisfying condition (N′), there is a Baire 2 function g equal to f almost everywhere and (2) implies that g satisfies condition (N). (Lusin's theorem is easily modified to this setting. That is, there are an increasing sequence of compact sets $\{X_n\}$ with $m(X-\bigcup X_n)=0$ and a sequence of functions $\{f_n\}$ with f_n defined and continuous on X_n such that f_n equals the restriction of f to X_n. Choose a point y_0 in Y and define $f_n(x)=y_0$ if $x\notin X_n$. Then f_n is a Baire 1 function on all of X since for a closed set $F\subset Y$ $f_n^{-1}(F)$ is a G_δ; in fact, $f_n^{-1}(F)$ is either a closed set or the union of the open set X_n^c with a closed set. Then $g=\lim f_n$ is a Baire 2 function which is equal to f almost everywhere.)

Assuming the continuum hypothesis an example of a function f can be constructed so that f satisfies condition (N′) but does not satisfy condition (N). Such a function can be chosen to be measurable, but it is not known whether such a function can be constructed without a condition like the continuum hypothesis.

The following simple lemma will be needed.

LEMMA. *If $f\colon X\to Y$, with Y a measure space and X a topological space, satisfies $f^{-1}(y)$ is closed for almost every $y\in Y$ (every $y\in Y$) and if $\{C_n\}$ is a decreasing sequence of compact sets such that $f(C_n)$ is measurable, then $m\big(\bigcap f(C_n)\big)=m\big(f(\bigcap C_n)\big)$ $\big(\bigcap f(C_n)=f(\bigcap C_n)\big)$.*

Proof. Since $\bigcap f(C_n)\supset f(\bigcap C_n)$ it follows that

$$m\big(\bigcap f(C_n)\big)\geqslant m\big(f(\bigcap C_n)\big).$$

If $Z=\{y\mid f^{-1}(y)$ is not closed$\}$ by hypothesis $m(Z)=0$ $(Z=\emptyset)$. If $y\in\bigcap f(C_n)-Z$ then $y=f(x_n)$ where $x_n\in C_n$. Then $x_{n+j}\in C_{n+j}\subset C_n$, $\{x_{n+j}\}_{j=1}^{\infty}\subset C_n$, and C_n compact implies there exists a limit point x_0 of $\{x_{n+j}\}_{j=1}^{\infty}$ in C_n. Then $x_0\in C_n$ for every n. Since $f^{-1}(y)$ is closed, it follows that $x_0\in f^{-1}(y)$, $f(x_0)=y$, and $y\in f(C_n)$. This being true for almost every y (every x) in $\bigcap f(C_n)$, it follows that

$$m\big(\bigcap f(C_n)\big)\leqslant m\big(f(\bigcap C_n)\big)\big[\bigcap f(C_n)\subset f(\bigcap C_n)\big].$$

THEOREM 1. *Let $f\colon X\to Y$ where X and Y have regular σ-finite measures. Suppose f satisfies:*

i) $f(C)$ is measurable for every compact set $C\subset X$,

ii) $f^{-1}(y)$ is closed for almost every $y\in Y$.

Then f satisfies condition (N) *provided f satisfies condition* (N′).

Proof. Suppose not. Then there is a set A of measure 0 contained in X (since X is regular A can be chosen to be a G_δ) such that $m\big(f(A)\big)>a>0$. There is a set $E\subset Y$ such that $\infty>m\big(f(A)\cap E\big)>a>0$. Let $A=\bigcap G_n$, G_n open. Let $G_1=\bigcup I_{1k}$, I_{1k} compact and of finite measure. Choose k_1 such that $m\big(f(A_1\cap A)\cap E\big)>a$ where $A_1=\bigcup_{k=1}^{k_1} I_{1k}$. This is possible since $m\big(f(A)\cap E\big)>a$ and

$$f(A)=f(A\cap\bigcup_k I_{1k})=f\big(\bigcup_k(A\cap I_{1k})\big)=\bigcup_k f(A\cap I_{1k}).$$

Note that A_1 is compact and $A_1\cap G_2$ is an F_σ. Let $A_1\cap G_2=\bigcup I_{2k}$ where each I_{2k} is compact. Choose k_2 so that $A_2=\bigcup_{k=1}^{k_2} I_{2k}$ satisfies

$$m\big(f(A\cap A_2)\cap E\big)>a.$$

This is again possible since $m\big(f(A\cap A_1)\cap E\big)>a$ and

$$f(A\cap A_1)=f(A\cap A_1\cap\bigcup_k I_{2k})=\bigcup_k f(A\cap A_1\cap I_{2k}).$$

Continuing this construction, if for $1<i<n$ $A_{i-1}\cap G_i=\bigcup_k I_{i-1,k}$ with each $I_{i-1,k}$ compact, k_i has been chosen so that

$$A_i=\bigcup_{k=1}^{k_i} I_{i-1,k}\quad\text{ and }\quad m\big(f(A_i\cap A)\cap E\big)>a,$$

then $A_{n-1}\cap G_n$ is an F_σ. Choose I_{nk} compact so that

$$A_{n-1}\cap G_n=\bigcup_k I_{nk}.$$

Choose k_n so that

$$A_n=\bigcup_{k=1}^{k_n} I_{nk}\quad\text{ satisfies }\quad m\big(f(A_n\cap A)\cap E\big)>a.$$

This again is possible since $m(f(A_{n-1}\cap A)\cap E)>a$ and $f(A\cap A_{n-1}) = \bigcup_k f(A\cap A_{n-1}\cap I_{nk})$. The induction proceeds yielding a decreasing sequence of compact sets A_n with $A_n\subset G_n$. Let $H=\bigcap A_n$. Then $H\subset\bigcap G_n$, so that $m(H)=0$, and H is compact since each A_n is compact. By the Lemma, $m(\bigcap f(A_n)\cap E)=m(f(\bigcap A_n)\cap E)$. Since $\infty>m(f(A_n)\cap E)>a$ it follows that $m(\bigcap f(A_n))\geqslant a$. Thus the image of the compact set H has positive measure, contrary to hypothesis.

THEOREM 2. *Let $f\colon X\to Y$ where X and Y are σ-compact spaces with regular σ-finite measures. Suppose further that the graph of f is an analytic set in $X\times Y$. Then, in order that f satisfy condition* (N) *it is sufficient that the image under f of compact sets of measure* 0 *be of measure* 0.

Proof. Suppose not. Then there is a function $f\colon X\to Y$ and a set X_0 with $m(X_0)=0$ but $m(f(X_0))>a>0$. Since X is regular the set X_0 can be chosen to be a G_δ. Then $f(X_0)$ is measurable since $f(X_0)=\mathrm{proj}_Y(B(f;X)\cap X_0\times Y)$ is an analytic set (it is the continuous image of the intersection of an analytic set with a G_δ set). Since Y is a regular σ-finite measure space, there is a closed set Y_0 such that $\infty>m(f(X_0)\cap Y_0)>a>0$. Let $A=f^{-1}(Y_0)\cap X_0$. Then $B(f;A)$ is an analytic set since $B(f;A) = B(f;X)\cap X_0\times Y_0$. Choose a determining system $\{A_{n_1\dots n_k}\}$ consisting of compact sets so that $A_{n_1n_2\dots n_kn_{k+1}}\subset A_{n_1n_2\dots n_k}$ and so that

$$B(f;A)=\bigcup(A_{n_1}\cap A_{n_1n_2}\cap\dots\cap A_{n_1n_2\dots n_k}\cap\dots),$$

where the union extends over all sequences $n_1,n_2,\dots,n_k\dots$ Choose h_1 so that $A_{h_1}=\bigcup(A_{n_1}\cap A_{n_1n_2}\cap\dots\cap A_{n_1n_2\dots n_k}\cap\dots)$ (where the union extends over all sequences with $n_1\leqslant h_1$) satisfies $m(\mathrm{proj}_Y(A_{h_1}))>a$. This is possible since $B(f;A)=\bigcup_{h=1}^{\infty}A_h$, the projection of the union is the union of the projections, and the projection of an analytic set is analytic (and thus measurable). In general choose h_j so that

$$A_{h_1h_2\dots h_j}=\bigcup(A_{n_1}\cap A_{n_1n_2}\cap\dots\cap A_{n_1n_2\dots n_k}\cap\dots)$$

(where the union extends over all sequences with $n_1\leqslant h_1,\ n_2\leqslant h_2,\dots,n_j\leqslant h_j$) satisfies $m(\mathrm{proj}_Y(A_{h_1h_2\dots h_j}))>a$. This is possible since

$$\bigcup_{h=1}^{\infty}A_{h_1\dots h_{j-1}h}=A_{h_1h_2\dots h_{j-1}}.$$

Let $A_1=\bigcup A_{n_1}$ where $n_1\leqslant h_1$. Then $A_{h_1}\subset A_1$. Let $A_j=\bigcup(A_{n_1}\cap A_{n_1n_2}\cap\dots\cap A_{n_1n_2\dots n_j})$ where $n_1\leqslant h_1,\ n_2\leqslant h_2,\dots,n_j\leqslant h_j$. Then $A_{h_1h_2\dots h_j}\subset A_j$ since each A_j is the finite union of compact sets, each A_j is compact. That $\bigcap_{j=1}^{\infty}A_j\subset B(f;A)$ is a set theoretic argument and can be found e.g. in [4]

p. 49. Since the A_j form a decreasing sequence of compact sets and proj_Y is a continuous function, by the lemma it follows that

$$\bigcap\mathrm{proj}_Y(A_j)=\mathrm{proj}_Y(\bigcap A_j).$$

Since $A_{h_1h_2\dots h_j}\subset A_j$, $\mathrm{proj}_Y(A_{h_1h_2\dots h_j})\subset\mathrm{proj}_Y(A_j)$ and $m(\mathrm{proj}_Y(A_j))>a$. Since $\mathrm{proj}_Y(A_j)$ forms a decreasing collection of measurable sets having finite measure $m(\bigcap\mathrm{proj}_Y(A_j))\geqslant a$. Letting $F=\mathrm{proj}_X(\bigcap A_j)$, it follows that F is compact, $F\subset A$, and

$$m(f(F))=m(\mathrm{proj}_Y(\bigcap A_j))=m(\bigcap\mathrm{proj}_Y(A_j))\geqslant a>0,$$

a contradiction.

THEOREM 3. *If $f\colon X\to Y$ where $X=Y=E_n$, endowed with Lebesgue measure, and if $B(f;X)$ has σ-finite Hausdorff n-measure and is n-measurable, then for f to satisfy condition* (N) *it is sufficient that f satisfy condition* (N′).

Proof. First observe that for any set A contained in $X\times Y$ with $n-m(A)=0$ one has $m(\mathrm{proj}_Y(A))=0$. This is immediate from the definition of n-measure. For if $n-m(A)=0$ then given $\varepsilon>0$ there is a cover U_ε of A such that $\sum_{I\in U_\varepsilon}(\mathrm{diam}(I))^n<\varepsilon$. But then the projection of U_ε into Y will have Lebesgue measure less than ε since $\mathrm{proj}_Y(I)$ is contained in an n-dimensional cube of side length $\mathrm{diam}(I)$. It follows that $m(\mathrm{proj}_Y(A))=0$.

Suppose the theorem is not true. Then there are a function f with $B(f;X)$ of σ-finite n-measure and a G_δ set E with $m(E)=0$ and $m^*(f(E))>0$. Since $f(E)=\mathrm{proj}_Y(B(f;E))$, it follows from the observation that $B(f;E)$ has n-outer measure >0. Since $B(f;X)$ is n-measurable, $B(f;E)=B(f;X)\cap E\times Y$ is also n-measurable and furthermore $B(f;E)$ has σ-finite n-measure. But then $B(f;E)=B(f;Z)\cup\bigcup_n B(f;X_n)$ where $B(f;X_n)$ are compact sets of finite n-measure and $B(f;Z)$ has n-measure 0 (see [3], pp. 90–99). Since $B(f;Z)$ has n-measure 0, from the observation it follows that $m(f(Z))=0$. But $m^*(\bigcup f(X_n))>0$ implies there is an N so that $m^*(f(X_N))>0$. However, X_N is compact, $X_N=\mathrm{proj}_X(B(f;X_N))$ and X_N is contained in E. Hence the Lebesgue measure of X_N is equal to 0 while $m^*(f(X_N))>0$ contrary to hypothesis. This contradiction proves Theorem 3.

The following example shows that it is possible, assuming the continuum hypothesis, for a measurable function to satisfy condition (N′) and not to satisfy condition (N).

Let A be a dense G_δ subset of the real numbers with $m(A)=0$. Let $F_1,F_2,\dots,F_\alpha,\dots,\alpha<\Omega$, be a well ordering of the closed nowhere dense subsets of the line. Choose $x_1\in A-F_1$ which is possible since A is residual and F_1 is 1st category. If, for every $\beta < \alpha$, x_β has been chosen such that $x_\beta \in A - \bigcup_{\gamma\leqslant\beta} F_\gamma$ and such that $x_\beta \neq x_\gamma$ for every $\gamma < \beta$, choose $x_\alpha \in A - \bigcup_{\beta<\alpha} F_\beta \cup \{x_\beta\}_{\beta<\alpha}$. Since $\bigcup_{\beta<\alpha} F_\beta$ is a countable union of first category sets, $\{x_\beta\}_{\beta<\alpha}$ a countable set, such a point x_α can be chosen and the induction is complete. Let $X = \{x_\alpha\}_{\alpha<\Omega}$ and let g be a map with domain X and image the real numbers. Let $f(x) = 0$ if $x \notin X$, $f(x) = g(x)$ if $x \in X$. Then f satisfies condition (N′) since if F is a closed set of measure 0, $F = F_{\alpha_0}$ for some $\alpha_0 < \Omega$, and $f(F)$ is an at most countable set. However f does not satisfy condition (N) since $f(A)$ is the real line.

This same example can be constructed if the continuum hypothesis is replaced by both of the following:

i) the union of fewer than τ sets of measure 0 is of measure 0,

ii) the union of fewer than τ sets of 1st category is of 1st category,

where τ is the power of the continuum.

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UNIVERSITY OF WISCONSIN-MILWAUKEE
Milwaukee, Wisconsin

Accepté par la Rédaction le 14. 3. 1974

A comment on Balbes' representation theorem for distributive quasi-lattices

by

R. Arthur Knoebel (Las Cruces, N. M.)

This short note points out that Balbes' [1] representation theorem for distributive quasi-lattices may be proven from rather general considerations. (A distributive quasi-lattice is an algebra with two semi-lattice operations connected by the distributive laws.) Recall his Theorem 4 (rewritten slightly): *an algebra $\mathfrak{D} = \langle D; +, \cdot\rangle$ with two binary operations is a distributive quasi-lattice iff there are two families Y, X of sets closed to intersection and union, respectively, and two one-to-one correspondences $\psi\colon D \leftrightarrow Y$ and $\varphi\colon D \leftrightarrow X$ such that*

$$a+b = \psi^{-1}(\psi a \cup \psi b)\,,$$

$$a\cdot b = \varphi^{-1}(\varphi a \cap \varphi b)\,,$$

$$a\cdot(b+c) = a\cdot b + a\cdot c\,,$$

$$a + b\cdot c = (a+b)\cdot(a+c)\,,$$

for all $a, b, c \in D$. This is true because any semi-lattice is isomorphic to a family of subsets closed to intersection (or union) [2].

Since this representation theorem for semi-lattices is equivalent to saying that each semi-lattice is a subdirect power of the two-element semi-lattice, the technique of Theorem 4 is generalizable to any algebra

$$\mathfrak{A} = \langle A; f_1, \ldots, f_k, g_1, \ldots, g_m, h_1, \ldots, h_n, \ldots\rangle$$

of which each reduct

$$\mathfrak{A}_f = \langle A; f_1, \ldots, f_k\rangle\,,$$

$$\mathfrak{A}_g = \langle A; g_1, \ldots, g_m\rangle\,,$$

$$\ldots\ldots\ldots\ldots\ldots$$

is representable as a subdirect power.