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A note on the Hurewicz isomorphism theorem in Borsuk's theory of shape

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Abstract. In shape theory, the role of the homotopy groups π_n is played by the so called fundamental groups $\underline{\pi}_n$, introduced by K. Borsuk, and the homology groups which are useful there, are of the Vietoris–Čech type. The classical Hurewicz isomorphism theorem gives a connection between the homotopy groups π_n and the singular homology groups H_n with integral coefficients. An example of a compactum X is constructed, showing that there is no exact analogue of the Hurewicz theorem in shape theory. The example is simple: X is the double suspension of the 3-adic solenoid. The compactum X is arcwise connected and it has the following properties: (i) $\underline{\pi}_q(X) \approx 0$, for $q = 1, 2, 3$, and (ii) $\underline{\pi}_4(X)$ and $\check{H}_4(X)$ are not isomorphic.

In the theory of shape of compacta K. Borsuk introduced the fundamental groups $\underline{\pi}_n$ (see [1], § 14) which are related to the usual homotopy groups π_n in a fashion similar to the way in which the Vietoris–Čech homology groups $\check{H}_n$ are related to the singular homology groups H_n. The natural question that arose then was: is there any isomorphism theorem of the Hurewicz type in shape theory? The following theorem, proved in [3] (Theorem 3.2), is one of that type.

THEOREM. *If the pointed compactum (X, x_0) is approximatively q-connected for $q = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, then the limit Hurewicz homomorphism $\underline{\varphi}\colon \underline{\pi}_n(X, x_0) \to \check{H}_n(X, x_0)$ is an isomorphism.*

The coefficient group for all homology groups considered in this note is the group of integers.

One may ask if the assumption of the approximative q-connectedness of (X, x_0) for $q = 0, 1, \ldots, n-1$ in the above theorem can be replaced by the weaker assumption $\underline{\pi}_q(X, x_0) \approx 0$ for $q = 0, 1, \ldots, n-1$, which would make the theorem completely analogous to the classical Hurewicz theorem. Obviously, the independence of $\underline{\pi}_q$ from the choice of the base point must be assured by an appropriate assumption. For some special classes of compacta, e.g. for movable pointed compacta, the answer is affirmative (see [3], Corollary 3.7), but as we shall prove, in general it is not the case. The aim of this note is to describe an arcwise connected pointed compactum (X, x_0) with the following properties:

(i) (X, x_0) is approximatively q-connected for $q = 0, 1, 2$ (in particular $\underline{\pi}_q(X, x_0) \approx 0$ for $q = 0, 1, 2$),

(ii) $\underline{\pi}_3(X, x_0) \approx 0$,

(iii) $\underline{\pi}_4(X, x_0) \not\approx 0$,

(iv) $\check{H}_4(X, x_0) \approx 0$.

Let us recall two well-known constructions.

Let $S^1 = \{z: |z| = 1\}$ be the unit circle on the complex plane. The p-adic solenoid (for an integer $p \geqslant 2$) is defined to be the inverse limit of the sequence

$$S^1 \xleftarrow{f} S^1 \leftarrow \ldots \leftarrow S^1 \xleftarrow{f} S^1 \leftarrow \ldots$$

where $f(z) = z^p$.

For a topological space T, the suspension of T, denoted by ΣT, is defined to be the quotient space of $T \times [0, 1]$ in which $T \times \{0\}$ is identified to one point and $T \times \{1\}$ is identified to another point.

Let Y be the 3-adic solenoid and define X as the double suspension of Y, $X = \Sigma(\Sigma Y)$. Assume that X is a subset of the Hilbert cube Q and let $x_0 \in X$. (S^k, a) will denote the k-dimensional sphere with the base point a.

Since X is the double suspension of a connected compactum, condition (i) is automatically satisfied.

To prove (ii) observe that X is the inverse limit of the sequence

$$S^3 \xleftarrow{g} S^3 \leftarrow \ldots \leftarrow S^3 \xleftarrow{g} S^3 \leftarrow \ldots$$

where g is a map of degree 3, i.e. the induced homomorphism $g_*: H_3(S^3) \to H_3(S^3)$ is $g_*(z) = 3z$. Hence $\check{H}_3(X, x_0) \approx 0$ as the inverse limit of the sequence

$$Z \xleftarrow{\beta} Z \leftarrow \ldots \leftarrow Z \xleftarrow{\beta} Z \leftarrow \ldots$$

where Z is the group of integers, and $\beta(z) = 3z$. By the theorem quoted above, $\underline{\pi}_3(X, x_0) \approx 0$.

Property (iv) is easily verified: $\check{H}_4(X, x_0) \approx 0$ since $\dim X = 3$.

It is well known that the 4th homotopy group of S^3, $\pi_4(S^3, a)$, is isomorphic to Z_2, the group of integers modulo 2. In other words, there is an essential map $h: (S^4, a) \to (S^3, a)$, and any two essential maps from (S^4, a) to (S^3, a) are homotopic. Now, let $g: (S^3, a) \to (S^3, a)$ be a map of degree 3. We claim that

(*) the homotopy classes $[h]$ of h and $[gh]$ of gh are identical.

The following, simple proof of (*) was suggested to the author by A. Trybulec:

The homotopy group $\pi_4(S^3, a)$ and the Borsuk's cohomotopy group $\pi^3(S^4, a)$ coincide (see for instance [4], §§ 1–6). Let $h^*: \pi^3(S^3, a) \to \pi^3(S^4, a)$ be the homomorphism induced by h. Since g is of degree 3, we have

$[g] = 3[i]$ in $\pi^3(S^3)$, where i is the identity map on S^3. Thus, we have $[gh] = h^*[g] = h^*(3[i]) = 3h^*[i] = h^*[i] = [ih] = [h]$.

To prove (iii) observe that X has compact neighborhoods A_k $(k = 1, 2, \ldots)$ in Q such that $A_{k+1} \subset A_k$, $\bigcap_{k=1}^{\infty} A_k = X$, each A_k is homeomorphic to $S^3 \times Q$, and the inclusion $j: A_{k+1} \to A_k$ is a map of degree 3.

Let $\xi_k: (S^4, a) \to (A_k, x_0)$ for $k = 1, 2, \ldots$ be an essential map. By (*), $j\xi_{k+1} \simeq \xi_k$ in (A_k, x_0). Hence $\underline{\xi} = [\xi_k, (S^4, a) \to (X, x_0)\}$ is an approximative map. Furthermore, for each k, ξ_k is essential in (A_1, x_0) which implies that $\underline{\xi}$ is essential. Therefore, $\underline{\pi}_4(X, x_0) \not\approx 0$ (as a matter of fact, $\underline{\pi}_4(X, x_0) \approx Z_2$), i.e. condition (iii) is satisfied.

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