Case 2. $y_0 > 0$. For each point w of

$$X-[l_1(w_0(1))\cup l_2(w_0(1))\cup l_1(w_0(2)(\cup l_2(w_0(2))]$$

define $\varphi(w, w_0, H) = 1$.

If $w_0(2)\in H$, then define $\varphi(w, w_0, H) = 1$ for each point w of $[l_2(w_0(2)\cup l_1(w_0(2))]-\{w_0\}$.

If $w_0(2)\notin H$, then define $\varphi(w, w_0, H) = \varphi(w, w_0(2), H)$ for each point w of $[l_2(w_0(2))\cup l_1(w_0(2))]-\{w_0\}$.

If $w_0(1)\in H$, define $\varphi(w, w_0, H) = 1$ for all points w of

$$[l_1(w_0(1))\cup l_2(w_0(1))]-\{w_0\}\,.$$

If $w_0(1)\notin H$, define $\varphi(w, w_0, H) = \varphi(w, w_0(1), H)$ for each point w of $[l_1(w_0(1))\cup l_2(w_0(1))]-\{w_0\}$.

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On an extremely restricted ω-rule

by

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Abstract. By an extremely restricted ω-rule (for Heyting's Arithmetic) we understand an ω-rule of the form:

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots} \\ \text{To conclude:} & \forall x Ax \\ \text{Provided} & \vdash_{HA} \forall x\dot{A}x\,. \end{array}$$

Although such a rule does not increase the class of theorems, it allows one to quickly obtain (infinite) derivations with the subformula property. From the subformula property many results can then be easily obtained.

§ 0. Introduction. From an intuitive point of view the ω-rule

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots, Ak, \ldots} \\ \text{To conclude:} & \forall x Ax \end{array}$$

is a much simpler rule to justify than its finitary cousin, the rule of induction:

$$\text{(IND)}\qquad \begin{array}{rl} \text{From:} & \underline{A0, \forall x(Ax\supset Ax')} \\ \text{To conclude:} & \forall x Ax\,. \end{array}$$

And yet the latter is usually preferred when considering formal systems. Probably the main objection against the ω-rule is that the derivations are then infinite trees of formulae and there is a natural distrust to using infinite sets when one is trying to better understand the infinite. This distrust is further enhanced by the fact that if to first order classical arithmetic, CA, one adds the ω-rule then one obtains a maximal system (i.e. for every sentence A, either A or $\neg A$ is derivable in CA+"ω-rule"); for then the fact that CA+"ω-rule" $\vdash A$ gives us no more information than $\mathfrak{N}\models A$ (A is true in the natural numbers).

On the other hand of instead of the (full) ω-rule one considers a restricted ω-rule, that is an infinitary rule of inference of the form

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots, Ak, \ldots} \\ \text{To conclude:} & \forall x Ax \\ \text{Provided that:} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

then it might be possible to extract some useful information from the fact that CA+"restricted ω-rule" $\vdash A$. How much more useful that information may be than $\mathfrak{N} \models A$ naturally depends on what restriction is imposed on the ω-rule.

One of the first restrictions considered was to require that there be a (*general*) *recursive function* f such that for each natural number n, $f(n)$ be the code of a (possibly infinite) derivation of $A\mathbf{n}$. In Shoenfield 1959 it is shown that CA+"recursively restricted ω-rule" is equivalent to CA+(full) ω-rule.

Other restrictions so far considered have usually been under one, or more of the following categories:

(Cat$_1$) requiring that there be a such and such function f such that for each n, $f(n)$ is the code of a derivation of $A\mathbf{n}$,

(Cat$_2$) requiring that the sentences $A\mathbf{0}, A\mathbf{1}, \ldots$ be atomic, or Σ_n^0 or ...,

(Cat$_3$) requiring that there be no more than α nestings of the ω-rule,

(Cat$_4$) requiring that the derivation be (provably) of such and such form, see, for example Feferman [2], Fenstad [4], Kent [5], Shoenfield [13].

On the whole the addition of such restricted ω-rules to CA, or even intuitionistic arithmetic, IA, result in stronger systems. The fact that the system with the (restricted) ω-rule is stronger is sometimes of no importance; for example if one wishes to show that $\mathbf{0} = \mathbf{1}$ is *not* provable in CA, one merely has to observe that $\mathbf{0} = \mathbf{1}$ is not provable in CA+"(restricted) ω-rule" because if it were then it would have a cut-free proof [using the cut-elimination theorem for CA+"(restricted) ω-rule"] and the latter is ridiculous. However it is of importance if one wishes to use the cut-elimination (or better the subformula property of the cut-free derivations) of systems with ω-rules to obtain results about the finitary system. For example, suppose that the sentence $A_1 \vee A_2$ is derivable in IA. Then it is simple to show that $A_1 \vee A_2$ is derivable in IA+"restricted ω-rule". Using the cut-elimination theorem one immediately obtains that either A_1 or A_2 is derivable in IA+"restricted ω-rule". However, unless we know that IA+restricted ω-rule is equivalent to IA (and it is not for most restrictions of type (Cat$_1$)-(Cat$_4$)) we cannot conclude that either A_1 or A_2 is derivable in IA.

Thus what is needed is an extremely restricted ω-rule such that

(A) its addition to IA results in a system equivalent to IA,

(B) the cut-elimination theorem holds in IA + the extremely restricted ω-rule.

Why do we need it? Because on the one hand one should be able to prove such well-known results as "If IA $\vdash A_1 \vee A_2$ then either IA $\vdash A_1$ or IA $\vdash A_2$" by *simple* observations about derivations and, on the other hand, there are results which seem to need such a cut-elimination; for example Kreisel's result that IA+"the reflection principle" is equivalent to IA+ε_0-induction (see Kreisel–Levy [10], § 10).

Kreisel has given lots of *hints* on how to obtain such an ω-rule, see for example pages 163–164 of Kreisel [9]. On the whole the hints have been such as to suggest that the required restrictions on the infinitary derivations (and hence on the ω-rule) would either be technically complicated or else very sophisticated; for example in page 140 of Kreisel–Levy [10] it is stated "The most delicate point is to set up the infinitary system to be *equivalent* to HA (= IA) ...".

In this paper it will be shown that neither sophistication nor technical dexterity is needed to obtain, and make use, of an extremely restricted ω-rule having properties (A), (B).

§ 1. The extremely restricted ω-rule for IA. It is simply the following:

$$\begin{array}{rl} \text{From:} & \dfrac{A\mathbf{0}, A\mathbf{1}, A\mathbf{2}, \ldots}{\forall x A x} \\ \text{To conclude:} & \\ \text{Provided that:} & \forall x A x \text{ is provable in IA}. \end{array}$$

§ 2. Syntactical details about the formal systems IA and ω-IA. We shall for the most part use the notations and conventions of IM (that is: Kleene [6]). The formulae of IA and ω-IA will simply be the formulae of the system of formal number theory given in Chapter IV of IM. The axiomatization for IA and ω-IA will be given in terms of sequents, that is expressions of the form

$$\Gamma \rightarrow \Theta$$

where Γ and Θ are finite (possible empty) sequences of formulae with the (intuitionistic) restriction that Θ should contain at most one formulae (see Lemma 32a of IM). In addition, for ω-IA we require that Γ, Θ consist exclusively of *sentences* (i.e. closed formulae).

2.1. Rules of inferences and aximos of IA.

2.1.1. Logical rules of inference for the propositional calculus. Exactly those given in IM for the intuitionistic system G2 (and G1), page 442.

2.1.2. Structural rules of inference. Exactly those given in IM for the intuitionistic system G2. That is those given on page 443 except that the following version of the cut-rule will be used:

$$\text{(CUT)} \qquad \frac{\Gamma \rightarrow M, \quad \Sigma \rightarrow \Omega}{\Gamma, \Sigma_M \rightarrow \Omega}$$

where Σ_M is the result of suppressing all occurrences of M in Σ and where it is assumed that Σ contains at least one occurrence of M. The formula M is called the cut formula of the inference.

2.1.3. Logical rules of inference for the predicate calculus. Exactly those given in IM for the system G2 (and G1), page 442.

2.1.4. Logical axiom schema. $C \to C$.

2.1.5. Arithemitical axioms.

$$a' = b' \to a = b\,,$$
$$a' = 0 \to$$
$$a = b,\ a = c \to b = c\,,$$
$$a = b \to a' = b'\,,$$
$$\to a + 0 = a\,,$$
$$\to a + b' = (a+b)'\,,$$
$$\to a \cdot 0 = 0\,,$$
$$\to a \cdot b' = a \cdot b + a\,.$$

2.1.6. Rule of induction.

$$\frac{\Gamma \to A0,\ \Gamma \to \forall x(Ax \supset Ax')}{\Gamma \to \forall x Ax}\,.$$

2.1.7. Derivations in IA. We assume the derivations in IA to be given in tree form (with a given analysis) and with axioms at the uppermost sequents. "$\mathrm{IA} \vdash \Gamma \to \Theta$" is used to express that the sequent $\Gamma \to \Theta$ is provable in the system IA. We shall also write "$\mathrm{IA} \vdash A$" for "$\mathrm{IA} \vdash \to A$".

2.2. Rules of inference and axioms of ω-IA. Recall that the sequents of ω-IA consist of only sentences.

2.2.1. Logical rules of inference for the propositional calculus. The same as for IA, i.e. 2.1.1.

2.2.2. Structural rules of inference. The same as for IA, i.e. 2.1.2.

2.2.3. Rules of inference for the quantifiers. $\forall\to$, $\to\exists$ and the following formulations of the extremely restricted ω-rule for IA

$$\to\forall_\omega \qquad \frac{\Gamma \to A0 \quad \Gamma \to A1 \ \ldots\ \Gamma \to Ak \ \ldots}{\Gamma \to \forall x Ax}$$

subject to the restriction that: $\mathrm{IA} \vdash \Gamma \to \forall x Ax$.

$$\exists_\omega\to \qquad \frac{A0, \Gamma \to \Theta \quad A1, \Gamma \to \Theta \ \ldots\ Ak, \Gamma \to \Theta}{\exists x Ax, \Gamma \to \Theta}$$

subject to the restriction that: $\mathrm{IA} \vdash \exists x Ax, \Gamma \to \Theta$.

2.2.4. No logical axiom schema.

2.2.5. Arithmetical axiom schemata. If t_1, t_2 are two closed terms and if under the canonical interpretation $t_1 = t_2$, then $\to t_1 = t_2$ is an arithmetical axiom. On the other hand if under the canonical interpretation $t_1 \neq t_2$ then $t_1 = t_2 \to$ is an axiom.

2.2.6. Derivations is ω-IA. We assume the derivations in ω-IA to be given in tree form and with (arithmetical) axioms at the uppermost sequents. An "analysis" of a derivation in ω-IA consists in assigning to each node n of the tree:

1. One of the expressions $\to\supset, \supset\to, \ldots, \to\forall_\omega, \exists_\omega\to$, ax in such a way that if $\to\supset, \ldots, ax$ respectively is assigned to the node n, then the sequent at the node n has been obtained from those immediately above it by the rule $\to\supset, \ldots, ax$, respectively (in the case of ax it is understood that the sequent at n is an uppermost sequent and an arithmetical axiom).

2. An ordinal α such that the ordinal assigned to the node n is strictly greater that the ordinals assigned to the nodes immediately above n. The ordinal assigned to the end-node (root) of the tree is the *ordinal* of the derivation w.r.t. the given analysis.

3. A natural number k such that in all applications of the cut rule at nodes above, or at, the node n the natural number k is strictly greater than the degree of the cut sentence (the degree of a formula is defined to be the number of occurrences of $\forall$, $\exists$, $\wedge$, $\vee$, and $\supset$ in the formula). The natural number assigned to the end-node is called the cut degree of the derivation.

"$\omega\text{-IA} \vdash \Gamma \to \Theta[\alpha, k]$" is used to express that there is a ω-IA derivation of the sequent $\Gamma \to \Theta$ of ordinal $\leqslant \alpha$ and cut degree $\leqslant k$. We shall also use the following conventions

$\omega\text{-IA} \vdash \Gamma \to \Theta$	for	$(E\alpha)(Ek)\,\omega\text{-IA} \vdash \Gamma \to \Theta[\alpha, k]$,
$\omega\text{-IA} \vdash A$	for	$\omega\text{-IA} \vdash \to A$,
$\omega\text{-IA}^- \vdash \Gamma \to \Theta$	for	$(E\alpha)\,\omega\text{-IA} \vdash \Gamma \to \Theta[\alpha, 0]$,
$\omega\text{-IA}^- \vdash A$	for	$\omega\text{-IA}^- \vdash \to A$.

Note that $\omega\text{-IA}^- \vdash \Gamma \to \Theta$ iff there is a cut free derivation of $\Gamma \to \Theta$.

§ 3. Equivalence of IA with ω-IA. The proof of the following theorems require neither too much work nor ingenuity.

3.1. THEOREM. *If* $\omega\text{-IA} \vdash A$ *then* $\mathrm{IA} \vdash A$.

Proof. One simply shows by induction on the ordinal α that

$$\text{if} \quad (Ek)\ \omega\text{-IA} \vdash A[\alpha, k] \quad \text{then} \quad \mathrm{IA} \vdash A\,.$$

The only steps that might require some thought are (a) the axioms of ω-IA; but they have been shown to be provable in IA by IM and (b) the infinitary rules $\rightarrow\forall_\omega$ and $\exists_\omega\rightarrow$; however because of the way they are formulated we do not even need the induction hypothesis for these rules!

3.2. THEOREM. *If A is a sentence and* $\mathrm{IA}\vdash A$, *then* ω-$\mathrm{IA}\vdash A$.

Proof. This time one uses induction on the height of the derivation in IA. The only non-trivial steps are the axioms, the rules $\rightarrow\forall$, $\exists\rightarrow$ and the induction rule. Let us consider the induction rule. That is assume that we have a derivation in IA which ends as follows

$$\text{(i)}\qquad \frac{\Gamma\rightarrow A\mathbf{0} \qquad \Gamma\rightarrow\forall x(Ax\supset Ax')}{\Gamma\rightarrow\forall x Ax}\,.$$

Then using the induction hypothesis we then obtain that

$$\text{(ii)}\qquad \omega\text{-IA}\vdash\Gamma\rightarrow A\mathbf{0}\,,$$

$$\text{(iii)}\qquad \omega\text{-IA}\vdash\Gamma\rightarrow\forall x(Ax\supset Ax')\,.$$

Then we observe (i.e. proof by induction on the degree of A) that for all natural numbers n,

$$\text{(iv)}\qquad \omega\text{-IA}\vdash\forall x(Ax\supset Ax')\rightarrow A\mathbf{n}\supset A\mathbf{n}'\,.$$

Applying the cut rule to (iii) and (iv) we obtain

$$\text{(v)}\qquad \omega\text{-IA}\vdash\Gamma\rightarrow A\mathbf{n}\supset A\mathbf{n}'\,.$$

Now we can obtain (again a proof by induction on the degree of A) that

$$\text{(vi)}\qquad \omega\text{-IA}\vdash A\mathbf{n},\ A\mathbf{n}\supset A\mathbf{n}'\rightarrow A\mathbf{n}'\,.$$

Applying the cut rule to (vi) and (v) we obtain

$$\text{(vii)}\qquad \omega\text{-IA}\vdash\Gamma,\ A\mathbf{n}\rightarrow A\mathbf{n}'\,.$$

Starting with (ii) and using a lot of cuts with (vii) we finally conclude that

$$\text{(viii)}\qquad (n)[\omega\text{-IA}\vdash\Gamma\rightarrow A\mathbf{n}]\,.$$

But (i) tolds us that $\mathrm{IA}\vdash\Gamma\rightarrow\forall xAx$, so we may apply $\rightarrow\forall_\omega$ to (viii) to obtain

$$\omega\text{-IA}\vdash\Gamma\rightarrow\forall xAx\,.$$

It should now be clear that similar methods can be applied to the other non-trivial steps.

Combining 3.1 and 3.2 we obtain the following.

3.3. THEOREM (Equivalence Theorem). *For any sentence A,* $\mathrm{IA}\vdash A$ *iff* ω-$\mathrm{IA}\vdash A$.

3.4. Remarks. (i) There is nothing magical about intuitionism, the same result, mutatis mutandis, would apply to classical arithmetic (and other systems).

(ii) By keeping a check on the growths of the ordinals of the derivations we see that 3.2 could be also stated:

$$\text{if}\quad \mathrm{IA}\vdash A\quad \text{then}\quad (Ea)_{a<\omega^2}(Ek)\,\omega\text{-IA}\vdash A[\alpha,k]$$

and also a bound for the k can be (primitively recursive) determined from the derivation of A in IA.

§ 4. Cut elimination in ω-IA. The literature is full of hints *and* examples of how to carry out cut-elimination for infinitary systems. The basic idea goes back to Gentzen in the case of the predicate calculus; the effect of the ω-rule is to replace the induction rule by a rule which is similar in form to the rule for conjunction. Thus all one has to do is take Gentzen's proof of cut-elimination for the predicate calculus and add two more cases to correspond to the rules $\rightarrow\forall_\omega$ and $\exists_\omega\rightarrow$ which are formally similar to $\rightarrow\wedge$ and $\vee\rightarrow$. One also has to keep track of the ordinal growth. The basic lemma is the following:

4.1. LEMMA. *If* ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta\,[\alpha,k+1]$ *then* ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta\,[2^\alpha,k]$.

Proof. Simply make the suggested changes to the proof given in IM to the basic lemma (or else rewrite the proof in Schütte [12] into the sequent formalism). Note that because of Theorem 3.1, every sequent occurring in the ω-IA derivation of $\Gamma\rightarrow\Theta$ is also IA derivable and so the restriction on the applications of the ω-rule can certainly be met when constructing the derivation of smaller cut degree.

4.2. THEOREM. (i) *If* ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta$ *then for some* $\alpha<\varepsilon_0$, ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta\,[\alpha,0]$, (ii) ω-$\mathrm{IA}\vdash A$ *iff* ω-$\mathrm{IA}^-\vdash A$.

Proof. Clearly (ii) is a consequence of (i). To prove (i) assume that ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta$. From Theorem 3.2 and Remark 3.4 we get that $(E\beta)_{\beta<\omega^2}(Ek)$ ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta[\beta,k]$. Then applying 4.1 k times we obtain and $\alpha<\varepsilon_0$ such that ω-$\mathrm{IA}\vdash\Gamma\rightarrow\Theta\,[\alpha,0]$.

§ 5. Immediate applications of the cut-elimination and equivalence theorem.

5.1. THEOREM. *Suppose that A_1, A_2 and $\exists xAx$ are sentences then*

(i) *If* $\mathrm{IA}\vdash\exists xAx$, *then for some n,* $\mathrm{IA}\vdash A\mathbf{n}$.

(ii) *If* $\mathrm{IA}\vdash A_1\vee A_2$, *then either* $\mathrm{IA}\vdash A_1$ *or* $\mathrm{IA}\vdash A_2$.

Proof of (i). Suppose $\mathrm{IA}\vdash\exists xAx$. Then ω-$\mathrm{IA}\vdash\exists xAx$ (by 3.1), and by 4.2 we then obtain ω-$\mathrm{IA}^-\vdash\exists xAx$. Consider a cut-free proof of $\exists xAx$. The last rule of inference applied must have been $\rightarrow\exists_\omega$, from which it follows that for some n, ω-IA $\vdash A\mathbf{n}$. Using then the equivalence theorem we obtain that IA $\vdash A\mathbf{n}$. The proof of (ii) is similar.

5.2. THEOREM. *Suppose that A, $\exists xBx$ are sentences.*

$$\text{IA} \vdash \neg A \supset \exists xBx \quad \textit{iff} \quad \text{IA} \vdash \exists x(\neg A \supset Bx)\,.$$

Proof. Consider a cut-free proof of $\neg A \supset \exists xBx$, i.e. of $\rightarrow \neg A \supset \exists xBx$. By arguments similar to those used in the proof of 5.1 we obtain that ω-IA $\vdash \exists x(\neg A \supset Bx)$. Using the equivalence theorem we obtain that IA $\vdash \exists x(\neg A \supset Bx)$.

5.3. THEOREM. *Suppose that Ax is a primitive recursive formula, and hence decidable in* IA. *Then*

$$\text{IA} \vdash (\neg \forall xAx \supset \exists x \neg Ax) \quad \textit{iff either } \text{IA} \vdash \forall xAx \textit{ or } \text{IA} \vdash \exists x \neg Ax\,.$$

Proof. Suppose IA $\vdash (\neg \forall xAx \supset \exists x \neg Ax)$. Then by 5.2 we obtain that IA $\vdash \exists x(\neg \forall xAx \supset \neg Ax)$. 5.1 then leads to

$$(En)[\text{IA} \vdash \neg \forall xAx \supset \neg A\mathbf{n}]$$

which in turn give us (because Ax is primitive recursive) that

$$(En)[\text{IA} \vdash A\mathbf{n} \supset \forall xAx]$$

and form the latter we obtain, using again the decidability of $A\mathbf{n}$ that IA $\vdash \forall xAx \vee \exists x \neg Ax$. Result follows then using 5.1.

§ 6. Formal equivalence of IA with ω-IA. In order to be able to carry out more refined applications of the equivalence and cut-elimination theorems we must consider what assumptions were used in their proofs. To express that the sequent $\Gamma \rightarrow \Theta$ is provable in IA is no problem; let Prov(x, y) be the formula of first-order number theory that (canonically) expresses the condition that x is the Gödel number of a derivation in IA of the sequent with Gödel number y. We now wish to find an arithmetical formula Der(x, y) which (canonically) expresses the condition that x is the code of a derivation in ω-IA of the sequent whose Gödel number is y so that we can then consider the status of the first-order sentence

$$\forall y\big(\exists x\text{Prov}(x, y) \equiv \exists x\text{Der}(x, y)\big)\,.$$

6.1. THE ARITHMETIZATION OF ω-IA. A derivation (with a given analysis) consists of a tree at whose nodes are assigned sequents, ordinals and natural numbers. Now sequents can be arithmetized so that to all intents and purposes they may considered natural numbers. The ordinals involved, in view of Theorem 4.2 can be restricted to being $< \varepsilon_0$ and fortunately for us Schütte has developed in Schütte [12] a system of

unique notations for ordinals well beyond ε_0 and has shown that many of the laws of ordinal arithmetic can be formally proven in IA for the notations (actually in Schütte [12] it is done for CA, but the methods used are intuitionistically valid). More specifically he defines

a p.r. linear ordering $\prec$,
a p.r. binary function $\oplus$,
a p.r binary function $\odot$,
a p.r. unary function $*$,
a p.r. unary function E,
a p.r. binary function $\#$,
terms, 0, 1, $\boldsymbol{\omega}$, $\boldsymbol{\varepsilon}_0$,

for which it is proven (in IA) that $\oplus$, 0, $*$, E, $\#$ satisfy the usual defining conditions for addition, ordinal multiplication, ordinal successor, ordinal exponentiation to base 2 and Hessenberg's natural sum of ordinals respectively. Furthermore the usual monotonicity conditions with respect to the ordering (e.g. $x \prec y \supset E(x) \prec E(y)$) are also shown to be provable. The terms 0, 1, $\boldsymbol{\omega}$, $\boldsymbol{\varepsilon}_0$ represent respectively the ordinals, 0, 1, ω and ε_0. In addition it is shown that transfinite induction w.r.t. $\prec$ is provable in IA in the following form:

Given a formula Ax let $\mathfrak{T}_xA(x)$ be defined by

$$\mathfrak{T}_xAx = \forall x\big(\forall y(y \prec x \supset Ay) \supset Ax\big)\,.$$

Then given any term t, let $\mathfrak{J}_x(A_x, t)$ be the formula

$$\mathfrak{J}_x(Ax, t) = \mathfrak{T}_xAx \supset \forall x(x \prec t \supset Ax)\,.$$

6.2. THEOREM (Schütte [12]). *If $\mathbf{k}$ is the notation for an ordinal smaller than ε_0 then for any formula Ax of first order arithmetic*:

$$\text{IA} \vdash \mathfrak{J}_x(Ax, \mathbf{k})\,.$$

In view of Schütte's results instead of requiring that ordinals be assigned to the nodes of the derivation tree we shall require that ordinal notations be assigned in their place. Furthermore using a standard coding of the set of natural numbers onto the set of finite sequences of natural numbers we see that we can consider a derivation in ω-IA to be nothing more (nor less) than a (special) number theoretic function. Or being a little more specific:

6.3. DEFINITION. A number theoretic function φ is a (code for *a*) *derivation in* ω-IA if

$$(m)\big(\varphi m = 0 \rightarrow (n)(\varphi(m^\frown n) = 0)\big),$$

$(m)\big(\varphi m \neq 0 \to (\varphi m)_0$ is the Gödel number of a sequent of ω-IA$\big)$,

$(m)\big(\varphi m \neq 0 \to (\varphi m)_1 \in \{\ulcorner \to \supset \urcorner, \ulcorner \to \supset \urcorner, \ldots, \ulcorner ax \urcorner\}\big)$,

$(m)\big(\varphi m \neq 0 \to (\varphi m)_0$ is related to $\{\varphi(m^\frown\langle i\rangle)\colon\ i = 0, 1, \ldots\}$ by the rule $(\varphi m)_1\big)$,

$(m)\Big(\varphi m \neq 0 \to (i)\big((\varphi(m^\frown\langle i\rangle))_2 \prec (\varphi m)_2\big)\Big)$,

$(\varphi(0))_2 \prec \varepsilon_0$,

$(m)\Big((\varphi m)_1 \in \{\ulcorner \to \forall_\omega \urcorner \ulcorner \exists_\omega \to \urcorner\} \to \mathrm{Prov}\big((\varphi m)_3, (\varphi m)_0\big)\Big)$.

Observe that the last condition of 6.3 has the effect of assigning to those nodes of a derivation in which the extremely restricted ω-rule has been applied a proof that the restriction on the ω-rule has been met.

6.4. Definition. $\mathrm{DEV}(\varphi, \ulcorner \Gamma \to \Theta \urcorner, u)$ iff φ is a code for a derivation in ω-IA, $(\varphi 0)_0 = \ulcorner \Gamma \to \Theta \urcorner$ and $(\varphi 0)_2 = u$.

It is clear from 6.3 that the formula $\mathrm{DEV}(\varphi, y, u)$ is arithmetical in the function φ and thus if the function φ is itself arithmetical then we would obtain an arithmetical formulae. The obvious collections of functions to consider are the class GR of general recursive functions and PR of primitive recursive functions.

6.5. Definitions.

(i) $\mathrm{DER}_{\mathrm{GR}}(e, y, u)$ iff e is the Gödel number of a general recursive function and $\mathrm{DEV}(\{e\}, y, u)$.

(ii) $\mathrm{DER}_{\mathrm{PR}}(b, y, u)$ iff b is the index of a primitive recursive function and $\mathrm{DEV}([b], y, u)$.

The advantage of using indeces of primitive recursive functions (for a definition of index see either Kleene [7] or Feferman [3]) instead of Gödel numbers is that the set of indices form a primitive recursive set. Let $\mathrm{Der}_{\mathrm{GR}}$ and $\mathrm{Der}_{\mathrm{PR}}$ be the formal counterparts to $\mathrm{DER}_{\mathrm{GR}}$, $\mathrm{DER}_{\mathrm{PR}}$ respectively.

$\mathrm{DER}_{\mathrm{GR}}$ is the analogue of Shoenfield's recursively restricted ω-rule. However the equivalence of $\omega\text{-IA} \vdash \Gamma \to \Theta$ with

$$(Ex)(Eu)\mathrm{DER}_{\mathrm{GR}}(e, \ulcorner \Gamma \to \Theta \urcorner, u)$$

is, unlike Shoenfield's result, nothing more than a simple observation. For suppose $\omega\text{-IA} \vdash \Gamma \to \Theta$. Then by the equivalence theorem, $\mathrm{IA} \vdash \Gamma \to \Theta$. But then the proof of 3.2 shows that $(Eb)(Eu)\mathrm{DER}_{\mathrm{PR}}(b, \ulcorner \Gamma \to \Theta \urcorner, u)$ from which it immediately follows that $(Ee)(Eu)\mathrm{DER}_{\mathrm{GR}}(e, \ulcorner \Gamma \to \Theta \urcorner, u)$.

The last remarks would suggest that it does not make much difference whether one uses $\mathrm{DER}_{\mathrm{GR}}$ of $\mathrm{DER}_{\mathrm{PR}}$, and that is the case provided one remembers to include the following as a rule of inference:

$$\begin{array}{ll} \text{From:} & \Gamma \to \Theta \\ \text{To conclude:} & \overline{\Gamma \to \Theta}\,, \end{array}$$

which for a lack of a better name we shall call the rule of repetition (Rep). For if one has such a rule then any GR derivation in ω-IA can be "strechted out" to a PR-derivation using the same rules of inference [plus (Rep)]. In particular if the GR derivation was cut-free so will be the stretched out PR derivation. Now simple applications of the fixed point theorem show that the cut-elimination holds for the GR-derivations. Thus it holds for the PR derivations if we include (Rep). On the other had considering $\forall x \exists y T(\mathbf{e}, x, y)$, where $\mathbf{e}$ is the Gödel number of a provably recursive function which is not primitive recursive, one can see the that, in our particular formalization, the cut elimination theorem does not hold for the PR derivations of ω-IA if (Rep) is not included. Thus we shall henceforth assume that (Rep) is one of the (structural) rules of inference of ω-IA and IA.

In the system ω-IA we had placed no restriction on the ω-rule other than the requirement that the conclusion be provable in IA. In particular, a derivation in ω-IA need not be a primitive recursive tree. Thus $\mathrm{Der}_{\mathrm{PR}}$ is not the formalization of the proof predicate of ω-IA but rather of a system in which there is the *added* restriction that the proof trees be primitive recursive. However we have already shown that the same theorems are provable in either system so we shall take the liberty of letting ω-IA be, from now on, the system with the added restriction that the proof trees be primitive recursive. Thus we may now claim that $\mathrm{Der}_{\mathrm{PR}}$ is a formalization of the proof predicate for ω-IA.

We are now ready to consider the formal equivalence of IA with ω-IA.

6.6. Theorem. *There is a term $t(x)$ for a primitive recursive function such that*

$$\mathrm{IA} \vdash \forall y\big[\mathrm{Prov}(x, y) \supset \exists u\big(\mathrm{Der}_{\mathrm{PR}}(t(x), y, u) \wedge u \prec \boldsymbol{\omega} \odot \boldsymbol{\omega}\big)\big]\,.$$

Proof. A straightforward (albeit long and boring) formalization of the proof of Theorem 3.2, using of course the formal rule (IND) of induction.

6.7. Theorem. *Let k be a notation for an ordinal $< \varepsilon_0$. Then*

$$\mathrm{IA} \vdash \forall y \forall u[\mathrm{Der}_{\mathrm{PR}}(x, y, u) \wedge u \prec \mathbf{k} \supset \exists z \mathrm{Prov}(z, y)]\,.$$

Proof. A formalization of the proof of Theorem 3.1, this time using transfinite induction up to $\mathbf{k}$, which by Schutte's theorem is derivable in IA.

Let "IA + $\mathrm{TI}_{\varepsilon_0}$" denote the system obtained from IA by adding as axioms all sequents of the form:

$$\mathcal{T}_x Ax \to \forall x(x \prec \boldsymbol{\varepsilon}_0 \supset Ax)\,.$$

Then the proof of Theorem 6.7 can be adapted to give:

6.8. Theorem. $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash \forall y[\exists x \exists u \mathrm{Der}_{\mathrm{PR}}(x, y, u) \supset \exists z \mathrm{Prov}(z, y)]$.

Thus combining Theorem 6.8 and 6.6 we obtain that the equivalence of IA with ω-IA can be proven in $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0}$. In order to simplify the notation let us agree to the following abbreviations

$$\mathrm{Pr}(y) \equiv \exists x \mathrm{Prov}(x, y),$$

$$\mathrm{Pr}_\omega(y) \equiv \exists x \exists u \mathrm{Der}_{\mathrm{PR}}(x, y, u),$$

$$\mathrm{Pr}_\omega^-(y) \equiv \exists x \exists u \mathrm{Der}_{\mathrm{PR}}^-(x, y, u).$$

In terms of the above notation the equivalence of IA and ω-IA can be stated as follows:

6.9. $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash (\mathrm{Pr}(y) \equiv \mathrm{Pr}_\omega(y))$.

The cut-elimination can be stated:

6.10. $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash (\mathrm{Pr}_\omega(y) \supset \mathrm{Pr}_\omega^-(y))$.

§ 7. An application. We shall now use the formal equivalence of IA with IA + "the extremely restricted ω-rule" to give a proof of Kreisel's result that $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0}$ is equivalent to IA + "the uniform reflection principle for IA".

In order to state the uniform reflexion principle we need some further notation. Let *num* be the term (of a definitional extension of) IA which represents the primitive recursive function which maps a natural number n to the Gödel number of the numeral **n**. Then given a formula A whose free variables are included in $\{x\}$ let $\overline{A}$ be the term (of a definitional extension of) IA representing the primitive recursive function which maps a natural number n to the Gödel number of the sentence obtained by substituting the nth numeral for the variable x in the formula A. Finally, instead of writing $\overline{A}(\mathbf{num}(x))$ we shall use the more suggestive notation: $\overline{A}(\dot{\bar{x}})$. We naturally extend the notation to sequents: $\overline{\Gamma}(\dot{\bar{x}}) \to \overline{\Theta}(\dot{\bar{x}})$.

The uniform reflection principle for IA is then the schema:

$$(R_{\mathrm{IA}}^+)\ \mathrm{Prov}\big(y, \overline{\Gamma}(\dot{\bar{x}}) \to \overline{\Theta}(\dot{\bar{x}})\big),\ \Gamma \to \Theta$$

provided $\Gamma \to \Theta$ is a sequent such that the free variables of $\Gamma \to \Theta$ are contained in $\{x\}$.

The schema of transfinite induction $\mathrm{TI}_{\varepsilon_0}$ is derivable in CA from the uniform reflection principle is given in enough detail in Kreisel–Levy [10] to see that the argument is equally applicable to IA. It is for the converse that, as observed by Kreisel, that some kind of ω-rule appears to be needed. However, contrary to the remarks made by Kreisel a delicate ω-rule is not needed for our ω-rule will do.

The only extra piece of information needed (in addition to 6.9 and 6.10) to carry out the derivation of R_{IA}^+ from $\mathrm{TI}_{\varepsilon_0}$ is that sequents occurring

in a cut free derivation of ω-IA are "true". That is let T_n be the partial truth definitions such that for all formulae A of degree $\leqslant n$:

7.1. $\mathrm{IA} \vdash T_n(\overline{A}(\dot{\bar{x}})) \supset A(x)$.

For a definition of T_n (which is by induction on n) and for the proof of 7.1 see page 35 of Troelstra [14].

Finally let, for each natural number k

$$k\text{-}\mathrm{Der}_{\mathrm{PR}}^-(x) \equiv (Ey)(Eu)(\mathrm{Der}_{\mathrm{PR}}^-(x, y, u) \wedge \text{"every sentence of the sequent } y \text{ is of degree } k\text{"}).$$

7.2. Lemma. *For every k,*

$$\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash k\text{-}\mathrm{Der}_{\mathrm{PR}}(x) \wedge [x](z) \neq \mathbf{0} \supset T_k^*\big(([x](z))_0\big).$$

where T_k^* is the natural extension of T_k to sequents.

Proof. By transfinite induction on the formula $A(w)$, where

$$A(w) \equiv \forall z \forall x \Big(k\text{-}\mathrm{Der}_{\mathrm{PR}}^-(x) \wedge [x](z) \neq \mathbf{0} \wedge ([x](z))_2 \leqslant w \supset T_k^*\big(([x](z))_0\big)\Big).$$

7.3. Corollary. *For a sentence A, if the degree of $A \leqslant k$ then*

$$\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash \mathrm{Pr}_\omega^-(\ulcorner A \urcorner) \supset T_k(\ulcorner A \urcorner).$$

7.4. Theorem (Kreisel). $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0} \vdash R_{\mathrm{IA}}^{+}$.

Proof. Suppose $A(x)$ is a formula of degree $\leqslant k$. Then applying 6.9, 6.10, 7.3 and 7.1 we obtain that the following sentences are provable in $\mathrm{IA}+\mathrm{TI}_{\varepsilon_0}$:

$$\forall x\big(\mathrm{Pr}(\overline{A}(\dot{\bar{x}})) \supset \mathrm{Pr}_\omega(\overline{A}(\dot{\bar{x}}))\big),$$

$$\forall x\big(\mathrm{Pr}(\overline{A}(\dot{\bar{x}})) \supset \mathrm{Pr}_\omega^-(\overline{A}(\dot{\bar{x}}))\big),$$

$$\forall x\big(\mathrm{Pr}(\overline{A}(\dot{\bar{x}})) \supset T_k(\overline{A}(\dot{\bar{x}}))\big),$$

$$\forall x\big(\mathrm{Pr}(\overline{A}(\dot{\bar{x}})) \supset A(x)\big).$$

Added in proof. The author would like to call the reader's attention to the article: *The use of abstract language in elementary metamathematics: some pedagogic examples*, by G. Kreisel, G. E. Mints and S. G. Simpson, in Springer Lecture Notes, vol. 453, where an analysis is made of the role played by what in the present fortuitous terminology could be called "extremely restricted rules".

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A note on the Hurewicz isomorphism theorem in Borsuk's theory of shape

by

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Abstract. In shape theory, the role of the homotopy groups π_n is played by the so called fundamental groups $\underline{\pi}_n$, introduced by K. Borsuk, and the homology groups which are useful there, are of the Vietoris–Čech type. The classical Hurewicz isomorphism theorem gives a connection between the homotopy groups π_n and the singular homology groups H_n with integral coefficients. An example of a compactum X is constructed, showing that there is no exact analogue of the Hurewicz theorem in shape theory. The example is simple: X is the double suspension of the 3-adic solenoid. The compactum X is arcwise connected and it has the following properties: (i) $\underline{\pi}_q(X) \approx 0$, for $q = 1, 2, 3$, and (ii) $\underline{\pi}_4(X)$ and $\check{H}_4(X)$ are not isomorphic.

In the theory of shape of compacta K. Borsuk introduced the fundamental groups $\underline{\pi}_n$ (see [1], § 14) which are related to the usual homotopy groups π_n in a fashion similar to the way in which the Vietoris–Čech homology groups $\check{H}_n$ are related to the singular homology groups H_n. The natural question that arose then was: is there any isomorphism theorem of the Hurewicz type in shape theory? The following theorem, proved in [3] (Theorem 3.2), is one of that type.

THEOREM. *If the pointed compactum (X, x_0) is approximatively q-connected for $q = 0, 1, \ldots, n-1$ $(n \geqslant 2)$, then the limit Hurewicz homomorphism $\underline{\varphi}$: $\underline{\pi}_n(X, x_0) \to \check{H}_n(X, x_0)$ is an isomorphism.*

The coefficient group for all homology groups considered in this note is the group of integers.

One may ask if the assumption of the approximative q-connectedness of (X, x_0) for $q = 0, 1, \ldots, n-1$ in the above theorem can be replaced by the weaker assumption $\underline{\pi}_q(X, x_0) \approx 0$ for $q = 0, 1, \ldots, n-1$, which would make the theorem completely analogous to the classical Hurewicz theorem. Obviously, the independence of $\underline{\pi}_q$ from the choice of the base point must be assured by an appropriate assumption. For some special classes of compacta, e.g. for movable pointed compacta, the answer is affirmative (see [3], Corollary 3.7), but as we shall prove, in general it is not the case. The aim of this note is to describe an arcwise connected pointed compactum (X, x_0) with the following properties: