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# Some continuous separation axioms

by

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**Abstract.** Let $\mathcal{F}X$ denote the space of closed subsets of $X$ with the Vietoris topology. A function $\varphi\colon X\times\mathcal{F}X\to[0,1]$ is a perfect normality operator (abbreviated PN-operator) if, for each $H\in\mathcal{F}X$, $H=\{x\in X\colon \varphi(x,H)=0\}$. $X$ is continuously perfectly normal if $X$ admits a continuous PN-operator. Notions of continuously normal and continuous complete regularity are defined in a similar fashion. It is shown that:

1. $X$ is metrizable $\Rightarrow X$ is continuously perfectly normal $\Rightarrow X$ is continuously normal $\Rightarrow X$ is continuously completely regular.

2. Every continuously perfectly normal space is a collectionwise normal Fréchet space.

3. The product of $X$ with the irrationals is continuously completely regular iff $X$ is continuously perfectly normal.

4. Every locally compact continuously completely regular space is first countable.

5. $X$ is metrizable if and only if $X$ admits a continuous PN-operator, $\varphi$, such that if $K$ is a finite subset of $X$ and if $x\in K$, then $\varphi(y,\{x\})\geqslant\varphi(y,K)$ for every $y\in X$.

6. Every $w\Delta$ continuously perfectly normal space is metrizable.

G. Gruenhage recently showed the author an example of a continuously perfectly normal, stratifiable, first countable space that is not metrizable. It is not known if every continuously perfectly normal space is metrizable.

In [14], the author shows that the $T_1$-space $X$ is metrizable if and only if there is a continuous function $\alpha$ from $\mathcal{F}X$, the space of closed subsets of $X$ with the Vietoris topology [1] into $CX$, the space of continuous, non-negative, realvalued functions defined on $X$ with the compact-open topology, such that

(a) if $H\in\mathcal{F}X$, then $H=\{x \mid \alpha(H)(x)=0\}$ and

(b) if $K$ is a finite subset of $X$ and if $x\in K$, then

$$\big(\alpha(\{x\})\big)(y)\geqslant\big(\alpha(K)\big)(y)\quad\text{for all } y\in X\,.$$

The author's attempts to decide if (b) of this theorem could be removed led to the notions of continuous perfect normality, continuous normality,

[1] If $X$ is a space and $U$ is a finite collection of subsets of $X$, then $RU$ will denote the set $\{F\in\mathcal{F}X \mid F\subset\bigcup U$ and $F$ intersects each member of $U\}$. Then the collection $\{RU \mid U$ is a finite collection of open subsets of $X\}$ forms a basis for a topology on $X$. The topology so induced is often called the Vietoris topology, the finite topology, or the exponential topology. Good studies of the Vietoris topology can be found in [7] and in [8].

and continuous complete regularity. Any continuously perfectly normal space $X$ admits a continuous function $\alpha\colon \mathcal{F}X \to CX$ satisfying condition (a) mentioned above. Recently, Gruenhage [3] has displayed an example of a continuously perfectly normal space that is not metrizable. In general, we have that $X$ is metrizable $\Rightarrow X$ is continuously perfectly normal $\Rightarrow X$ is continuously normal $\Rightarrow X$ is continuously completely regular. In Section 5, we give an example of a continuously completely regular space that is not continuously normal. The author does not know of a continuously normal space that is not continuously perfectly normal. In Section 2, the general properties of continuously perfectly normal spaces are investigated; in Section 3, continuously perfectly normal spaces and continuously completely regular spaces are studied and Section 4 is devoted to metrization theorems.

**1. Definitions and conventions.** All of our spaces are at least $T_1$. If $X$ is a space, then $\mathcal{F}X$ will denote the space of closed subsets of $X$ with the Vietoris topology (see footnote (1)), $\mathcal{M}X$ will denote the space

$$\{(H, K) \in \mathcal{F}X \times \mathcal{F}X\colon H \cap K = \emptyset\},$$

and $\mathcal{D}X$ will be the space $\{(x, K) \in X \times \mathcal{F}X\colon x \notin K\}$.

1.1. Definition. A function $\varphi\colon X \times \mathcal{F}X \to [0, 1]$ is called a *perfect normality operator* (abbreviated by PN-operator) if for each $H \in \mathcal{F}X$ it is true that $H = \{x\colon \varphi(x, H) = 0\}$. A space is said to be *continuously perfectly normal* (CPN) if $X$ admits a continuous PN-operator.

Clearly, a space $X$ is perfectly normal provided that it admits a PN-operator that is continuous in the first variable.

1.2. Definition. A function $\varphi\colon X \times \mathcal{M}X \to [0, 1]$ is a *normality operator* (N-operator) if it is true that if $(H, K) \in \mathcal{M}X$, then

$$H \subset \{x \in X\colon \varphi(x, (H, K)) = 0\} \quad \text{and} \quad K \subset \{x \in X\colon \varphi(x, (H, K)) = 1\}.$$

$X$ is said to be *continuously normal* (CN) if $X$ admits a continuous N-operator.

1.3. Definition. A function $\varphi\colon X \times \mathcal{D}X \to [0, 1]$ is a *complete regularity operator* (CR-operator) if for each $(x, H) \in \mathcal{D}X$, $\varphi(x, (x, H)) = 0$ and $H \subset \{y \in X\colon \varphi(y, (x, H)) = 1\}$. A space that admits a continuous CR-operator will be said to be *continuously completely regular* (CCR).

Clearly, a space that admits an N-operator (CR-operator) which is continuous in the first variable is normal (completely regular). Also, it is clear that a CN-space is a CCR-space.

**2. Some fundamental properties of CPN-spaces.**

2.1. Theorem. *Every metrizable space is a CPN-space and every CPN-space is CN.*



Proof. Let $d$ be a metric for the space $X$ such that $d(x, y) < 1$ for all $(x, y) \in X \times X$. Let $\varphi\colon X \times \mathcal{F}X \to [0, 1]$ be defined by $\varphi(x, H) = \mathrm{glb}\{d(x, y)\colon y \in H\}$. To see that $\varphi$ is continuous, let $\varepsilon > 0$, $x \in X$, and $H \in \mathcal{F}X$. Let $U = \{y\colon \varphi(y, H) < \frac{1}{4}\varepsilon\}$ and let $x'$ be a point of $H$ such that

$$|\varphi(x, H) - d(x, x')| < \tfrac{1}{4}\varepsilon.$$

Let $V = \{y\colon d(y, x') < \frac{1}{4}\varepsilon\}$ and let $W = \{y\colon d(y, x) < \frac{1}{4}\varepsilon\}$. Clearly, $x \in W$ and $H \in R\{U, V\}$ (see footnote (1)). Suppose that $y \in W$ and $K \in R\{U, V\}$. Then

$$\begin{aligned}\varphi(y, K) = \mathrm{glb}\{d(y, z)\colon z \in K\} &\leqslant \mathrm{glb}\{d(y, z)\colon z \in H\} + \tfrac{1}{4}\varepsilon\\ &\leqslant d(x, y) + \mathrm{glb}\{d(x, z)\colon z \in H\} + \tfrac{1}{4}\varepsilon\\ &\leqslant \varphi(x, H) + \tfrac{1}{2}\varepsilon.\end{aligned}$$

Now, let $k$ be a point of $K \cap V$. Then

$$\begin{aligned}\varphi(y, K) &\geqslant d(k, y) - \tfrac{1}{4}\varepsilon \geqslant d(k, x) - \tfrac{1}{2}\varepsilon \geqslant d(x', x) + d(x', k) - \tfrac{1}{2}\varepsilon\\ &\geqslant d(x', k) - \tfrac{3}{4}\varepsilon \geqslant \varphi(x, H) - \varepsilon.\end{aligned}$$

Thus, $|\varphi(y, K) - \varphi(x, H)| < \varepsilon$ and $\varphi$ is continuous.

Suppose now that $X$ is a CPN-space. Let $\varphi$ be a continuous PN-operator for $X$. Let $\varphi'$ be the function defined on $X \times \mathcal{M}X$ defined by $\varphi'(x, (H, K)) = \varphi(x, H)/(\varphi(x, H) + \varphi(x, K))$. Clearly, $\varphi'$ is a continuous $N$-operator.

Greunhage's example [3] is not first countable; however, we have the following Theorem:

2.2. Theorem. *Every* CPN-*space is a Fréchet space* (2).

Proof. Suppose $x_0$ is a limit point of the set $H$. Let $\varphi$ be a continuous PN-operator for $X$. Using induction, we will choose, for each $n > 0$, a point $x_n$ and a collection $\{U(i, n)|\ i = 0, \ldots, n\}$ of open sets such that

(1) for each $i \geqslant 1$, $x_i \in H$,

(2) $U(i, n)$ contains $x_i$,

(3) if $\{k_1, \ldots, k_j\} \subset \{1, \ldots, n\}$ and if $K \in R\{U(0, n), U(k_1, n), \ldots$ $\ldots, U(k_j, n)\}$, then $\varphi(x_0, K) < 1/2^n$),

(4) $\mathrm{cl}\, U(i, n+1) \subset U(i, n)$ for all $i \leqslant n$,

(5) $\mathrm{cl}\, U(n+1, n+1) \subset U(0, n)$ and

(6) $U(i, j) \cap U(i, k) \neq \emptyset$ implies $i = k$.

Before demonstrating the inductive construction let us show that if we have a sequence $x_1, x_2, \ldots$ and a sequence of collections $\{U(i, n)|\ i = 0, \ldots, n\}_n$ of open sets satisfying (1)-(6), then $x_1, x_2, \ldots$

(2) $X$ is a Fréchet space if limit points are determined by convergent sequences.

converges to $x_0$. To this end, suppose that $x_1, x_2, \ldots$ does not converge to $x_0$. Then there are an open set $U$ containing $x_0$ and an infinite increasing sequence $t(1), t(2), \ldots$ of integers such that, for each $i$, $x_{t(i)}$ is not in $U$. Let $K = \mathrm{cl}\{x_{t(1)}, x_{t(2)}, \ldots\}$. Since $x_0$ is not in $K$, there is an integer $N$ such that $\varphi(x_0, K) > 1/N$. By (1), (4), and (5) we have that $\mathrm{cl}\{x_{t(N+1)}, x_{(N+2)}, \ldots$ $\ldots, U(t(N), t(N))$; thus, $K \in R\{U(0, t(N)), U(t(1), t(N)), \ldots, U(t(N), t(N))\}$. And so, by (3), $\varphi(x_0, K) < 1/2^n$, which is a contradiction from which the Theorem will follow.

Now, to demonstrate the inductive construction, note that we already have $x_0$. Let $U(0,0)$ be an open set containing $x_0$ such that if $K \in \mathcal{F}X$ and $K \in R\{U(0,0)\}$, then $\varphi(x_0, K) < 1$. Suppose that we have $\{x_0, \ldots, x_n\}$ and $\{U(i,j)|\ i = 0, \ldots, j\}$ for all $j < n$. For each subset $K$ of $\{x_1, x_2, \ldots, x_n\}$, let $\mathcal{U}K$ denote a collection of mutually exclusive open sets covering $K \cup \{x_0\}$ such that if $H \in R(\mathcal{U}K)$, then $\varphi(x_0, H) < 1/2^n$ and such that each member of $\mathcal{U}K$ contains exactly one point of $K \cup \{x_0\}$. For each $0 \leqslant j \leqslant n$, let $\mathcal{K}(j) = \{K \subset \{x_1, \ldots, x_n\}|\ x_j \in K\}$ and for each $K \in \mathcal{K}(j)$, let $U(K, j)$ denote that element of $\mathcal{U}K$ that contains $x_j$. For each $0 \leqslant j \leqslant n$, let

$$V(j, n+1) = \Big(\bigcap_{i=0}^{n} U(j,i)\Big) \cap \Big(\bigcap_{K \in \mathcal{K}(j)} U(K, j)\Big).$$

For each $0 > j > n$, let $U(j, n+1)$ be an open set containing $x_j$ such that $\mathrm{cl}\, U(j, n+1) \subset V(j, n+1)$. Finally, let $x_{n+1}$ be a point of $\big(V(0, n+1) - \{x_0\}\big) \cap H$ and let $U(0, n+1)$ and $U(n+1, n+1)$ be mutually exclusive open sets containing $x_0$ and $x_{n+1}$ respectively such that

$$\mathrm{cl}\big(U(0, n+1) \cup U(n+1, n+1)\big) \subset U(0, n).$$

2.3. Theorem. *Every* CPN*-space is hereditarily* CPN.

Proof. Let $\varphi$ be a continuous PN-operator on $X$ and let $Y$ denote a subspace of $X$. For each $H \in \mathcal{F}Y$ and each $x \in Y$, let

$$\varphi'(x, H) = \varphi(x, \mathrm{cl}_X H).$$

2.4. Theorem. *Every* CPN*-space is collectionwise normal.*

A couple of lemmas will facilitate the proof of 2.4.

2.5. Lemma. *The space $X$ is collectionwise normal if and only if it is true that if $\{H_a\colon a \in A\}$ is a discrete collection of closed sets, then there is a function $\gamma$ taking $A \times \mathcal{N}$* [3] *into the collection of open subsets of $X$ such that* (1) *for each $a \in A$, $\{\gamma(a, n)\colon n \in \mathcal{N}\}$ covers $H_a$ and* (2) *for each $a \in A$, and for each $n \in \mathcal{N}$, $H_a \cap \mathrm{cl}\big(\bigcup\{\gamma(b, n)\colon b \in A - a\}\big) = \emptyset$.*

[3] $\mathcal{N}$ denotes the set of positive integers.



Proof. For each $a \in A$, let

$$D(a, n) = \gamma(a, n) - \mathrm{cl} \bigcup \{\gamma(b, j)\colon b \in A - a, j \leqslant n\}$$

and

$$U_a = \bigcup_{n=1}^{\infty} D(a, n).$$

To see that $H_a \subset U_a$, let $x \in H_a$. Then there is a first integer $n$ such that $x \in \gamma(a, n)$. But according to (2) of the hypothesis, $x$ is not in $\mathrm{cl} \bigcup \{\gamma(b, j)\colon b \neq a; j \leqslant n\}$; and so, $x \in D(a, n)$. It is clear that $\{U_a\colon a \in A\}$ is a collection of mutually exclusive open sets.

2.6. Lemma. *The perfectly normal space $X$ is collectionwise normal if and only if it is true that if $\{H_a\colon a \in A\}$ is a discrete collection of closed sets, then there is a function $\beta$ from $A \times \mathcal{N}$ into the collection of subsets of $X$ such that* (1) *for each $a \in A$, $\{\beta(a, n)\colon n \in \mathcal{N}\}$ covers $H_a$ and* (2) *for each $n \in \mathcal{N}$, $\{\beta(a, n)\colon a \in A\}$ is a collection of mutually exclusive open sets.*

Proof. Suppose that $\beta$ satisfies (1) and (2) for the collection $\{H_a\colon a \in A\}$ of closed sets. For each $n \in \mathcal{N}$ and for each $a \in A$, let $\beta'(a, n) = \beta(a, n) - \bigcup_{b \neq a} H_a$. Since $X$ is perfectly normal and since $\bigcup\{\beta'(a, n)\colon a \in A\}$ is an open set, there is a sequence $D(1, n), D(2, n), \ldots$ of open sets such that

$$\bigcup_{j \in \mathcal{N}} D(j, n) = \bigcup_{j \in \mathcal{N}} \mathrm{cl} D(j, n) = \bigcup \{\beta'(a, n)\colon a \in A\}.$$

For each $(a, j, n) \in A \times \mathcal{N} \times \mathcal{N}$, let $\gamma'\big(a, (j, n)\big) = \beta'(a, n) \cap D(j, n)$. Let $g$ be a one-to-one function from $\mathcal{N}$ onto $\mathcal{N} \times \mathcal{N}$. For each integer $j$, let $\gamma(a, j) = \gamma'\big(a, g(j)\big)$. We need only show that $\gamma$ satisfies condition (2) of Lemma 2.5. To this end, suppose that there are an $a \in A$, a $j \in \mathcal{N}$, and an $x \in H_a$ such that $x \in \mathrm{cl} \bigcup \{\gamma(b, j)\colon b \in A - a\}$. Suppose that $g(j) = (i, n)$. Since the members of $\{\beta'(c, n)\colon c \in A\}$ are mutually exclusive, $x$ is not in $\bigcup \{\beta'(c, n)\colon c \in A\}$. But

$$\bigcup \{\gamma'\big(b, (i, n)\big)\colon b \in A - a\} \subset D(i, n) \subset \mathrm{cl} D(i, n) \subset \bigcup \{\beta'(c, n)\colon c \in A\};$$

and so, $x$ cannot be a point of $\mathrm{cl} \bigcup \{\gamma(b, j)\colon b \in A - a\}$ which is a contradiction from which the lemma follows.

2.7. Proof of Theorem 2.4. Let $\varphi$ be a continuous PN-operator for $X$. Let $\{H_a|\ a \in A\}$ be a discrete collection of closed sets in $X$.

We will construct a function $\beta$ taking $A \times \mathcal{N}$ into the collection of open subsets of $X$ satisfying the hypothesis of Lemma 2.6. To this end, we will assume that the indexing set $A$ is an initial segment of ordinal numbers. We will take $H_0 = \emptyset$ and, for each $n$, define $\varphi^{-1}(H_0, M) = \emptyset$ for each subset $M$ of the interval $[0, 1]$. For each $a$ in $A$, let:

(A) $\mathcal{H}_a = \bigcup_{b \leqslant a} H_b$,

(B) $D(a, n) = \{x \in X|\ \varphi(x, \mathcal{K}_a) < 1/n\} - \mathrm{cl}\big(\bigcup_{b<a} \{x \in X|\ \varphi(x, \mathcal{K}_b) < 1/n\}\big)$,

(C) $U(a)$ be an open set such that $H_a \subset U(a) \subset \mathrm{cl}\, U(a) \subset X - \bigcup_{b \neq a} H_b$,

and

(D) $\beta(a, n) = D(a, n) \cap U(a)$.

Clearly, for each $n$, the collection $\{\beta(a, n)|\ a \in A\}$ is a collection of mutually exclusive open sets. We need only show that $\{\beta(a, n)|\ n \in \mathcal{N}\}$ covers $H_a$. Suppose otherwise; then there is a point $x_0$ in $H_a$ such that, for each $n$, $x_0$ is a limit point of $\bigcup_{b<a} \{x \in X|\ \varphi(x, \mathcal{K}_b) < 1/n\}$. For each $n$, there must be an ordinal $a(n) < a$ such that $\varphi(x_0, \mathcal{K}_{a(n)}) < 1/n$. We may assume that if $n > m$, then $a(n) > a(m)$. Let $\mathcal{K} = \bigcup_{i \in \mathcal{N}} \mathcal{K}_{a(i)}$. Since $x$ is not in $\mathcal{K}$, there is an integer $M > 0$ such that $\varphi(x_0, \mathcal{K}) > 1/M$. Since $\varphi$ is continuous, there is an $M'$ such that if $n > M'$, then $|\varphi(x_0, \mathcal{K}_{a(n)})| > 1/M$. But this is a contradiction since $|\varphi(x_0, H_{a(M+M')})| < 1/(M+M') < 1/M$. Thus, $\beta$ satisfies the hypothesis of Lemma 2.6.

Note. It is not known if every CPN-space is paracompact. Indeed, it is apparently not known if every perfectly normal and collectionwise normal space is paracompact.

2.8. THEOREM. *If $\{X_i|\ i \in \mathcal{N}\}$ is a collection of spaces such that, for each $n$, the product $\prod_{i=1}^{n} X_i$ is a CPN-space, then $\prod_{i \in \mathcal{N}} X_i$ is a CPN-space.*

Proof. For each $n$, let $P_n$ denote the projection of $\prod_{i \in \mathcal{N}} X_i$ onto $\prod_{i=1}^{n} X_i$ and let $\varphi_n$ be a continuous PN-operator for $\prod_{i=1}^{n} X_i$. Let $\varphi$ be the function taking $(\prod_{i \in \mathcal{N}} X_i) \times \mathcal{F}(\prod_{i \in \mathcal{N}} X_i)$ into $[0, 1]$ defined by

$$\varphi(x, H) = \sum_{i \in \mathcal{N}} \Big[(1/2^i)\big(\varphi(P_i(x), H_i)\big)\Big], \quad \text{where} \quad H_i = \mathrm{cl}\, P_i(H).$$

It is easily verified that $\varphi$ is a continuous PN-operator for $\prod_{i \in \mathcal{N}} X_i$.

2.9. THEOREM. *If $X$ is a CPN-space then $X \times \Sigma$ is a CPN-space, where $\Sigma$ denotes the space of irrationals.*

Proof. For each $i$, let $\mathcal{N}_i$ denote a copy of $\mathcal{N}$. Then $\Sigma$ is homeomorphic to $\prod_{i=1}^{\infty} \mathcal{N}_i$. It is easy to see that, for each $n$, $X \times \prod_{i=1}^{n} \mathcal{N}_i$ is a CPN-space. Thus, 2.9 follows from 2.8.

In general, the continuous separation axioms are not preserved under mappings; indeed, we will see later that the closed continuous image of a metric space need not be even a CCR-space. However, we do have the following theorem:



2.10. THEOREM. *If $f\colon X \to Y$ is onto, continuous, open, and perfect* (4), *then $Y$ is a CPN-space (CN-space) provided that $X$ is a CPN-space (CN-space).*

Proof. We will show that if $X$ is continuously perfectly normal, then so is $Y$. The argument for continuous normality is much the same as we use here.

Let $\varphi$ be a continuous PN-operator for $X$. For each $(y, H) \in Y \times \mathcal{F}Y$, define $\varphi'(y, H) = \mathrm{glb}\{\varphi(x, f^{-1}(H))|\ x \in f^{-1}(y)\}$. Since $\varphi$ is bounded, $\varphi'(y, H)$ is defined; indeed, since $f^{-1}(y)$ is compact, $\varphi'(y, H) = 0$ if and only if $y \in H$. To see that $\varphi'$ is continuous, let $(y_0, H_0)$ be a point of $Y \times \mathcal{F}Y$ and let $\varepsilon > 0$. Let $x_0$ be a point of $f^{-1}(y_0)$ such that $\varphi'(y_0, H_0) = \varphi(x_0, f^{-1}(H_0))$ ($x_0$ exists since $f^{-1}(y_0)$ is compact). There are a basic open set $U = R\{U_1, \ldots, U_n\}$ in $\mathcal{F}X$ containing $f^{-1}(H_0)$ and an open set $V$ in $X$ containing $x_0$ such that if $K \in U$ and $x \in V$, then $|\varphi(x, K) - \varphi(x_0, f^{-1}(H_0))| < \varepsilon$. Let $W = Y - \big(f(X - \bigcup_{i=1}^{n} U_i)\big)$ and let $W_1 = R\{f(U_1) \cap W, \ldots, f(U_n) \cap W\}$. Then $\mathcal{W}_1 = f(V) \times W_1$ is an open set in $X \times \mathcal{F}X$ containing $(y_0, H_0)$. Let $(y, H) \in \mathcal{W}_1$. Then $f^{-1}(H) \in U$ and there is a point $x$ of $f^{-1}(y)$ in $V$. It follows that $\varphi(x, f^{-1}(H)) < \varphi(x_0, f^{-1}(H_0)) + \varepsilon$; and so, $\varphi'(y, H) < \varphi(H_0, x_0) + \varepsilon$.

Now, for each point $x$ of $f^{-1}(y_0)$, let $U_x$ be an open set in $\mathcal{F}X$ containing $f^{-1}(H_0)$ and let $V_x$ be an open set in $X$ containing $x$ such that if $H \in U_x$ and $x' \in V_x$, then $|\varphi(x', H) - \varphi(x, f^{-1}(H_0))| < \varepsilon$. Since $f^{-1}(y_0)$ is compact, there is a finite subset $\{x(1), \ldots, x(n)\}$ of $f^{-1}(y_0)$ such that

$$\{V_{x(1)}, V_{x(2)}, \ldots, V_{x(n)}\}$$

covers $f^{-1}(y_0)$. Let $\mathcal{U} = R\{U_1, U_2, \ldots, U_k\}$ be a basic open set in $\mathcal{F}X$ such that $f^{-1}(H_0) \in \mathcal{U} \subset \bigcap_{i=1}^{n} U_{x(i)}$. Let $W' = Y - f(X - \bigcup_{i=1}^{k} U_i)$, $W_2 = R\{f(U_1) \cap W', \ldots, f(U_k) \cap W'\}$, $V' = Y - f(X - \bigcup_{i=1}^{n} V_{x(i)})$ and $\mathcal{W}_2 = V' \times W_2$.

It is easily verfied that $(y_0, H_0) \in \mathcal{W}_2$. Let $(y, H) \in \mathcal{W}_2$. Then $f^{-1}(H) \in \mathcal{U}$ and $f^{-1}(y) \in U\{V_{x(1)}, \ldots, V_{x(n)}\}$. Let $x$ be a point of $f^{-1}(y)$ such that $\varphi'(y, H) = \varphi(x, f^{-1}(H))$. Then $x$ is in some member of $\{V_{x(1)}, \ldots, V_{x(n)}\}$, say $V_{x(j)}$. Then $\varphi(x, f^{-1}(H)) > \varphi(x(j), f^{-1}(H)) - \varepsilon \geqslant \varphi(x_0, f^{-1}(H_0))$. Thus, if $(y, H) \in \mathcal{W}_1 \cap \mathcal{W}_2$, then $|\varphi'(y, H) - \varphi'(y_0, H_0)| < \varepsilon$; and so, $\varphi'$ is continuous.

2.11. THEOREM. *If $X$ is a CPN-space, then the diagonal of $X$ in $X \times X$ is a zero-set (and hence a regular $G_\delta$-set).*

Proof. Let $f\colon X \times X \to [0, 1]$ be defined by $f(x, y) = \varphi(x, \{y\})$. Clearly, the diagonal of $X$ is the set $\{(x, y)|\ f(x, y) = 0\}$.

(4) The continuous function $f\colon X \to Y$ is *perfect* if it is closed and point inverses are compact.

**3. Continously normal spaces and continuously completely regular spaces.**

3.1. THEOREM. *If $X\times Y$ is a CCR-space, then either no countable subset of $X$ has a limit point or $Y$ is a CPN-space.*

Proof. Let $C=\{x_1, x_2, \ldots\}$ be a countable subset of $X$ with a limit point, say $x_0$, in $X-C$. For each $n$, let $C_n=\{x_1, \ldots, x_n\}$. Let $\varphi$ be a continuous CR-operator for $X\times Y$. For each $n$, let $\varphi_n$ be the function from $Y\times\mathcal{F}Y$ into $[0,1]$ defined by

$$\varphi_n(y, H)=1-\varphi\big((x_n, y), (x_0, y), C_n\times H\big).$$

Let $\varphi_0$ be the function from $Y\times\mathcal{F}Y$ into $[0,1]$ defined by

$$\varphi_0(y, H)=\sum_{i\in\mathcal{N}}(1/2^i)\varphi_i(y, H).$$

To see that $\varphi_0$ is continuous, we only need show that each $\varphi_i$ is continuous. To this end, let $\varepsilon>0$. Since $\varphi$ is continuous, there are open sets $U$ and $V$ in $Y$ containing $y$ and a basic open set $R\{U_1, \ldots, U_n\}$ in $\mathcal{F}(X\times Y)$ containing $C_i\times H$ such that if $y'\in U\cap V$ and $K\in R\{U_1, \ldots, U_n\}$, then

$$\Big|\varphi\big((x_i, y'), (x_0, y'), K\big)-\varphi\big((x_i, y), \big((x_0, y), C_i\times H\big)\big)\Big|<\varepsilon.$$

For $j<n$ and for each $k\leqslant i$, let $U(k, j)=\{w\in Y|\ (x_k, w)\in U_j\}$. Then if the closed subset $K$ of $Y$ is in $\bigcap_{k\leqslant 1}R\{U(k, j)|\ j\leqslant n\}$ and if $y'\in U\cap V$, then

$$\begin{aligned}&|\varphi_i(y, H)-\varphi_i(y', K)|\\ &\quad=\big|\varphi\big((x_i, y), (x_0, y), C_i\times H\big)-\varphi\big((x_i, y'), (x_0, y'), C_i\times K\big)\big|<\varepsilon.\end{aligned}$$

Thus, each $\varphi_i$ is continuous.

It remains to show that if $H\in\mathcal{F}Y$, then $H=\{y\in Y|\ \varphi_0(y, H)=0\}$. Clearly, $H\subset\{y\in Y|\ \varphi_0(y, H)=0\}$. Suppose that $y$ is not in $H$. Let $H^\infty=\mathrm{cl}\big(\bigcup_{i\in N}(C_i\times H)\big)$. Since $y$ is not in $H$, $(x_0, y)$ is not in $H^\infty$. Since $\varphi$ is continuous, there are open sets $U$ and $V$ in $X$ and $Y$ respectively such that $(x_0, y)\in U\times V$ and a basic open set $R\{U_1, \ldots, U_n\}$ in $\mathcal{F}(X\times Y)$ containing $H^\infty$ such that if $(x', y')\in U\times V$ and $H'\in R\{U_1, \ldots, U_n\}$, then $\varphi\big((x', y'), (x_0, y), H'\big)<\frac{1}{2}$. Since $(C_1\times H), (C_2\times H), \ldots$ converges to $H^\infty$ in $\mathcal{F}X$ and since $x_0$ is a limit point of $C$, we may choose an integer $n$ such that $x_n\in U$ and $C_n\times H\in R\{U_1, \ldots, U_n\}$. Thus,

$$\begin{aligned}\varphi_0(y, H)&\geqslant 2^{-n}\varphi_n(y, H_n)=2^{-n}\Big(1-\varphi\big((x_n, y), (x_0, y), C_n\times H\big)\Big)\\ &>(2^{-n-1})>0.\end{aligned}$$

From Theorems 3.1 and 2.9, we have the following result:



3.2. THEOREM. *Let $\Sigma$ denote the space of irrational numbers. The following conditions for a space $X$ are equivalent:*

(1) *$X$ is a CPN-space.*

(2) *$X\times\Sigma$ is a CPN-space.*

(3) *$X\times\Sigma$ is a CN-space.*

(4) *$X\times\Sigma$ is a CCR-space.*

3.3. THEOREM. *Suppose that $\{X_i|\ i\in\mathcal{N}\}$ is a countable collection of nondegenerate spaces. The following conditions for $\{X_i|\ i\in\mathcal{N}\}$ are equivalent:*

(1) *For each $n$, $\prod_{i=1}^{n}X_i$ is a CPN-space.*

(2) *$\prod_{i\in\mathcal{N}}X_i$ is a CPN-space.*

(3) *$\prod_{i\in\mathcal{N}}X_i$ is a CN-space.*

(4) *$\prod_{i\in\mathcal{N}}X_i$ is a CCR-space.*

Proof. According to Theorem 2.8, we need only show that (4) implies (1). To this end, for each $i$, let $Y_i$ denote a subset of $X_i$ containing exactly two points. Then for each $n$, $\prod_{i=1}^{\infty}Y_i$ is a copy of the Cantor set. Since $(\prod_{i=1}^{n}X_i)\times(\prod_{i=n+1}^{\infty}Y_i)$ is a closed subspace of $\prod_{i\in\mathcal{N}}X_i$, $(\prod_{i=1}^{n}X_i)\times(\prod_{i=n+1}^{\infty}Y_i)$, it is continuously completely regular. Thus, according to Theorem 3.1, $\prod_{=1}^{n}X_i$ is a CPN-space.

Note. As a corollary to 3.1, it follows that the product of an uncountable collection of non-degenerate spaces cannot be continuously completely regular.

3.4. THEOREM. *A separable continuously completely regular space is perfectly normal.*

Proof. Let $\varphi$ be a continuous CR-operator for the separable space $X$. For each closed subset $H$ of $X$, let $N(H)$ denote a countable dense subset of $X-H$. For each $x\in N(H)$, let $D(x, H)=\{y|\ \varphi(y, x, H)>\frac{1}{2}\}$. According to Corollary 2 of [13], it is sufficient to show that $H=\bigcap_{x\in N(H)}D(x, H)$ $=\bigcap_{x\in N(H)}\mathrm{cl}\,D(x, H)$.

Clearly, $H\subset\bigcap_{x\in N(H)}D(x, H)$. Let $y\in\bigcap_{x\in N(H)}\mathrm{cl}\,D(x, H)$ and suppose $y$ is not in $H$. Then there is an open set $U$ containing $y$ such that if $w\in U$ and $z\in U$, then $\varphi(w, z, H)<\frac{1}{2}$. Let $x$ be a point of $N(H)$ in $U$. Then

$\{w \mid \varphi(w, x, H) < \frac{1}{2}\}$ is an open set containing $y$ that does not intersect $D(x, H)$ which contradicts the assumption that $y \in \operatorname{cl} D(x, H)$.

3.5. LEMMA. *If $f$ is a continuous one-to-one function from the space $X$ into the metric space $M$, then the diagonal of $X$ is a zero-set in $X \times X$.*

Proof. Let $f \times f$ denote the function taking $X \times X$ into $M \times M$ defined by $(f \times f)(x, y) = (f(x), f(y))$. Since $f$ is one-to-one and continuous, so is $f \times f$. Since the diagonal $\Delta_M$ of $M$ in $M \times M$ is a zero-set, there is a function $g$: $M \times M \to [0, 1]$ such that $\Delta_M = \{x \in M \times M \mid g(x) = 0\}$. Then $\Delta_X$, the diagonal of $X$ in $X \times X$, is precisely the set $\{x \in X \times X \mid g((f \times f)(x)) = 0\}$.

3.6. THEOREM. *If $X$ is a separable CCR-space, then diagonal of $X$ is a zero set in $X \times X$.*

Proof. Let $N$ denote a countable dense subset of $X$ and let $\varphi$ be a continuous CR-operator for $X$. For each $(x, y) \in (N \times N) - \Delta_N$, let $f_{(x,y)}$: $X \to [0, 1]$ be defined by $f_{(x,y)}(z) = \varphi(z, x, \{y\})$. Then

$$\{f_{(x,y)} \mid (x, y) \in (N \times N) - \Delta_N\}$$

is a countable collection of continuous functions from $X$ into $[0, 1]$; and so $\{f_{(x,y)} \mid (x, y) \in N \times N\}$ induces a continuous function $F$ from $X$ into the Hilbert cube.

According to the Embedding Lemma [5, p. 116], if

$$\{f_{(x,y)} \mid (x, y) \in N \times N\}$$

separates points, then $F$ is one-to-one and our theorem will then follow from Lemma 3.5. To this end, let $(w, z) \in X \times X$ with $w \neq z$. There are mutually exclusive open sets $U$ and $V$ containing $w$ and $z$ respectively such that if $x \in U$ and $y \in V$, then $\varphi(w, x, \{y\}) < \frac{1}{2}$ and $\varphi(z, x, \{y\}) > \frac{1}{2}$. Then choose an $x \in U \cap N$ and a $y \in V \cap N$. Then $f_{(x,y)}(w) \neq f_{(x,y)}(z)$.

3.7. LEMMA. *The one-point compactification of an uncountable discrete space is not a CCR-space.*

Proof. Let $M$ denote an uncountable discrete space and let $Y = M \cup \{y_0\}$ be the one-point compactification of $M$ and suppose that $\varphi$ is a continuous CR-operator for $Y$. Let $m_1$ be a point of $M$. There is an open set $U_1$ containing $y_0$ such that if $x$ and $y$ are points of $U_1$, then $\varphi(x, y, \{m_1\}) < \frac{1}{2}$. Having $m_1, m_2, \ldots, m_n$ and $U_1, \ldots, U_{n-1}$, let $U_n$ be an open set containing $y_0$ such that (1) $U_n \subset \bigcap_{i=1}^{n-1} U_i$ and (2) if $x$ and $y$ are points of $U_n$, then $\varphi(x, y, \{m_n\}) < 1/2^n$. Since, for each $n$, $Y - U_n$ is finite, there is a point $x_0$ of $\bigcap_{i \in \mathcal{N}} U_i$ distinct from $y_0$. There is an open set $V$ containing $y_0$ such that if $x$ and $y$ are in $V$, then $\varphi(x, x_0, \{y\}) > \frac{1}{2}$. Let $n$ be an integer greater than 1 such that $m_n \in V$ and let $x_1$ be a point of $V \cap (\bigcap_{i \in \mathcal{N}} U_i)$. Then $\varphi(x_1, x_0, \{m_n\}) > \frac{1}{2}$ but at the same time $\varphi(x_1, x_0, \{m_n\}) < 1/2^n$ which is a contradiction from which the lemma follows.



3.8. THEOREM. *Every compact CCR-space is first countable.*

Proof. Let $\varphi$ be a continuous CR-operator for the compact space $X$, let $x \in X$, and let $\leqslant$ be a well-ordering for $X - x$. By induction, choose a subset $H$ of $X - x$ such that

(1) $\bigcup_{h \in H} \{y \in X - x \mid \varphi(y, h, \{x\}) < \frac{1}{2}\} = X - x$,

(2) if $h \in H$, then $h$ is not in $\bigcup_{h' < h} \{y \in X - x \mid \varphi(y, h', \{x\}) < \frac{1}{2}\}$.

We will first show that if $y$ is a limit point of $H$, then $y = x$. To this end, suppose otherwise; let $h_0$ denote the first element of $H$ so that $y \in \{z \mid \varphi(z, h_0, \{x\}) < \frac{1}{2}\}$. Then $y$ is a limit point of $\{h \in H \mid h < h_0\}$. There is an open set $U$ containing $y$ such that if $y' \in U$, then $\varphi(y', y, \{x\}) < \frac{1}{2}$. Thus, there is an $h < h_0$ such that $\varphi(h, y, \{x\}) < \frac{1}{2}$ which is a contradiction from which it follows that $x$ is the only limit point of $H$.

If $H$ is finite, then $x$ is an isolated point of $X$; and so, suppose $H$ is infinite. It follows from Lemma 3.7 that $H$ is countable. For each $h \in H$, let $B_h = X - \{y \mid \varphi(y, h, \{x\}) < \frac{1}{2}\}^-$. Then $\{B_h \mid h \in H\}$ must be a countable sub-basis for $x$.

Note. The author does not know if a compact CCR-space is metrizable (see 4.4 for a related theorem).

## 4. Some metrization theorems.

4.1. LEMMA. *If $\varphi$ is a continuous PN-operator for $X$ and if $\gamma$: $\mathcal{F}X \to C(X, [0, 1])$ is the function defined by $[\gamma(H)](x) = \varphi(x, H)$, then $\gamma$ is continuous. ($C(X, [0, 1])$ is endowed with the compact-open topology.)*

Proof. Let $(C, U)$ be a subbasic open set in $C(X, [0, 1])$; i.e., $(C, U) = \{f \in C(X, [0, 1]) \mid f(C) \subset U\}$. Suppose that $H \in \mathcal{F}X$ is such that $\gamma(H) \in (C, U)$. Then for each $x \in C$, there are open sets $U_x$ in $X$ and $V_x$ in $\mathcal{F}X$ containing $x$ and $H$ respectively such that if $y \in U_x$ and $K \in V_x$, then $\varphi(y, K) \in U$. Since $C$ is compact, there is a finite subset $C'$ of $C$ such that $\{U_x \mid x \in C'\}$ covers $C$. Then $V = \bigcap \{V_x \mid x \in C'\}$ is an open set in $\mathcal{F}X$ such that if $K \in V$, then $[\gamma(K)](C) \subset U$.

As an immediate consequence of 4.1, 2.1, and the main result of [13], we have the following theorem:

4.2. THEOREM. *$X$ is metrizable if and only if $X$ admits a continuous PN-operator $\varphi$ such that if $K$ is a finite subset of $X$ and if $x \in K$, then $\varphi(y, \{x\}) \geqslant \varphi(y, K)$ for every $y \in X$.*

4.3. LEMMA. *The space $X$ is metrizable if and only if $X$ admits a semi-metric $d$ satisfying the following property:*

(*) *if $x$ is not a limit point of the set $H$, then there are an open set $U$ containing $x$ and a number $k>0$ such that if $y \in U$ and $w \in H-x$, then $d(y, w)>k$.*

Proof. We will show that if $d$ satisfies property (*), then $d$ satisfies the following property:

(W) if $\{x_n\}$ and $\{y_n\}$ are sequences of points of $X$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(y_n, x) = 0$.

In [11], Wilson shows us that any space that admits a semi-metric satisfying (W) is metrizable.

Suppose, then, that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y_n) = 0$ but $\{y_n\}$ does not converge to $x$. Then there are a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ and a number $k>0$ such that for each $i$, $d(x, y_{n_i})>k$.

Since $d$ has property (*) there is an open set $U$ containing $x$ such that if $y \in U$, then $d(y, y_{n_i})>k$ for each $i$. This is impossible however since all but finitely many of the points of $\{x_{n_i}\}$ are in $U$ and

$$\lim_{n\to\infty} d(x_{n_i}, y_{n_i}) = 0 .$$

4.4. THEOREM. *The separable space $X$ is metrizable if and only if $X$ admits a continuous CR-operator $\varphi$ such that if $K$ is a finite subset of $X$, $y \in K$ and $x \in X-K$, then for all $w \in X$ it is true that $\varphi(w, x, K) \geqslant \varphi(w, x, \{y\})$; in particular, then, every separable CMCR-space* (5) *is metrizable.*

Proof. Let $\varphi$ be a continuous CR-operator satisfying the hypothesis of our theorem. Let $\gamma\colon (X\times \mathfrak{D}X)\to[0,1]$ be defined by $\gamma(x, y, K) = 1-\varphi(x, y, K)$.

Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of $X$. For each pair $(x, y)\in X\times X$, let $\mathcal{N}(x, y) = \{n \in \mathcal{N}|\ x_n$ is neither $x$ nor $y\}$. For each $n\in\mathcal{N}$, let $d_n\colon [(X\times X)-(\{x_n\}\times X\cup X\times\{x_n\})]\to[0,2]$ be defined by $d_n(x,y) = (x, y)(x, x_n, \{y\}) + (x, y)(y, x_n, \{x\})$. Let $d\colon X\times X\to[0,2]$ be defined by $d(x, y) = \sum_{n\in\mathcal{N}} (x, y)2^{-n}d_n(x, y)$.

First, we will show that if $x$ is a limit point of the set $H$ and if $\varepsilon>0$, then there is a point $y$ of $H-\{x\}$ such that $d(x, y)<\varepsilon$. Let $N$ denote an integer such that $2^{-(N-2)}<\varepsilon$. There is a neighborhood $U_0$ of $x$ such that $U_0-\{x\}$ does not intersect $\{x_1, \ldots, x_N\}$.

Case 1. $x\in\{x_1, \ldots, x_N\}$, say $x = x_k$. Since, for each $n\leqslant N$, with $n\neq k$, $\gamma(x, x_n, \{x\}) = 0$, there is a neighborhood $U_n$ of $x$ such that if $K\subset U_n$ and $y\in U_n$, then $\gamma\big(y, (x_n, K)\big)<\varepsilon/4n$. Let

$$y\in H\cap\big(\bigcap\{U_n|\ 0\leqslant n\leqslant N,\ n\neq k\}.$$

(5) See 4.6 for definition.



Then

$$d(x, y) = \sum_{n\in\mathcal{N}(x,y)} 2^{-n}d_n(x, y)<\varepsilon .$$

Case 2. $x\notin\{x_1, \ldots, x_n\}$. The argument for this case is essentially the same as the argument for Case 1.

Now, we will show that $d$ satisfies the property (*) of our lemma. It will then follow that $d$ is a semi-metric for $X$ and, by our lemma, that $X$ is metrizable. Let $(x, H)\in\mathfrak{D}X$. Then there are an open set $U$ in $X$ containing $x$ and an open set $V$ in $\mathcal{F}X$ containing $H$ such that if $w$ and $z$ are in $U$ and $K\in V$, then $\gamma(w, z, K)>\frac{1}{2}$. Let $x_N\in U-\{x\}$ and let $U' = U-\{x_N\}$. Let $k = 2^{-(N+1)}$ and let $K$ denote a finite subset of $H$ that is in $V$. Let $y\in H$ and let $w\in U'$. Then

$$d(w, y)\geqslant 2^{-N}\gamma(w, x_N, \{y\})\geqslant 2^{-N}\gamma(w, x_N, K\cup\{y\}>2^{-(N+1)} .$$

In [15], it was shown that if $X$ is locally compact, connected and locally connected, and $X$ has a regular $G_\delta$-diagonal, then $X$ is metrizable; thus, from 2.11 and 3.6 we have the following result:

4.5. THEOREM. *The locally connected, locally peripherally compact, and connected space $X$ is metrizable if and only if $X$ is a CPN-space. Indeed, if $X$ is also separable, then $X$ is metrizable if and only if $X$ is a CCR-space.*

In [11], the author shows that an $M$-space (6) $X$ is metrizable if and only if $X$ has a regular $G_\delta$-diagonal. Also, it is shown in [2], that a $w\Delta$-space (7) $X$ is a Moore-space if $X$ has a regular $G_\delta$-diagonal. Recall that a collection-wise normal Moore space is metrizable [1]. Thus we have:

4.6. THEOREM. *The following conditions for a space $X$ are equivalent:*

a. *$X$ is metrizable.*

b. *$X$ is a CPN-$M$-space.*

c. *$X$ is a CPN-$w\Delta$-space.*

d. *$X$ is either an $M$-space or a $w\Delta$-space and $X\times X$ is a CCR-space.*

e. *$X$ is either an $M$-space or a $w\Delta$-space and $X\times M$ is a CCR-space for every metric space $M$.*

4.7. THEOREM. *The following conditions for a separable metric space $X$ are equivalent:*

a. *$X$ is metrizable.*

b. *$X$ is either an $M$-space or a $w\Delta$-space and $X$ is a CCR-space.*

(6) $X$ is said to be an *$M$-space* if there is a normal sequence $\{\mathfrak{U}_n\}$ of open covers of $X$ such that if $x\in X$, and if, for each $i$, $x_i\in \mathrm{St}(x, \mathfrak{U}_i)$, then the sequence $x_1, x_2, \ldots$ has a cluster point [9].

(7) According to Borges [2], $X$ is a *$w\Delta$-space* if there is a sequence $\{\mathfrak{U}_n\}$ of open covers of $X$ such that if $x\in X$ and if, for each $i$, $x_i\in\mathrm{St}(x, \mathfrak{U}_i)$, then $x_1, x_2, \ldots$ has a cluster point.

4.8. Definition. An N-operator $\varphi$ for $X$ is said to be *monotone* if it is true that if $(H, K)$ and $(H', K')$ are in $\mathcal{M}X$ such that $H \subset H'$ and $K' \subset K$, then $\varphi(x, H, K) \geqslant \varphi(x, H', K')$ for every $x \in X$. $X$ is *continuously monotonically normal* (CMN) if $X$ admits a continuous monotone N-operator. *Monotone* CR-*operators* and *continuously monotonically completely regular* (CMCR)-*spaces* are similarly defined.

In [5], it is shown that a space $X$ is monotonically normal if and only if $X$ admits a monotone N-operator which is continuous in the second variable. In the next theorem we see that CMN-spaces (in fact, CMCR-spaces) are related to metrizable spaces much in the same way that monotonically normal spaces are related to stratifiable spaces.

4.9. Theorem. *If $X \times Y$ is a CMCR-space then either $X$ is metrizable or every countable subset of $Y$ is discrete.*

Proof. Let $\varphi$ be a continuous MCR-operator for $X \times Y$. Using the techniques of the proof of Theorem 3.1, obtain the continuous PN-operator for $X$, $\varphi_0$. But by the construction of $\varphi_0$, it is the case that if $H \subset K$ are in $\mathcal{F}X$, then $\varphi_0(x, H) \geqslant \varphi_0(x, K)$ for every $x \in X$. It then follows from 4.2 that $X$ is metrizable.

4.10. Corollary. *The following conditions on a space $X$ are equivalent:*

a. *$X$ is metrizable.*

b. *$X$ is either an $M$-space or a $w\Delta$-space such that $X^2$ is $\omega$CMCR-space.*

c. *$X$ is a Fréchet space such that $X^2$ is a* CMN-*space.*

d. *$X$ is a Fréchet space such that $X^2$ is a* CMCR-*space.*

e. *$X^\omega$ is a* CMN-*space.*

f. *$X^\omega$ is a* CMCR-*space.*

g. *$X \times M$ is a* CMCR-*space for every metric space $M$.*

4.11. Theorem. *The* CCR-*space $X$ is metrizable if it is the closed continuous image of a metrizable space.*

Proof. Suppose the contrary; i.e., suppose that $X$ is a nonmetrizable CCR-space that is the closed continuous image of a metric space. Let $W = \mathcal{N} \cup \{\omega\}$ denote the one-point compactification of the integers. Let $M = W \times \mathcal{N}$, let $Z = M/\{(\omega, n)|\ n \in \mathcal{N}\}$, and let $\pi$ denote the canonical projection of $M$ onto $Z$. Let $Z$ be endowed with the quotient topology; i.e., $U$ is open in $Z$ if and only if $\pi^{-1}(U)$ is open in $M$. In [10], Van Doren has shown that $X$ must contain a copy of $Z$ (since $X$ is a non-metrizable closed continuous image of a metric space). Since $Z$ is homeomorphic to a subspace of $X$, $Z$ must be a CCR-space; and so, let $\varphi$ be a continuous CR-operator for $Z$. Since $\varphi(\pi(2, w), \pi(1, 1), \{\pi(1, \omega)\}) = 1$, there is an integer $n_2$ such that $\varphi(\pi(2, w), \pi(1, 1), \{\pi(2, n_2)\}) > \frac{1}{2}$. Suppose we have $\{n_2, n_3, \ldots, n_j\}$ such that

$$\varphi(\pi(j, w), \pi(1, j), \{\pi(2, n_2), \pi(3, n_3), \ldots, \pi(j, n_j)\}) > 1 - 1/j\,,$$



since

$$\varphi(\pi(j, w), \pi(1, j+1), \{\pi(2, n_2), \pi(3, n_3), \ldots, \pi(j, n_j), \pi(j+1, w)\}) = 1\,,$$

there is an $n_{j+1}$ such that

$$\varphi(\pi(j, w), \pi(1, j+1), \{\pi(2, n_2), \pi(3, n_3), \ldots, \pi(j+1, n_{j+1})\}) > 1 - 1/(j+1)\,.$$

Let $H = \{\pi(j, n_j)|\ j \geqslant 2\}$ and let $H_i = \{\pi(j, n_j|\ i \geqslant j \geqslant 2\}$ for each $i$. Note that $H$ is a closed set not containing $\pi(1, )$. Thus,

$$\varphi(\pi(1, \omega), \pi(1, \omega), H) = 0\,.$$

Since $H_1, H_2, \ldots$ converges to $H$ and since $\pi(1, 1), \pi(1, 2), \ldots$ converges to $\pi(1, \omega)$, it must be true that there is an $N$ such that if $n > N$, then $\varphi(\pi(1, \omega), \pi(1, n), H_n) < \frac{1}{4}$ which is a contradiction since

$$\lim_{n\to\infty} \varphi(\pi(1, \omega), \pi(1, n), H_n) = 1\,.$$

## 5. An example.

5.1. Example. There is a continuously monotonically completely regular Moore space that is not normal. The space $X$ is Heath's plane, [4], which is described in the following fashion:

Let $X = \{(x, y) \in E^2|\ y \geqslant 0\}$.

For convenience, the coordinates of the points $w$ and $w_0$ of $X$ will be denoted by $(x, y)$ and $(x_0, y_0)$ respectively. For each pair of real numbers $r$ and $\alpha$, let $l_1(\alpha, r) = \{(x, y) \in X|\ y = x - r$ and $|x - r| < \alpha\}$ and let $l_2(\alpha, r) = \{(x, y) \in X|\ y = r - x$ and $|x - r| < \alpha\}$. The statement that $B$ is a basic open set means that either

(1) there is a $w = (x, y) \in X$ such that $y > 0$ and $B = \{w\}$ or

(2) there are real numbers $\alpha$ and $r$ such that $B = l_1(\alpha, r) \cup l_2(\alpha, r)$.

For each $w = (x, y) \in X$ with $y = 0$, let $l_1(w) = \bigcup_{\alpha>0} l_1(\alpha, x)$ and $l_2(w) = \bigcup_{\alpha>0} l_2(\alpha, x)$ for each $w = (x, y) \in X$ with $y > 0$, let $w(1)$ denote the point of the $X$-axis such that $w \in l_1(w(1))$ and let $w(2)$ denote the point of the $X$-axis such that $w \in l_2(w(2))$. To construct a continuous monotone complete regularity operator $\varphi$, let $(w_0, H) \in \mathfrak{D}X$. Define $\varphi(w_0, w_0, H) = 0$.

Case 1. $y_0 = 0$. Let

$$\varphi(w, w_0, H) = \begin{cases} 1 & \text{if} \quad w \in [X - (l_1(w_0) \cup l_2(w_0))] \cup H\,, \\ 0 & \text{otherwise.} \end{cases}$$

Case 2. $y_0 > 0$. For each point $w$ of

$$X - [l_1(w_0(1)) \cup l_2(w_0(1)) \cup l_1(w_0(2)( \cup l_2(w_0(2))]$$

define $\varphi(w, w_0, H) = 1$.

If $w_0(2) \in H$, then define $\varphi(w, w_0, H) = 1$ for each point $w$ of $[l_2(w_0(2) \cup l_1(w_0(2))] - \{w_0\}$.

If $w_0(2) \notin H$, then define $\varphi(w, w_0, H) = \varphi(w, w_0(2), H)$ for each point $w$ of $[l_2(w_0(2)) \cup l_1(w_0(2))] - \{w_0\}$.

If $w_0(1) \in H$, define $\varphi(w, w_0, H) = 1$ for all points $w$ of

$$[l_1(w_0(1)) \cup l_2(w_0(1))] - \{w_0\} .$$

If $w_0(1) \notin H$, define $\varphi(w, w_0, H) = \varphi(w, w_0(1), H)$ for each point $w$ of $[l_1(w_0(1)) \cup l_2(w_0(1))] - \{w_0\}$.

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# On an extremely restricted $\omega$-rule

by

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**Abstract.** By an extremely restricted $\omega$-rule (for Heyting's Arithmetic) we understand an $\omega$-rule of the form:

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots} \\ \text{To conclude:} & \forall x Ax \\ \text{Provided} & \vdash_{\mathrm{HA}} \forall x Ax . \end{array}$$

Although such a rule does not increase the class of theorems, it allows one to quickly obtain (infinite) derivations with the subformula property. From the subformula property many results can then be easily obtained.

§ 0. **Introduction.** From an intuitive point of view the $\omega$-rule

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots, Ak, \ldots} \\ \text{To conclude:} & \forall x Ax \end{array}$$

is a much simpler rule to justify than its finitary cousin, the rule of induction:

$$\text{(IND)} \qquad \begin{array}{rl} \text{From:} & \underline{A0, \forall x(Ax \supset Ax')} \\ \text{To conclude:} & \forall x Ax . \end{array}$$

And yet the latter is usually preferred when considering formal systems. Probably the main objection against the $\omega$-rule is that the derivations are then infinite trees of formulae and there is a natural distrust to using infinite sets when one is trying to better understand the infinite. This distrust is further enhanced by the fact that if to first order classical arithmetic, CA, one adds the $\omega$-rule then one obtains a maximal system (i.e. for every sentence $A$, either $A$ or $\neg A$ is derivable in CA+"$\omega$-rule"); for then the fact that CA+"$\omega$-rule" $\vdash A$ gives us no more information than $\mathfrak{N} \models A$ ($A$ is true in the natural numbers).

On the other hand of instead of the (full) $\omega$-rule one considers a restricted $\omega$-rule, that is an infinitary rule of inference of the form

$$\begin{array}{rl} \text{From:} & \underline{A0, A1, \ldots, Ak, \ldots} \\ \text{To conclude:} & \forall x Ax \\ \text{Provided that:} & \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$