Tightly packed families of sets

by

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Abstract. Let m, n, p, q be infinite cardinals with $m \geqslant n \geqslant p$ and $m \geqslant q$. Necessary and sufficient conditions are given for the existence of a family $\mathcal{A}$ of m sets each of cardinality n with $|A_1 \cap A_2| < p$ for each pair A_1, A_2 from $\mathcal{A}$, such that each subses of $\bigcup \mathcal{A}$ of power q has an intersection of cardinality at least p with m different membert of the family $\mathcal{A}$.

This paper is devoted to a proof of the following theorem. (The notation is explained below.) The Generalized Continuum Hypothesis is assumed throughout.

THEOREM 0.1. *Let m, n, p, q be infinite cardinals with $m \geqslant n \geqslant p$ and $m \geqslant q$. The conditions*

(1) $$n < q$$

and

(2) *either* $m = q$ *or else* $m = q^+$ *and* $p' = q'$

are necessary and sufficient for the existence of a family $\mathcal{A}$ of m sets each of cardinality n with $|A_1 \cap A_2| < p$ for each pair A_1, A_2 from $\mathcal{A}$, such that

(3) $$S \in [\bigcup \mathcal{A}]^q \Rightarrow \operatorname{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} = m .$$

Notation. Cardinal numbers are identified with the initial ordinals. Small Greek letters always denote ordinal numbers, and small Roman letters cardinal numbers. The cardinality of a set X is denoted either by $\operatorname{card} X$ or $|X|$, and $[X]^r$ is the set of subsets of X of cardinality r. The symbols $[X]^{\leqslant r}, [X]^{>r}$ have the obvious meanings. For any cardinal r, by r^+ is denoted the least cardinal larger than r and by r' the cofinality cardinal of r, that is, the least cardinal s for which r can be written as the sum of s cardinals all less than r. When $r' = r$ then r is said to be regular, and otherwise singular.

DEFINITION 0.2. A family of sets $\mathcal{A}$ is called an (m, n, p)-*family* if $|\mathcal{A}| = m$, $|A| = n$ for each A in $\mathcal{A}$, and $|A_1 \cap A_2| < p$ for each pair A_1, A_2 from $\mathcal{A}$. We define $(\leqslant m, n, p)$-*family* and $(m, \geqslant n, p)$-*family* analogously.

We shall make use of the following results of Tarski [1, Théorème 5].

PROPOSITION 0.3. *Let* $\mathcal{A}$ *be an* $(m, \geqslant n, p)$*-family, where* $m > n \geqslant p$. *Then* $m = |\bigcup \mathcal{A}|$ *unless perhaps* $n = p$ *and* $p' = |\bigcup \mathcal{A}|'$, *in which case* $m = |\bigcup \mathcal{A}|^+$ *is also a possibility.*

PROPOSITION 0.4. *Let* $q > q' = p'$ *and* $q \geqslant p$. *Given* X *with* $|X| = q$ *there is a* (q^+, p, p)*-family* $\mathcal{B}$ *with* $\bigcup \mathcal{B} \subseteq X$.

§ 1. Theorem 0.1, necessity. The condition (1) is easily seen to be necessary if (3) is to hold, for if $n \geqslant q$ then by choosing for S a subset of some A_1 in $\mathcal{A}$, only possibly for $A = A_1$ is $|A \cap S| \geqslant p$ true (by the condition $|A_1 \cap A_2| < p$).

On the other hand, suppose there is S with $|S| = q$ for which $\mathrm{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} = m$. Then the family $\mathcal{B} = \{A \cap S;\ A \in \mathcal{A}$ and $|A \cap S| \geqslant p\}$ is an $(m, \geqslant p, p)$-family with $|\bigcup \mathcal{B}| \leqslant |S| = q$, and so if $m > q$ we must have $m = q^+$ and by Proposition 0.3 also $p' = q'$. Thus the condition (2) is necessary for (3).

§ 2. Theorem 0.1, sufficiency. We now suppose conditions (1) and (2) to hold and deduce (3). Note that from (1) it follows that $m > n$. Consider first the case $m = q$.

THEOREM 2.1. *Let* $m > n \geqslant p$. *Then there is an* (m, n, p)*-family* $\mathcal{A}$ *such that*

$$S \in [\bigcup \mathcal{A}]^m \Rightarrow \mathrm{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} = m\,. \tag{4}$$

Proof. Suppose that m is regular. Using Zorn's Lemma, let $\mathcal{A}$ be a maximal $(\leqslant m^+, n, p)$-family of subsets of m. Then by Proposition 0.3, in fact $|\mathcal{A}| \leqslant m$. And we have

$$S \in [\bigcup \mathcal{A}]^{>n} \Rightarrow \mathrm{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} \geqslant |S|\,. \tag{5}$$

For suppose on the contrary that there is S with $S \in [m]^{>n}$ for which $\mathrm{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} < |S|$. Then $|\bigcup\{A \in \mathcal{A};\ |A \cap S| \geqslant p\}| < |S|$. But if X is chosen so $X \subseteq S - \bigcup\{A \in \mathcal{A};\ |A \cap S| \geqslant p\}$ with $|X| = n$, then $\mathcal{A} \cup \{X\}$ is a $(\leqslant m^+, n, p)$-family contradicting the maximality of $\mathcal{A}$. From (5) it follows that (4) holds for $\mathcal{A}$.

Now suppose that m is singular. Choose regular cardinals m_σ for $\sigma < m'$ so that $n < m_0 < m_1 < \ldots < m$ and $m = \sum\{m_\sigma;\ \sigma < m'\}$. For each σ with $\sigma < m'$ take an (m_σ, n, p)-family $\mathcal{A}_\sigma$ with the property (5), and further ensure that the sets $\bigcup \mathcal{A}_\sigma$ are pairwise disjoint. Put $\mathcal{A} = \bigcup\{\mathcal{A}_\sigma;\ \sigma < m'\}$, so $\mathcal{A}$ is an (m, n, p)-family.

Take S in $[\bigcup \mathcal{A}]^m$. For $\sigma < m'$ put $S_\sigma = S \cap \bigcup \mathcal{A}_\sigma$. Then $|\bigcup\{S_\sigma;\ |S_\sigma| > n\}| = m$. By the property (5) for $\mathcal{A}_\sigma$, if $|S_\sigma| > n$ then $\mathrm{card}\{A \in \mathcal{A}_\sigma;\ |A \cap S_\sigma| \geqslant p\} \geqslant |S_\sigma|$. Hence $\mathrm{card}\{A \in \mathcal{A};\ |A \cap S| \geqslant p\} \geqslant \sum\{|S_\sigma|;\ |S_\sigma| > n\} = m$. Thus (4) holds for $\mathcal{A}$. This completes the proof.

Consider next the case $m > q$. Here $m = q^+$ and $p' = q'$. Since $q > n \geqslant p \geqslant p' = q'$, in fact q is singular.

LEMMA 2.2. *Let* $q > q' = p'$ *and* $q > n \geqslant p$. *Given* X *with* $|X| = q$ *and a* $(\leqslant q, n, p)$*-family* $\mathfrak{D}$ *there is a* (q^+, p, p)*-family* $\mathcal{B}$ *with* $\bigcup \mathcal{B} \subseteq X$ *such that* $|B \cap D| < p$ *for* $B \in \mathcal{B}$, $D \in \mathfrak{D}$.

Proof. Take a family $\mathcal{B}$ as given by Proposition 0.4. Note that $|\bigcup\{[D]^p;\ D \in \mathfrak{D}\}| \leqslant q \cdot n^p = q$, and so we may omit from $\mathcal{B}$ any set B for which there is D in $\mathfrak{D}$ with $|B \cap D| \geqslant p$ and still retain a (q^+, p, p)-family.

LEMMA 2.3. *Given* Z *with* $|Z| = q^+$ *and a family* $\mathcal{S} = \{S_\nu;\ \nu < q^+\}$ *where always* $S_\nu \in [Z]^{\leqslant q}$, *then there is a* $(q^+, q^+, 1)$*-family* $\mathcal{C} = \{C_\nu;\ \nu < q^+\}$ *such that*

$$\mu \leqslant \nu \Rightarrow C_\nu \cap S_\mu = \emptyset\,. \tag{6}$$

Proof. Write $Z = \bigcup\{Z_\nu;\ \nu < q^+\}$ where the Z_ν are pairwise disjoint with $|Z_\nu| = q^+$. Put $C_\nu = Z_\nu - \bigcup\{S_\mu;\ \mu \leqslant \nu\}$.

THEOREM 2.4. *Let* $q > q' = p'$ *and* $q > n \geqslant p$. *Then there is a* (q^+, n, p)*-family* $\mathcal{A}$ *such that*

$$S \in [\bigcup \mathcal{A}]^q \Rightarrow \mathrm{card}\{A \in \mathcal{A};\ [A \cap S| \geqslant p\} = q^+\,. \tag{7}$$

Proof. Let $\mathcal{S} = [q^+]^q = \{S_\nu;\ \nu < q^+\}$ and take a $(q^+, q^+, 1)$-family $\mathcal{C} = \{C_\nu;\ \nu < q^+\}$ as given by Lemma 2.3 for $\mathcal{S}$. Choose (q^+, n, p)-families $\mathcal{C}_\nu = \{C_{\nu\alpha};\ \alpha < q^+\}$ such that $\bigcup \mathcal{C}_\nu \subseteq C_\nu$.

Construct by transfinite induction (q^+, n, p)-families $\mathcal{A}_\mu$ for $\mu < q^+$ such that

$$\mathcal{A}_\mu = \mathcal{A}_\nu \quad \text{for some } \nu < \mu, \quad \text{or else} \quad \bigcup \mathcal{A}_\mu \subseteq S_\mu \cup C_\mu\,, \tag{8}$$

and

$$A_1, A_2 \in \bigcup\{\mathcal{A}_\nu;\ \nu \leqslant \mu\} \Rightarrow |A_1 \cap A_2| < p \quad (\text{if } A_1 \neq A_2)\,, \tag{9}$$

as follows. Take ξ with $\xi < q^+$ and suppose the $\mathcal{A}_\nu$ for $\nu < \xi$ have already been satisfactorily defined. If there is $\nu < \xi$ for which

$$\mathrm{card}\{A \in \mathcal{A}_\nu;\ |A \cap S_\xi| \geqslant p\} = q^+\,,$$

put $\mathcal{A}_\xi = \mathcal{A}_\nu$ for such a ν. (Then (8) and (9) hold when $\mu = \xi$.) Otherwise, put $\mathfrak{D}_\xi = \{A;\ A \in \mathcal{A}_\nu$ for some $\nu < \xi$, and $|A \cap S_\xi| \geqslant p\}$, so $|\mathfrak{D}_\xi| \leqslant q$. Let $\mathcal{B}_\xi = \{B_{\xi\alpha};\ \alpha < q^+\}$ be a (q^+, p, p)-family with $\bigcup \mathcal{B}_\xi \subseteq S_\xi$, for which $|B \cap D| < p$ when $B \in \mathcal{B}_\xi$, $D \in \mathfrak{D}_\xi$ (as provided for by Lemma 2.2). Put $A_{\xi\alpha} = B_{\xi\alpha} \cup C_{\xi\alpha}$ and finally $\mathcal{A}_\xi = \{A_{\xi\alpha};\ \alpha < q^+\}$. Then $\bigcup \mathcal{A}_\xi \subseteq S_\xi \cup C_\xi$ and (8) holds with $\mu = \xi$. Since $\bigcup \mathcal{B}_\xi \subseteq S_\xi$ and $S_\xi \cap C_\xi = \emptyset$, when $\alpha \neq \beta$ we have $A_{\xi\alpha} \cap A_{\xi\beta} = (B_{\xi\alpha} \cap B_{\xi\beta}) \cup (C_{\xi\alpha} \cap C_{\xi\beta})$ so that $|A_{\xi\alpha} \cap A_{\xi\beta}| < p$. Thus $\mathcal{A}_\xi$ is a (q^+, n, p)-family. Also, by (6), if $\nu < \xi$ then $C_\xi \cap (S_\nu \cup C_\nu) = \emptyset$. So if $A \in \mathcal{A}_\nu$ for some $\nu < \xi$ then $A \cap C_\xi = \emptyset$ by (8). Thus $A \cap A_{\xi\alpha}$

$\subseteq A \cap (S_\xi \cup C_\xi) \subseteq A \cap S_\xi$. But now either $|A \cap S_\xi| < p$, or else $A \in \mathfrak{D}_\xi$. In either case, $|A \cap B_{\xi\alpha}| < p$. Since $A \cap A_{\xi\alpha} \subseteq A \cap (B_{\xi\alpha} \cup C_\xi) \subseteq A \cap B_{\xi\alpha}$, we have $|A \cap A_{\xi\alpha}| < p$. Thus (9) holds when $\mu = \xi$. This completes the construction.

Now put $\mathcal{A} = \bigcup \{\mathcal{A}_\mu;\ \mu < q^+\}$, so $\mathcal{A}$ is a (q^+, n, p)-family, by (9). We show (7) holds for $\mathcal{A}$. Take S in $[\bigcup \mathcal{A}]^q$. Then $S = S_\mu$ for some $\mu < q^+$; yet $\mathrm{card}\{A \in \mathcal{A}_\mu;\ |A \cap S_\mu| \geqslant p\} = q^+$, and $\mathcal{A}_\mu \subseteq \mathcal{A}$. Thus (7) holds, and the proof is complete.

Together, Theorems 2.1 and 2.4 establish the sufficiency of (1) and (2).

References

[1] A. Tarski, *Sur la décomposition des ensembles en sous-ensembles presque disjoints*, Fund. Math. 14 (1929), pp. 205–215.

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A note on Lusin's condition (N)

by

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Abstract. A function is said to satisfy condition (N′) provided the image of closed sets of measure 0 is of measure 0. In this paper it is shown that for several classes of functions (N′) implies Lusin's condition (N). The Baire functions (in a general setting) are one such class. Using the continuum hypothesis, a real valued function is constructed which satisfies (N′) but does not satisfy Lusin's condition (N).

A function $f\colon X \to Y$, where X and Y are measure spaces, is said to *satisfy condition* (N) if the image under f of each set of measure 0 in X is of measure 0 in Y. Condition (N) arises quite naturally in the study of integrals. (See, e.g., [4], p. 224ff.) A function $f\colon X \to Y$ will be said to *satisfy condition* (N′) if X is a topological space and the image under f of compact sets of measure 0 is of measure 0. The purpose of this note is to show that for several classes of functions condition (N′) implies condition (N); that is, the compact sets of measure 0 are a determining class for condition (N).

Although greater generality is attainable, the spaces X and Y will always be σ-compact metric spaces. The following notation and definitions will be used:

1) $B(f; A)$ will denote the *graph of f on the set A*, i.e., if $f\colon X \to Y$, $B(f; A) = \{(x, y) \in X \times Y \mid x \in A,\ y = f(x)\}$.

2) $m(E)$ will denote the *measure of E* when it is clear which measure this is.

3) $n-m(E)$ will denote *Hausdorff n-measure*. Briefly, a set B has n-measure less than or equal to b if for any given $\varepsilon > 0$ there is a cover U_ε of B with each set $I \in U_\varepsilon$ having diameter less than ε and $\sum_{I \in U_\varepsilon} (\operatorname{diam} I)^n < b + \varepsilon$.

4) proj_Y will denote the *projection map* from $X \times Y$ to Y; similarly, proj_X denotes the projection map from $X \times Y$ to X.

5) A measure space X is of *σ-finite measure* if X is the countable union of sets of finite measure.