Tightly packed families of sets

by

N. H. Williams (Brisbane)

Abstract. Let \( m, n, p, q \) be infinite cardinals with \( m \geq n \geq p \) and \( m \geq q \). Necessary and sufficient conditions are given for the existence of a family \( A \) of \( m \) sets each of cardinality \( n \) with \( |A_1 \cap A_2| < p \) for each pair \( A_1, A_2 \) from \( A \), such that each subset of \( \bigcup A \) of power \( q \) has an intersection of cardinality at least \( p \) with \( m \) different members of the family \( A \).

This paper is devoted to a proof of the following theorem. (The notation is explained below.) The Generalized Continuum Hypothesis is assumed throughout.

**Theorem 6.1.** Let \( m, n, p, q \) be infinite cardinals with \( m \geq n \geq p \) and \( m \geq q \). The conditions

1. \( n < q \)
2. either \( m = q \) or else \( m = q^+ \) and \( p' = q' \)

are necessary and sufficient for the existence of a family \( A \) of \( m \) sets each of cardinality \( n \) with \( |A_1 \cap A_2| < p \) for each pair \( A_1, A_2 \) from \( A \), such that

\[
S \in \bigcup A \Rightarrow \text{card}(A \times A; |A \cap S| \geq p) = m
\]

Notation. Cardinal numbers are identified with the initial ordinals. Small Greek letters always denote ordinal numbers, and small Roman letters cardinal numbers. The cardinality of a set \( X \) is denoted either by \( \text{card}X \) or \( |X| \), and \( [X]^r \) is the set of subsets of \( X \) of cardinality \( r \). The symbols \( [X]^{<r}, [X]^{\geq r} \) have the obvious meanings. For any cardinal \( r \), by \( r^+ \) is denoted the least cardinal larger than \( r \) and by \( r' \) the cofinality cardinal of \( r \), that is, the least cardinal \( s \) for which \( r \) can be written as the sum of \( s \) cardinals all less than \( r \). When \( r' = r \) then \( r \) is said to be regular, and otherwise singular.

**Definition 6.2.** A family of sets \( A \) is called an \( (m, n, p) \)-family if \( |A| = m \), \( |A| = n \) for each \( A \) in \( A \), and \( |A_1 \cap A_2| < p \) for each pair \( A_1, A_2 \) from \( A \). We define \( (n, p) \)-family and \( (m, \geq n, p) \)-family analogously.
We shall make use of the following results of Tarski [1, Théorème 5].

**Proposition 0.3.** Let \( A \) be an \((m, p, g, p', q')\)-family, where \( m > q > p' \). Then \( m = |A| + |A^*| \) unless perhaps \( n = p \) and \( p' = |A| \), in which case \( m = |A| \).

**Proposition 0.4.** Let \( g > q' = p > p' \) and \( q > p' \). Given \( X \) with \( |X| = q \) there is an \((q, p, p', q')\)-family \( B \) with \( \cup B \subseteq X \).

### § 1. Theorem 0.1. Necessity

The condition (1) is easily seen to be necessary if \( S \) is to hold, for if \( m > n \) then by choosing for \( S \) a subset of some \( A \), then only possible for \( A \), is \(|A \cap S| \geq p \) true (by the condition \(|A \cap A^*| \leq p \)).

On the other hand, suppose there is \( S \) with \( |S| = q \) for which \( \text{card}(A \in A; \ |A \cap S| \geq p) = m \). Then the family \( B = \{A \in A; \ |A \cap S| \geq p\} \) is an \((m, p, p', q')\)-family with \( |\cup B| \leq |S| = q \), and so if \( m > q \) we must have \( m = q^* \) and by Proposition 0.3 also \( p = q' \).

Thus the condition (2) is necessary for \( S \).

### § 2. Theorem 0.1. Sufficiency

We now suppose conditions (1) and (2) to hold and deduce (3). Note that from (1) it follows that \( m > n \). Consider first the case \( m = g \).

**Theorem 2.1.** Let \( m > n > p \). Then there is an \((m, n, p)\)-family \( A \) such that

\[
S \in \{A \mid \text{card}(A \in A; \ |A \cap S| \geq p) = m \}.
\]

Proof. Suppose that \( m \) is regular. Using Zorn's Lemma, let \( A \) be a maximal \((\leq m^*, n, p)\)-family of subsets of \( m \). Then by Proposition 0.3, in fact \( |A| \leq m \). And we have

\[
S \in \{A \mid \text{card}(A \in A; \ |A \cap S| \geq p) \geq |S| \}.
\]

For suppose on the contrary that there is \( S \) with \( S \in \{A \mid \text{card}(A \in A; \ |A \cap S| \geq p) = m \} \) for which \( \text{card}(A \in A; \ |A \cap S| \geq p) \geq |S| \). Then \( \text{card}(A \in A; \ |A \cap S| \geq p) \geq |S| \).

But if \( X \) is chosen so \( X \subseteq S \cup \{A \in A; \ |A \cap S| \geq p\} \) with \( |X| = n \), then \( A \cup X \) is an \((\leq m^*, n, p)\)-family contradicting the maximality of \( A \). From (3) it follows that (4) holds for \( A \).

Now suppose that \( m \) is singular. Choose regular cardinals \( m^*_n \) for \( \sigma < \kappa \) so that \( m^*_n < m^*_n < m^*_n < \ldots < m \) and \( m = \sum \{m^*_n; \sigma < \kappa \} \).

For each \( \sigma < \kappa \) take an \((m^*_n, n, p)\)-family \( A^*_\sigma \) with the property (0), and further ensure that the sets \( \cup A^*_\sigma \) are pairwise disjoint. Put \( A = \cup \{A^*_\sigma; \sigma < \kappa\} \), so \( A \) is an \((m, n, p)\)-family.

Take \( S \) in \( \{A \mid \text{card}(A \in A; \ |A \cap S| \geq p) \leq |S| \} \). For \( \sigma < \kappa \) put \( S^*_\sigma = S \cap \cup A^*_\sigma \). Then \( \cup \{S^*_\sigma; |S^*_\sigma| \geq n\} = m \). By the property (0) for \( A^*_\sigma \), if \( |S^*_\sigma| > n \) then \( \text{card}(A \in A; \ |A \cap S^*_\sigma| \geq p) > \sum |S^*_\sigma| > |S^*_\sigma| = m \). Hence \( \text{card}(A \in A; \ |A \cap S| \geq p) = \sum |S^*_\sigma| > |S^*_\sigma| = m \). Thus (i) holds for \( A \). This completes the proof.

Consider next the case \( m > q \). Here \( m = q^* \) and \( p' = q' \). Since \( q > m > n > p' = q' \), in fact \( q \) is singular.

**Lemma 2.2.** Let \( q > g = p' \) and \( q > n > p \). Given \( X \) with \( |X| = q \) and \( a = \{g, q, n, p\} \)-family \( D \) there is a \((g^*, q, p, D)\)-family \( B \) with \( \cup B \subseteq X \) such that \( |B \cap D| < p \) for \( B \in B \).

**Proof.** Take a family \( S \) as given by Proposition 0.4. Note that \( |\cup \{B \mid |B \cap D| < p \} = q \). And so we may omit from \( S \) any set \( B \) for which \( B \in D \) and still retain a \((g^*, q, p, D)\)-family.

**Lemma 2.3.** Given \( Z \) with \( |Z| = q^* \) and a family \( S = \{S_i; n < q^* \} \) where always \( S_i \in \{Z_i; n < q^* \} \), then there is a \((q^*, q^*, 1)\)-family \( C = \{C_i; n < q^* \} \)

\[
\text{such that } \mu \leq n = C_i \cap S_i = \emptyset.
\]

**Proof.** Write \( Z = \cup \{Z_i; n < q^* \} \) where the \( Z_i \) are pairwise disjoint with \( |Z_i| = q^* \). Put \( C_i = Z_i \cup \{C_i; n < q^* \} \).

**Theorem 2.4.** Let \( q > g = p' \) and \( q > n > p \). Then there is a \((g^*, n, p)\)-family \( A \) such that

\[
S \in \{A \mid \text{card}(A \in A; \ |A \cap S| \geq p) = q^* \}.
\]

**Proof.** Let \( S = \{S^*_n; n < q^* \} \) and take a \((q^*, q^*, 1)\)-family \( C = \{C_i; n < q^* \} \) as given by Lemma 2.3 for \( S \). Choose \( \{q^*, n, p\} \)-families \( C_i = \{C_i; n < q^* \} \) such that \( \cup C_i \subseteq C_i \).

Construct by transfinite induction \((q^*, n, p)\)-families \( A_\xi \) for \( \mu < q^* \) such that

\[
A_\xi = \text{for some } \nu < \mu, \text{ or else } \cup A_\xi \subseteq S_\nu \cup C_\nu,
\]

and

\[
A_\xi, A_\lambda \in \cup \{A_\xi; n < q^* \} \Rightarrow |A_\xi \cap A_\lambda| < p \text{ if } A_\xi \neq A_\lambda.
\]

As follows. Take \( \xi \) with \( \xi < q^* \) and suppose the \( A_\xi \), for \( n < \xi \) have already been satisfactorily defined. If there is \( \nu < \xi \) for which

\[
\text{card}(A \in A; \ |A \cap S_\nu| \geq p) = q^*;
\]

put \( A_\xi = A_\nu \) for such a \( \nu \). (Then (8) and (9) hold when \( \mu = \xi \).) Otherwise, put \( D_\nu = \{A \in A; \ x < \xi, \ |A \cap S_\nu| \geq p\} \), so \( |D_\nu| = q^* \). Let \( S_\xi = \{S_\mu; \ x < \xi\} \). Then \( |\cup S_\xi| \subseteq C_\xi \subseteq C_\lambda \), for which \( |B \cap D| < p \) when \( B \in B \). (As provided for by Lemma 2.2.) Put \( A_\xi = B_\xi \cup C_\xi \) and finally \( A_\xi = \{A_\xi; n < q^* \} \). Then \( \cup A_\xi \subseteq S_\xi \cup C_\xi \) and (8) holds with \( \mu = \xi \). Since \( \cup C_\xi \subseteq S_\xi \) and \( S_\xi \cap C_\xi = \emptyset \), when \( A_\xi \neq B_\lambda \), we have \( A_\xi \cap A_\lambda = (B_\xi \cap B_\lambda) \cup (C_\xi \cap C_\lambda) \) so that \( |A_\xi \cap A_\lambda| < p \). Thus \( A_\xi \) is a \((q^*, n, p)\)-family. Also, by (6), if \( n < \xi \) then \( C_\xi \cap (S_\nu \cup C_\nu) = \emptyset \). So if \( A \in A_\xi \) for some \( n < \xi \), then \( A \cap C_\xi = \emptyset \). Thus \( A \cup A_\xi \)

5 — Fundamenta Mathematica Issue XC
A note on Lusin's condition (N)

by

James Foran (Milwaukee, Wis.)

Abstract. A function is said to satisfy condition (N') provided the image of closed sets of measure 0 is of measure 0. In this paper it is shown that for several classes of functions (N') implies Lusin's condition (N). The Baire functions (in a general setting) are one such class. Using the continuum hypothesis, a real valued function is constructed which satisfies (N') but does not satisfy Lusin's condition (N).

A function \( f : X \to Y \), where \( X \) and \( Y \) are measure spaces, is said to satisfy condition (N) if the image under \( f \) of each set of measure 0 in \( X \) is of measure 0 in \( Y \). Condition (N) arises quite naturally in the study of integrals. (See, e.g., [1], p. 224ff.) A function \( f : X \to Y \) will be said to satisfy condition (N') if \( X \) is a topological space and the image under \( f \) of compact sets of measure 0 is of measure 0. The purpose of this note is to show that for several classes of functions condition (N') implies condition (N); that is, the compact sets of measure 0 are a determining class for condition (N).

Although greater generality is attainable, the spaces \( X \) and \( Y \) will always be \( \sigma \)-compact metric spaces. The following notation and definitions will be used:

1) \( B(f; A) \) will denote the graph of \( f \) on the set \( A \), i.e., if \( f : X \to Y \), \( B(f; A) = \{(x, y) \in X \times Y | x \in A, y = f(x)\} \).

2) \( m(E) \) will denote the measure of \( E \) when it is clear which measure this is.

3) \( n-m(E) \) will denote \textit{Hausdorff} \( n \)-measure. Briefly, a set \( B \) has \( n \)-measure less than or equal to \( b \) if for any given \( \varepsilon > 0 \) there is a cover \( U \) of \( B \) with each set \( I \in U \) having diameter less than \( \varepsilon \) and \( \sum (\text{diam}(I))^n < b + \varepsilon \).

4) \( \text{proj}_{Y} \) will denote the projection map from \( X \times Y \) to \( Y \); similarly, \( \text{proj}_{X} \) denotes the projection map from \( X \times Y \) to \( X \).

5) A measure space \( X \) is of \( \sigma \)-finite measure if \( X \) is the countable union of sets of finite measure.