A note on the Hurewicz isomorphism theorem in Borsuk's theory of shape

by

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Abstract. In shape theory, the role of the homotopy groups $\pi_n$ is played by the so called fundamental groups $\pi_n$, introduced by K. Borsuk, and the homology groups which are useful there, are of the Vicktor-Veech type. The classical Hurewicz isomorphism theorem gives a connection between the homotopy groups $\pi_n$ and the singular homology groups $H_n$ with integral coefficients. An example of a compactum $X$ is constructed, showing that there is no exact analogue of the Hurewicz theorem in shape theory. The example is simple: $X$ is the double suspension of the 3-adic solenoid. The compactum $X$ is arcwise connected and it has the following properties: (i) $\pi_q(X) \neq 0$, for $q = 1, 2, 3$, and (ii) $\pi_3(X)$ and $H_3(X)$ are not isomorphic.

In the theory of shape of compacta K. Borsuk introduced the fundamental groups $\pi_n$ (see [1], § 14) which are related to the usual homotopy groups $\pi_n$ in a fashion similar to the way in which the Vicktor-Veech homology groups $H_n$ are related to the singular homology groups $H_n$.

The natural question that arose then was: is there any isomorphism theorem of the Hurewicz type in shape theory? The following theorem, proved in [3] (Theorem 3.2), is one of that type.

**Theorem.** If the pointed compactum $(X, x_0)$ is approximatively $q$-connected for $q = 0, 1, \ldots, n-1$ $(n \geq 2)$, then the limit Hurewicz homomorphism

$$\varphi: \pi_q(X, x_0) \rightarrow H_q(X, x_0)$$

is an isomorphism.

The coefficient group for all homology groups considered in this note is the group of integers.

One may ask if the assumption of the approximative $q$-connectedness of $(X, x_0)$ for $q = 0, 1, \ldots, n-1$ in the above theorem can be replaced by the weaker assumption $\pi_q(X, x_0) \approx 0$ for $q = 0, 1, \ldots, n-1$, which would make the theorem completely analogous to the classical Hurewicz theorem. Obviously, the independence of $\pi_q$ from the choice of the base point must be assured by an appropriate assumption. For some special classes of compacta, e.g. for movable pointed compacta, the answer is affirmative (see [3], Corollary 3.7), but as we shall prove, in general it is not the case. The aim of this note is to describe an arcwise connected pointed compactum $(X, x_0)$ with the following properties:
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(i) $(X, x_0)$ is approximatively $g$-connected for $g = 0, 1, 2$ (in particular $\pi_2(X, x_0) \cong 0$ for $g = 0, 1, 2$).
(ii) $\pi_3(X, x_0) \cong 0$.
(iii) $\pi_4(X, x_0) \cong 0$.
(iv) $H_4(X, x_0) \cong 0$.

Let us recall two well-known constructions.

Let $S^0 = \{z : |z| = 1\}$ be the unit circle on the complex plane. The $p$-adic solenoid (for an integer $p \geq 2$) is defined to be the inverse limit of the sequence

$$S^0 \leftarrow S^0 \leftarrow \ldots \leftarrow S^0 \leftarrow S^0 \leftarrow \ldots$$

where $f(z) = z^p$.

For a topological space $T$, the suspension of $T$, denoted by $\Sigma T$, is defined to be the quotient space of $T \times [0,1]$ in which $T \times \{0\}$ is identified to one point and $T \times \{1\}$ is identified to another point.

Let $Y$ be the $3$-adic solenoid and define $X$ as the double suspension of $Y$, $X = \Sigma (\Sigma Y)$. Assume that $X$ is a subset of the Hilbert cube $Q$ and let $x_0 \in X$. $(S^0, a)$ will denote the $k$-dimensional sphere with the base point $a$.

Since $X$ is the double suspension of a connected compactum, condition (i) is automatically satisfied.

To prove (ii) observe that $X$ is the inverse limit of the sequence

$$S^0 \leftarrow S^0 \leftarrow \ldots \leftarrow S^0 \leftarrow S^0 \leftarrow \ldots$$

where $g$ is a map of degree $3$, i.e. the induced homomorphism $g_* : H_4(S^0) \rightarrow H_4(S^0)$ is $g(x) = 3x$. Hence $H_4(X, x_0) \cong 0$ as the inverse limit of the sequence

$$Z \leftarrow Z \leftarrow \ldots \leftarrow Z \leftarrow Z \leftarrow \ldots$$

where $Z$ is the group of integers, and $\beta(x) = 3x$. By the theorem quoted above, $\pi_4(X, x_0) \cong 0$.

Property (iv) is easily verified: $H_4(X, x_0) \cong 0$ since $\dim X = 3$.

It is well known that the 4th homotopy group of $S^0$, $\pi_4(S^0, a)$, is isomorphic to $Z_2$, the group of integers modulo 2. In other words, there is an essential map $h : (S^0, a) \rightarrow (S^0, a)$, and any two essential maps from $(S^0, a)$ to $(S^0, a)$ are homotopic. Now, let $g : (S^0, a) \rightarrow (S^0, a)$ be a map of degree 3. We claim that

(*) the homotopy classes $[h]$ of $h$ and $[gh]$ of $gh$ are identical.

The following, simple proof of (*) was suggested to the author by A. Trybulec:

The homotopy group $\pi_4(S^0, a)$ and the Borsuk's cohomotopy group $\pi_4(S^0, a)$ coincide (see for instance [4], §1–6). Let $h^* : \pi_4(S^0, a) \rightarrow \pi_4(S^0, a)$ be the homomorphism induced by $h$. Since $g$ is of degree 3, we have

$$[gh] = h^*[g] = h^*[3i] = 3h^*[i] = 3h^*[i] = [h] = [h].$$

To prove (iii) observe that $X$ has compact neighborhoods $A_k (k = 1, 2, \ldots)$ in $Q$ such that $\bigcup_{k=1}^{\infty} A_k = X$, each $A_k$ is homeomorphic to $S^0 \times Q$, and the inclusion $j : A_{k+1} \rightarrow A_k$ is a map of degree 3.

Let $\xi_k : (S^0, a) \rightarrow (A_k, x_0)$ for $k = 1, 2, \ldots$ be an essential map. By (*), $j_{k+1} \xi_k \cong \xi_k$ in $(A_k, x_0)$. Hence $\xi = \bigcup_{k=1}^{\infty} \xi_k$ in $(A_k, x_0)$ is an approximative map. Furthermore, for each $k$, $\xi_k$ is essential in $(A_k, x_0)$ which implies that $\xi$ is essential. Therefore, $\pi_4(X, x_0) \cong 0$ (as a matter of fact, $\pi_4(X, x_0) \cong Z_2$), i.e. condition (ii) is satisfied.

References


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