Positive definite functions and coincidences

by

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Abstract. The compactness and the completeness of a metric space X is first characterized by the behaviour of certain types of non-negative real-valued functions on X. These new characterizations are then used to obtain a general coincidence theorem for maps $S, T: X \to Z$ (Z an arbitrary space) which contains the usual topological fixed-point theorems (essentially as special cases of more general coincidence theorems). A generalization of the coincidence theorem, from which one can obtain "coincidence" theorems for pairs of upper, and for pairs of lower, semi-continuous set-valued maps $S, T: X \to 2^Z$ is also given.

Given two maps $S, T: X \to Z$ of metric spaces, we seek conditions on S and T that will assure they have a coincidence (i.e., $S(\xi) = T(\xi)$ for some $\xi \in X$).

Rather than attack this question by considering the fixed-point problem for the associated set-valued map $S^{-1} \circ T: X \to 2^X$, we present a method that reduces the coincidence problem to one of constructing certain types of real-valued functions on X, or on $X \times X$; indeed, the existence of an appropriate such function is shown to be necessary and sufficient for a coincidence. This approach extends also to suitable pairs of semicontinuous set-valued maps $S, T: X \to 2^Z$, again giving necessary and sufficient conditions for the existence of a coincidence in terms of the existence of an appropriate real-valued function. Thus, one can regard each of the classical fixed-point theorem as resulting from the existence of a suitable real-valued function; and, by so doing, it is possible to start with a fixed-point theorem and construct, fairly mechanically, a coincidence theorem for set-valued maps that reduces to it when one (or both) of S, T are single-valued maps.

A non-negative real-valued function on a metric space X is called *positive definite* mod $A \subset X$ if it has a positive lower bound outside each ε-neighborhood of A. Positive definiteness of a given function clearly depends on the metric used in X; in Sec. 2, we give simple characterizations, which are apparently new, for the compactness of A and for the completeness of X in terms of the behaviour of certain such functions. In Sec. 3, we show that each coincidence is characterized by a suitable positive definite function; the general coincidence theorems are 3.1, 3.2.

These results are applied in Sec. 4 to obtain the Browder [4], Edelstein [7] and Belluce–Kirk [2] fixed-point theorems by constructing appropriate positive definite functions. In Sec. 5, we give a method for constructing coincidence analogs of fixed-point theorems (i.e. coincidence theorems producing the given fixed-point theorem when applied to the case of T, id: $X\to X$); in particular, we find a fixed-point principle (5.2) which is formally similar to (and includes) the Banach contraction principle, but which is applicable also to non-contractive maps. In the final section, the results of Sec. 3 are extended to provide general "coincidence" theorems for two lower (resp. upper) semicontinuous set-valued maps; in this extension, it appears more appropriate to call $\xi\in X$ a "coincidence" of two lower (resp. upper) semicontinuous maps $S, T\colon X\to 2^Z$ if $S(\xi)\subset T(\xi)$ (resp. $S(\xi)\cap T(\xi)\neq\emptyset$).

1. Positive definite functions. In all that follows, $\boldsymbol{R}_0$ will denote the subspace $\{x\in\boldsymbol{R}|\ x\geqslant 0\}$ of the real line $\boldsymbol{R}$.

1.1. Definition. Let (X, d) be a metric space and let $A\subset X$. A map [1] $P\colon X\to\boldsymbol{R}_0$ is called *positive definite* mod A if

$$\inf\{P(x)|\ d(x, A)\geqslant\varepsilon\}>0 \quad \text{for each } \varepsilon>0 .$$

It is clear that if P is positive definite mod A, then $P^{-1}(0)\subset\bar{A}$; the possibility $P^{-1}(0)=\emptyset\subset A$ is of course not excluded. However, in general, even the strong condition $P^{-1}(0)=\emptyset$ does not assure positive definiteness mod A: in $X=\boldsymbol{R}_0$, the (continuous) function $x\mapsto e^{-x}$ vanishes nowhere on $\boldsymbol{R}_0$ yet is not positive definite mod$[0,1]$. Note that [2] when A is closed, if either P is a closed map, or if P is lower semicontinuous and X is compact, or if P is arbitrary and $A=\emptyset$, then P will be positive definite mod A if and only if $P|(X-A)$ has no zero.

It is evident that, if P is positive definite mod A, then so also is $P+Q$ for any $Q\colon X\to\boldsymbol{R}_0$; the converse need not be true: in $\boldsymbol{R}$, the function $x\mapsto e^x+e^{-x}$ is positive definite mod$[-1,+1]$ although neither term is. Equally evident are (a): If P, Q are positive definite mod A, so also is $P\cdot Q$ and (b): If $\alpha\colon(\boldsymbol{R}_0, 0)\to(\boldsymbol{R}_0, 0)$ is continuous at 0, and if $\alpha\circ P$ is positive definite mod A, then so also is P.

It is simple to see that, if P is positive definite mod A, then (a): P is positive definite mod any $A'\supset A$, and (b): Any $S\colon X\to\boldsymbol{R}_0$ satisfying $S(x)\geqslant P(x)$ on $X-A$ is also positive definite mod A. In particular, if $d(x, A)\leqslant S(x)$ on $X-A$, then S is positive definite mod A.

Observe that the positive definiteness mod A of a given $P\colon X\to\boldsymbol{R}_0$ depends on the metric used in X: for example, with $X=\boldsymbol{R}_0$ and $A=\{$all positive integers$\}$, the function $P(x)=e^{-x}$, which is not positive definite mod A when the Euclidean metric is used in $\boldsymbol{R}_0$, becomes positive definite mod A when the equivalent metric $d(x, y)=|e^{-x}-e^{-y}|$ is used [3].

We shall have to consider cartesian products of metric spaces. For all cartesian products $(X, d_X)\times(Y, d_Y)$ in this paper, we will always use the metric $D[(x, y), (x', y')]=d_X(x, x')+d_Y(y, y')$. The diagonal in $(X, d)\times(X, d)$ will be denoted by $\Delta(X)$; noting that the metric D used in $X\times X$ has the feature that $D[(x, x'), \Delta]=d(x, x')$ immediately gives then the following result which we will use frequently:

1.2. $P\colon X\times X\to\boldsymbol{R}_0$ is positive definite mod $\Delta(X)$ if and only if for each $\varepsilon>0$ there is a $\beta>0$ such that $d(x, y)<\varepsilon$ whenever $P(x, y)<\beta$.

2. Positive definiteness, compacteness, and completeness. In this section, we characterize compactness and completeness by the behaviour of appropriate positive definite functions.

2.1. Theorem. *Let (X, d) be a metric space and $A\subset X$ closed. The following two properties are equivalent:*

1. *A is compact.*
2. *For every lower semi-continuous $V\colon X\to\boldsymbol{R}_0$ with $\inf V|X=0$: if V is positive definite* mod A, *then $V(a)=0$ for some $a\in A$.*

Proof. $1\Rightarrow 2$. The function $V|A$, being lower semicontinuous on the compact A, attains its infimum α [6, p. 227]. If $\alpha>0$, then $U=\{x\in X|\ V(x)>\alpha/2\}$ would be an open set containing the compact A, so that $(A, \varepsilon)\subset U$ for some $\varepsilon>0$. Since $\inf V|X=0$, it would follow that V does not have a positive lower bound outside (A, ε), contradicting its positive definiteness.

$2\Rightarrow 1$. Assume that A is not compact; then A contains an infinite discrete closed subset $\{a_i|\ i=1,2,\ldots\}$ which, since A is closed in X, is closed discrete in X. By Tietze's theorem [6, p. 151], there is a continuous $f\colon X\to]0,1]$ with $f|X-(A,1)\equiv 1$ and $f(a_n)=1/n$ for each $n=1,2,\ldots$ Then the function $V(x)=f(x)+d(x, A)$ is positive definite mod A (since $V(x)\geqslant d(x, A)$), has $\inf V|X=0$ (since $V(a_n)=1/n$), yet vanishes nowhere on A (since $V(a)\geqslant f(a)>0$). This completes the proof.

The d-completeness of (X, d) is characterized by using functions $V\colon X\to\boldsymbol{R}_0$ for which the associated map $P\colon X\times X\to\boldsymbol{R}_0$, given by $(x, y)\mapsto V(x)+V(y)$, is positive definite mod the diagonal; we do not impose any positive definite requirement on the functions V themselves [4].

(1) Unless specifically stated, a map $S\colon X\to Y$ is not required to be continuous.

(2) We will always denote $\{x\in X|\ d(x, A)<\epsilon\}$ by (A, ϵ). With this notation, 1.1 is equivalent to: For each (A, ϵ), there is a $\beta>0$ such that $P^{-1}[0, \beta[\subset(A, \epsilon)$.

(3) It is easy to see, however, that if the subset $A\subset X$ is compact, a function positive definite mod A using one metric in X will remain positive definite mod A using any equivalent metric in X.

(4) Recall that positive definiteness of a sum does not imply that of each summand.

2.2. THEOREM. *Let (X, d) be a metric space. The following two properties are equivalent:*

1. *(X, d) is d-complete.*

2. *For every lower semicontinuous $V: X \to R_0$ with $\inf V|X = 0$: if the map $P: X \times X \to R_0$, defined by $(x, y) \mapsto V(x)+V(y)$ is positive definite* $\operatorname{mod}\Delta(X)$, *then $V(\xi) = 0$ for some $\xi \in X$.*

Furthermore, whenever the conditions in (2) are satisfied: if $\{\xi_n\}$ is any sequence in X with $V(\xi_n) \to 0$, then $\xi_n \to \xi$.

Proof. $1 \Rightarrow 2$. Choose any sequence $\{x_n\}$ in X with $V(x_n) \downarrow 0$ and, for each $n = 1, 2, \ldots$, let $A_n = \{x \in X|\ V(x) \leqslant V(x_n)\}$. By lower semicontinuity, each A_n is closed; by their definition, each $A_n \neq \emptyset$ and $A_1 \supset A_2 \supset A_3 \supset \ldots$ We now show $\inf\{\operatorname{diam} A_n|\ n = 1, 2, \ldots\} = 0$: for, choose any $\varepsilon > 0$; then, by 1.2, find $\beta > 0$ so that $d(x, y) < \varepsilon$ whenever $P(x, y) < \beta$, and finally pick N so large that $V(x_N) < \beta/2$; now, for any pair $x, y \in A_N$ we find $P(x, y) = V(x)+V(y) < 2\beta/2$, consequently $d(x, y) < \varepsilon$ and therefore $\operatorname{diam} A_N \leqslant \varepsilon$.

This established, Cantor's intersection theorem gives a single point $\xi \in \bigcap_n A_n$; since $V(\xi) = 0$ because $V(\xi) \leqslant V(x_n)$ for all n, this proves $1 \Rightarrow 2$. To establish the additional conclusion [5], it suffices to observe that, if $V(\xi_i) < \beta$ for all $i \geqslant n$, then from the formula $P(\xi_k, \xi) = V(\xi_k)+V(\xi) \equiv V(\xi_k)$ it follows that $d(\xi_i, \xi) < \varepsilon$ for all $i \geqslant n$.

$2 \Rightarrow 1$. Assume that (X, d) is not complete. Letting $(\hat{X}, \hat{d})$ be its completion, choose a $\hat{\xi} \in \hat{X} - X$ and define $V: X \to R_0$ by $V(x) = \hat{d}(x, \hat{\xi})$. Since $\hat{d}|X \times X = d$, we find that P is positive definite $\operatorname{mod}\Delta(X)$ (because $P(x, y) = \hat{d}(x, \hat{\xi})+\hat{d}(y, \hat{\xi}) \geqslant \hat{d}(x, y) = d(x, y) = D[(x, y), \Delta]$) and that $\inf V|X = 0$ because X is dense in $\hat{X}$. However, V vanishes nowhere on X, and this completes the proof.

3. Positive definitess and coincidences. Given $S, T: X \to Z$, a coincidence of S and T is a point $\xi \in X$ such that $S(\xi) = T(\xi)$; obviously, a coincidence of T, id: $X \to X$ is simply a fixed point for T. We will now reduce the problem of finding coincidences to one of constructing positive definite functions and show, in fact, that these problems are equivalent in the category of metric spaces.

The graph of a map $S: X \to Z$ will be denoted by $G(S)$.

3.1. THEOREM. *Let (X, d) be complete, Z a topological space, and $S, T: X \to Z$ continuous. In order that S, T have at least one coincidence ξ,*

[5] The proof given can be easily modified to show, somewhat more generally: Let X be d-complete and let $V: X \to R_0$ be lower semicontinuous with $\inf V|X = 0$. Assume that there is some map $\alpha: (R_0 \times R_0, (0, 0)) \to (R_0, 0)$ continuous at $(0, 0)$ and such that $\alpha \circ (V \times V): X \times X \to R_0$ is positive definite $\operatorname{mod}\Delta(X)$. Then $V(\xi) = 0$ for some $\xi \in X$, and if $V(\xi_n) \to 0$ then $\xi_n \to \xi$.

it is sufficient, and if Z is metrizable also necessary, that there exist a V: $X \times Z \to R_0$ such that

1. *V is lower semicontinuous and $V^{-1}(0) \subset G(S)$.*

2. $\inf\{V(x, Tx)|\ x \in X\} = 0$.

3. *The function $P: X \times X \to R_0$ given by $(x, y) \mapsto V(x, Tx)+V(y, Ty)$ is positive definite* $\operatorname{mod}\Delta(X)$.

Moreover, when such a V exists:

(a) *If $V(x_n, Tx_n) \to 0$ for any sequence $\{x_n\}$ in X, then $x_n \to \xi$.*

(b) *If $V^{-1}(0) = G(S)$ then ξ is the only coincidence of S and T.*

Proof. Sufficiency. Define $L: X \to R_0$ by $x \mapsto V(x, Tx)$. An application of 2.2 gives a ξ with $V(\xi, T\xi) = 0$ and, from the hypothesis 1, that $T\xi = S\xi$. This proves the sufficiency. The additional conclusion (a) being immediate from 2.2, we now prove (b): If $V^{-1}(0) = G(S)$ and S, T had two coincidences $\xi \neq \eta$, then

$$P(\xi, \eta) \equiv V(\xi, T\xi)+V(\eta, T\eta) = V(\xi, S\xi)+V(\eta, S\eta) = 0$$

and, since $(\xi, \eta) \notin \Delta$, this contradicts the positive definiteness $\operatorname{mod}\Delta$ of P.

Necessity. Assume that $S(\xi) = T(\xi)$. Letting d_Z be a metric in Z, define $V: X \times Z \to R_0$ by $V(x, z) = d(x, \xi)+d_Z(z, S\xi)$; this vanishes only at $(\xi, S\xi) \in G(S)$. Because $P(x, y) \geqslant d(x, \xi)+d(y, \xi) \geqslant d(x, y) = D[(x, y), \Delta]$, this shows P is positive definite mod $\Delta(X)$ and, since $V(\xi, T\xi) = 0$, the proof is complete [6].

Note that, if S, T have more than one coincidence, different functions V must be used to locate the different coincidences.

The hypothesis of completeness can be removed by relying on 2.1 rather than on 2.2, giving

3.2. THEOREM. *Let (X, d) be a metric space, Z a topological space, and $S, T: X \to Z$ continuous. In order that S, T have at least one coincidence ξ, it is sufficient, and if Z is metrizable also necessary, that there exist a V: $X \times Z \to R_0$ such that*

1. *V is lower semicontinuous, and $V^{-1}(0) \subset G(S)$.*

2. $\inf\{V(x, Tx|\ x \in X\} = 0$.

3. *The map $x \mapsto V(x, Tx)$ is positive definite* mod *some compact $A \subset X$.*

The proof is omitted, since it is similar to that of 3.1, and uses the same function V for the necessity.

It is useful to observe that, if $W: Z \times Z \to R_0$ vanishes only at points on $\Delta(Z)$, then $V(x, z) = W(Sx, z)$ is a function on $X \times Z$ with $V^{-1}(0) \subset G(S)$; using such functions W gives a more symmetric formulation of the sufficiency in 3.1, 3.2, which is frequently more convenient in applications:

[6] Instead of the sum function in hypothesis 3, one can use $\alpha[V(x, Tx), V(y, Ty)]$ where $\alpha: (R_0 \times R_0, (0, 0)) \to (R_0, 0)$ is any function continuous at $(0, 0)$. Cf. footnote [5].

3.3. THEOREM. *Let (X, d) be a metric space, Z a topological space, and $S, T: X \to Z$ continuous. Then S, T will have a coincidence ξ whenever there is a lower semi-continuous $W: Z\times Z \to \boldsymbol{R}_0$ such that*

1. $W^{-1}(0) \subset \Delta(Z)$,
2. $\inf\{W(Sx, Tx)|\ x \in X\} = 0$, *and*

3′. $x \mapsto W(Sx, Tx)$ *is positive definite* mod *some compact* $A \subset X$ *or*

3″. (X, d) *is complete and* $(x, y) \mapsto W(Sx, Tx)+W(Sy, Ty)$ *is positive definite* mod $\Delta(X)$.

4. Application to fixed-point theorems. In this section, we illustrate the use of positive definite functions, deriving some of the standard fixed-point theorems by producing an appropriate function.

4.1. THEOREM. *Let (X, d) be complete and $T: X\to X$ such that $d(Tx, Ty) \leqslant d(x, y)$. If the map $L: X\times X \to \boldsymbol{R}_0$ given by $(x, y) \mapsto d(x, y)-d(Tx, Ty)$ is positive definite* mod $\Delta(X)$, *then T has a unique fixed point ξ, and $T^n x \to \xi$ for each $x \in X$.*

Proof. We show 3.1 is satisfied with $V = d$ and $S = \mathrm{id}$ (so that $G(S)$ is $\Delta(X)$). Since

$$L(x, y) \equiv d(x, y)-d(Tx, Ty) \leqslant d(x, Tx)+d(y, Ty) \equiv P(x, y),$$

it follows that P is positive definite mod $\Delta(X)$, so it remains only to show that $\inf\{d(x, Tx)|\ x \in X\} = 0$. To this end, let $\varepsilon > 0$ be given, and let $\beta = \inf\{L(x, y)|\ d(x, y) \geqslant \varepsilon\}$; by positive definiteness, β is positive. Because $L(x, y) \geqslant 0$ for all (x, y) we find, fixing any (x, Tx), that the sequence $\{d(T^n x, T^{n+1}x)\}$ is necessarily monotone non-increasing, hence Cauchy, consequently $d(T^n x, T^{n+1}x)-d(T^{n+1}x, T^{n+2}x) < \beta$ for all $n \geqslant N$. This says $L(T^N x, T(T^N x)) < \beta$, therefore $d(T^N x, T^{N+1}x) < \varepsilon$. Thus, $\inf d(x, Tx) = 0$ and, since d vanishes on the entire diagonal, the conclusions all follow from 3.1.

This result contains Browder's fixed-point theorem [4]; for if $\omega: [0, M] \to \boldsymbol{R}_0$ is monotone non-decreasing right continuous, with $\omega(r) < r$ for all $r > 0$, it is easy to see that $r-\omega(r)$ is positive definite mod 0; thus, if (X, d) is bounded and complete, and if $T: X\to X$ satisfies $d(Tx, Ty) \leqslant \omega[d(x, y)]$, then the required positive definiteness in 4.1 follows from $d(x, y)-\omega[d(x, y)] \leqslant d(x, y)-d(Tx, Ty)$.

Note also that if X is compact, the positive definiteness requirement in 4.1 is satisfied if only $d(x, y)-d(Tx, Ty) > 0$ whenever $x \neq y$; for the general case, we prove a version of Edelstein's theorem [7] in arbitrary metric spaces [8,3.2] rather than complete spaces.

4.2. THEOREM. *Let (X, d) be a metric space and $T: X\to X$ continuous. Assume that*

1. $d(x, y)-d(Tx, Ty) > 0$ *whenever* $x \neq y$.
2. *For some $p \in X$, some subsequence $\{T^{n_i}p\}$ of its iterates converges to a point ξ.*

Then ξ is a fixed point for T.

Proof. We shall use 3.2. Define $V: X\times X \to \boldsymbol{R}_0$ by

$$V(x, y) = d(x, y)-d(Tx, Ty)+d(x, \xi);$$

then $V^{-1}(0) \subset \Delta$ and $V(x, Tx)$ is positive definite mod ξ, so it remains to show that $\inf\{V(x, Tx)|\ x \in X\} = 0$. Choose any $\varepsilon > 0$; again as in 4.1, it follows that $\{d(T^n p, T^{n+1}p)\}$ is a Cauchy sequence, so that $|d(T^n p, T^{n+1}p)-d(T^s p, T^{s+1}p)| < \frac{1}{2}\varepsilon$ for all $n, s \geqslant$ some N; selecting an $n_i \geqslant N$ such that $d(T^{n_i}p, \xi) < \frac{1}{2}\varepsilon$ then gives $V(T^{n_i}p, T(T^{n_i}p)) < \varepsilon$. Thus, $\inf\{V(x, Tx)|\ x \in X\} = 0$ and, by 3.2, the proof is complete.

In case $T: X\to X$ is non-expansive, i.e. $d(Tx, Ty) \leqslant d(x, y)$, it is easy to verify (see [8], for example) that if any one orbit is bounded, then the function $x \mapsto \delta(x) \equiv$ diameter $\{T^n x|\ n = 0, 1, \ldots\}$ is uniformly continuous on X. The map $T: X\to X$ is said to have *shrinking orbits* if, for each x with $\delta(x) > 0$, there is an n such that $\delta(T^n x) < \delta(x)$. We will use 2.1 to establish the Belluce–Kirk theorem [2], [9] for arbitrary (rather than complete) metric spaces.

4.3. THEOREM. *Let (X, d) be a metric space, and let $T: X\to X$ be a non-expansive map with shrinking orbits. If for some $p \in X$ some subsequence $\{T^{n_i}p\}$ of its iterates converges to a point $\xi \in X$, then ξ is a fixed point for T.*

Proof. For any fixed integer $s > 0$, consider the function

$$V(x) = \delta(x)-\delta(T^s x)+d(x, \xi)$$

which is positive definite mod ξ. Since the sequence $\{\delta(T^n p)|\ n = 0, 1, \ldots\}$ is monotone non-increasing it follows again, as in 4.1, that it is Cauchy so, given any $\varepsilon > 0$ we have $|\delta(T^i p)-\delta(T^j p)| < \varepsilon$ for all $i, j \geqslant$ some N. Choosing $n_i \geqslant N$ to satisfy $d(T^{n_i}p, \xi) < \varepsilon$ gives $V(T^{n_i}p) < 2\varepsilon$. Thus, $\inf V(x) = 0$; consequently, by 2.1, $V(\xi) = 0 = \delta(\xi)-\delta(T^s\xi)$. Since s is arbitrary, this shows that $\delta(\xi) = \delta(T^s\xi)$ for all $s > 0$ so, by the shrinking orbit property, $\delta(\xi) = 0$ and the proof is complete.

As one more illustration of this technique, we establish

4.4. THEOREM. *If (X, d) is complete and a continuous $T: X\to X$ satisfies*

$$d(Tx, Ty) \leqslant a\max[d(x, Tx), d(y, Ty)], \qquad a < 1,$$

then T has a unique fixed point ξ, and $T^n x \to \xi$ for each $x \in X$.

Proof. From

$$D[(x, y), \Delta] = d(x, y) \leqslant d(x, Tx)+d(Tx, Ty)+d(Ty, y) \leqslant 2[d(x, Tx)+d(y, Ty)]$$

follows that $(x, y)\mapsto d(x, Tx)+d(y, Ty)$ is positive definite mod $\Delta(X)$. We next show that $d(T^nx, T(T^nx))\to 0$ for each x: Assuming that no T^nx is a fixed point, from $d(Tx, T^2x)\leqslant \alpha\max[d(x, Tx), d(Tx, T^2x)]$ and $\alpha<1$, we must have $d(Tx, T^2x)<d(x, Tx)$ and therefore $d(Tx, T^2x)\leqslant \alpha d(x, Tx)$. Thus, $d(T^nx, TT^nx)\leqslant \alpha^n d(x, Tx)\to 0$, as asserted. An application of 3.1, taking $V=d$, now completes the proof.

In the same way, one obtains the above conclusion under the more general hypothesis

$$d(Tx, Ty)\leqslant \alpha\max[d(x, Tx), d(y, Ty)]+\beta d(x, y)+\gamma[d(x, Ty)+d(y, Tx)]$$

where $\alpha, \beta, \gamma\geqslant 0$ and $\alpha+\beta+2\gamma<1$. This relates to some of Reich's results [10] but with the additional assumption that T be continuous.

5. Coincidence analogs. We will illustrate some methods for generating coincidence theorems from fixed-point theorems.

5.1. Definition. Let $S, T\colon X\to Z$. An *$S^{-1}T$-orbit* of $x_0\in X$ is any sequence $\{x_n\}$ in X such that $Tx_n=Sx_{n+1}$ for each $n\geqslant 0$.

A given $x_0\in X$ may not have any $S^{-1}T$-orbit; however, if S is surjective, then each $x_0\in X$ has at least one $S^{-1}T$-orbit, determined inductively by choosing an $x_{n+1}\in S^{-1}(Tx_n)$ for each $n\geqslant 0$.

It is clear that (a): If $x_0\in S^{-1}T(x_0)$ then x_0 has an $S^{-1}T$-orbit (x_n) with $x_n=x_0$ for each n (it may also have other orbits) and (b): If $\{x_n\}$ is an $S^{-1}T$-orbit of x_0, then $\{x_{n+k}|\ n=0,1,2,\dots\}$ is an $S^{-1}T$-orbit of x_k.

For id, $T\colon X\to X$, each $x\in X$ has the unique $(\mathrm{id})^{-1}T$-orbit $\{T^nx\}$; the following coincidence theorem therefore reduces to that of Browder–Petryshyn [5]:

5.1. Theorem. *Let (X, d), (Z, ϱ) be metric spaces, and $S, T\colon X\to Z$ continuous. If there is an $S^{-1}T$-orbit $\{x_n\}$ having a subsequence $\{x_{n_i}\}$ such that*

1. $x_{n_i}\to\xi$,
2. $d(x_{n_i}, x_{1+n_i})\to 0$,

then S and T have the coincidence ξ.

Proof. Define $V\colon X\times Z\to \boldsymbol{R}_0$ by

$$V(x, z)=d(x, \xi)+\varrho(z, T\xi)$$

which vanishes only at $(\xi, T\xi)\in G(T)$. Since $V(x, Sx)$ is positive definite mod ξ, the result will follow from 3.2 once we establish that $\inf V(x, Sx)=0$; and, since

$$V(x_{1+n_i}, Sx_{1+n_i})\leqslant d(x_{1+n_i}, x_{n_i})+d(x_{n_i}, \xi)+\varrho(Tx_{n_i}, T\xi),$$

this conclusion follows at once from the hypotheses and the continuity of T. This completes the proof.

Another simple method to generate coincidence theorems from fixed-point theorems is to replace the metric by a suitable function W and use 3.3; the following result specializes to the Banach contraction principle:

5.2. Theorem. *Let (X, d) be complete, Z a topological space, and $S, T\colon X\to Z$ continuous, with S surjective. Assume that there is some $W\colon Z\times Z\to \boldsymbol{R}_0$ such that*

1. *W is lower semicontinuous and $W^{-1}(0)\subset\Delta(Z)$.*
2. *There is an $\alpha<1$ such that $W(Tx, Ty)\leqslant \alpha W(Sx, Sy)$ for all $x, y\in X$.*
3. *$W(Sx, Tx)+W(Sy, Ty)$ is positive definite* mod $\Delta(X)$.

Then S, T have at least one coincidence.

Proof. According to 3.3, we need show only that $\inf W(Sx, Tx)=0$. Consider any $S^{-1}T$-orbit $\{x_n\}$; since

$$W[Sx_{n+1}, Tx_{n+1}]=W[Tx_n, Tx_{n+1}]\leqslant \alpha W[Sx_n, Sx_{n+1}]=\alpha W[Sx_n, Tx_n]$$

it follows by induction that $W[Sx_n, Tx_n]\leqslant \alpha^n W[Sx_0, Tx_0]$ and, since $\alpha<1$, this completes the proof.

Assuming the Z in 5.2 to be metrizable, note that if S is not bijective, then the function $W=d_Z$ cannot satisfy all the requirements; and if W can in fact be taken to be d_Z, then 5.2(3) is redundant.

It should be noted that the form 5.2 of the Banach contraction principle may be applicable to find fixed points of maps $T\colon X\to X$ in instances where the classical version is not. As a transparent such example, let $X=[0, \frac{3}{4}]\subset \boldsymbol{R}$ and let $T\colon X\to X$ be $x\mapsto x^2$. Clearly, T is not contractive (nor even non-expansive): however, using the function $W(x, y)=x+y$ on $X\times X$, it is easy to see that $W(Tx, Ty)\leqslant \frac{1}{3}W(x, y)$ and that W satisfies the remaining requirements in 5.2.

There is no difficulty in obtaining a coincidence version of 4.4, by replacing d with a suitable W as in 5.2.

A coincidence theorem that reduces to the Bailey fixed-point theorem [1] is

5.3. Theorem. *Let (X, d) be complete, Z an arbitrary space, and $S, T\colon X\to Z$ continuous, with S surjective. Assume that there is a $W\colon Z\times Z\to \boldsymbol{R}_0$ such that*

1. *W is lower semicontinuous and $W^{-1}(0)\subset\Delta(Z)$.*
2. *For each $\varepsilon>0$ and each pair of $S^{-1}T$-orbits $\{x_n\}$, $\{y_n\}$, there is some $n=n(x, y, \varepsilon)$ such that $W[Sx_n, Sy_n]<\varepsilon$.*
3. *$W(Sx, Tx)+W(Sy, Ty)$ is positive definite* mod $\Delta(X)$.

Then S, T have a coincidence.

Proof. Again by 3.3, we need only to show that $\inf W(Sx, Tx) = 0$; and since $W[Sx_n, Tx_n] = W[Sx_n, Sx_{n+1}]$ this follows by considering the $S^{-1}T$-orbits $\{x_0, x_1, x_2, \ldots\}$ and $\{x_1, x_2, x_3, \ldots\}$. This completes the proof.

To develop an analog of 4.1, we first formalize the essential observation used in its proof (and that of 4.2, 4.3):

5.4. Lemma. *Let (X, d) be a metric space, Z an arbitrary space, and $S, T\colon X \to Z$ two maps. Assume that there is a $W\colon Z \times Z \to \mathbf{R}_0$ such that $L(x, y) \equiv W(Sx, Sy) - W(Tx, Ty)$ is positive definite* $\operatorname{mod} \Delta(X)$. *Then for any two $S^{-1}T$-orbits $\{x_n\}$, $\{y_n\}$ and for each $\varepsilon > 0$,*

1. $|W(Sx_n, Sy_n) - W(Sx_k, Sy_k)| < \varepsilon$ *for all large* n, k,
2. $d(x_n, y_n) \to 0$.

Proof. 1. Since $W[Sx_n, Sy_n] \geqslant W[Tx_n, Ty_n] \equiv W[Sx_{n+1}, Sy_{n+1}]$, the sequence $\{W(Sx_n, Sy_n)\}$ is monotone non-increasing, hence covergent, hence Cauchy.

2. Choose any $\varepsilon > 0$; by positive definiteness, $d(x, y) < \varepsilon$ whenever $L(x, y) < \beta$, where $\beta > 0$. Choosing n so large that $W[Sx_n, Sy_n] - W[Sx_{n+1}, Sy_{n+1}] < \beta$ for all large n, we have $L(x_n, y_n) < \beta$ so $d(x_n, y_n) < \varepsilon$ for all large n. This completes the proof.

This leads to

5.5. Theorem. *Let (X, d), (Z, ϱ) be metric spaces, and $S, T\colon X \to Z$ continuous. Assume that there is a $W\colon Z \times Z \to \mathbf{R}_0$ such that*

1. *W is lower semicontinuous and $W^{-1}(0) \subset \Delta(Z)$.*
2. *$W(Sx, Sy) - W(Tx, Ty)$ is positive definite* $\operatorname{mod} \Delta(X)$.

If some $p \in X$ has an $S^{-1}T$-orbit $\{p_n\}$ containing a convergent subsequence, then S, T have a coincidence (7).

Proof. Letting $\{p_n\}$ be the given orbit of p, then $\{p_{n+1}\}$ is an orbit of p_1 and, according to the Lemma, $d(p_n, p_{n+1}) \to 0$. Since some subsequence of $\{p_n\}$ converges, an application of 5.1 completes the proof.

6. Application to set-valued maps. The set of all non-empty closed subsets of a given space Z will be denoted by 2^Z; the graph of a map $T\colon X \to 2^Z$ is $G(T) = \bigcup_x \{(x, z) \mid z \in Tx\} \subset X \times Z$.

If $V\colon X \times Z \to \mathbf{R}_0$ and $S\colon X \to 2^Z$ are given, we define $V^S, V_S\colon X \to \mathbf{R}_0$ by

$$V^S(x) = \sup\{V(x, z) \mid z \in Sx\}, \qquad V_S(x) = \inf\{V(x, z) \mid z \in Sx\}.$$

It is well-known [3, p. 121] that if V is lower semicontinuous and if S is a lower (resp. point-compact upper) semicontinuous set map, then V^S (resp. V_S) is lower semicontinuous.

(7) Note that, even using $W = \mathrm{id}$, this does not follow from [8, 3.2] by considering the set-valued map $S^{-1}T\colon X \to 2^X$, since in 5.5, the map $S^{-1}T$ is not required to be point-compact.

We give two general coincidence theorems for maps $S, T\colon X \to 2^Z$, one when both S, T are lower semicontinuous, the other when both are upper semicontinuous; observe that, for pairs of upper semicontinuous set maps, the notion of coincidence natural in our approach is slightly different from that for pairs of lower semicontinuous set maps.

6.1. Theorem. *Let (X, d) be a metric space, Z an arbitrary space, and $S, T\colon X \to 2^Z$ lower semicontinuous. In order that there be a point $\xi \in X$ with $S(\xi) \subset T(\xi)$ it is sufficient, and if Z is metrizable also necessary, that there exist a $V\colon X \times Z \to \mathbf{R}_0$ such that*

1. *V is lower semicontinuous and $V^{-1}(0) \subset G(T)$,*
2. $\inf\{V^S(x) \mid x \in X\} = 0$, *and*

3′. *V^S is positive definite* mod *some compact $A \subset X$, or*

3″. *X is complete and $(x, y) \mapsto V^S(x) + V^S(y)$ is positive definite* $\operatorname{mod} \Delta(X)$.

6.2. Theorem. *Let (X, d) be a metric space, Z an arbitrary space, and $S, T\colon X \to 2^Z$ upper semicontinuous maps, with S also point-compact. In order that there be a point $\xi \in X$ with $S(\xi) \cap T(\xi) \neq \emptyset$ it is sufficient, and if Z is metrizable also necessary, that there exist a $V\colon X \times Z \to \mathbf{R}_0$ such that*

1. *V is lower semicontinuous and $V^{-1}(0) \subset G(T)$,*
2. $\inf\{V_S(x) \mid x \in X\} = 0$, *and*

3′. *V_S is positive definite* mod *some compact $A \subset X$, or*

3″. *X is complete and $(x, y) \to V_S(x) + V_S(y)$ is positive definite* $\operatorname{mod} \Delta(X)$.

Proofs. We shall prove only 6.1, since the proof of 6.2 is similar. Sufficiency: According to 2.1 and 2.2, there is a $\xi \in X$ with $V^S(\xi) = 0$; this implies that $V(\xi, z) = 0$ for all $z \in S(\xi)$ and, since $V^{-1}(0) \subset G(T)$, it follows that $S(\xi) \subset T(\xi)$. Necessity: For the function $V(x, z) = d(x, \xi) + \varrho(z, T\xi)$, ϱ a metric for Z, the corresponding V^S is positive definite mod the compact $A = \xi$ (resp. $V^S(z) + V^S(y)$ is positive definite $\operatorname{mod} \Delta(X)$) and is readily verified to have the desired properties. This completes the proof.

Each of Theorems 6.1, 6.2 clearly reduces to 3.1, 3.2 if S, T are continuous point-maps. Taking $S, T\colon X \to 2^X$, with $S(x) =$ singleton x, the theorems describe the real-valued functions necessary and sufficient to assure that $T\colon X \to 2^X$ has a fixed point. Application yielding the results in, say, [8], and their extension to "coincidence" versions, can be obtained in the manner of Sections 4 and 5.

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Some continuous separation axioms

by

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Abstract. Let $\mathcal{F}X$ denote the space of closed subsets of X with the Vietoris topology. A function $\varphi\colon X\times\mathcal{F}X\to[0,1]$ is a perfect normality operator (abbreviated PN-operator) if, for each $H\in\mathcal{F}X$, $H=\{x\in X\colon \varphi(x,H)=0\}$. X is continuously perfectly normal if X admits a continuous PN-operator. Notions of continuously normal and continuous complete regularity are defined in a similar fashion. It is shown that:

1. X is metrizable $\Rightarrow X$ is continuously perfectly normal $\Rightarrow X$ is continuously normal $\Rightarrow X$ is continuously completely regular.

2. Every continuously perfectly normal space is a collectionwise normal Fréchet space.

3. The product of X with the irrationals is continuously completely regular iff X is continuously perfectly normal.

4. Every locally compact continuously completely regular space is first countable.

5. X is metrizable if and only if X admits a continuous PN-operator, φ, such that if K is a finite subset of X and if $x\in K$, then $\varphi(y,\{x\})\geqslant\varphi(y,K)$ for every $y\in X$.

6. Every $w\Delta$ continuously perfectly normal space is metrizable.

G. Gruenhage recently showed the author an example of a continuously perfectly normal, stratifiable, first countable space that is not metrizable. It is not known if every continuously perfectly normal space is metrizable.

In [14], the author shows that the T_1-space X is metrizable if and only if there is a continuous function α from $\mathcal{F}X$, the space of closed subsets of X with the Vietoris topology [1] into CX, the space of continuous, non-negative, realvalued functions defined on X with the compact-open topology, such that

(a) if $H\in\mathcal{F}X$, then $H=\{x\mid \alpha(H)(x)=0\}$ and

(b) if K is a finite subset of X and if $x\in K$, then

$$\big(\alpha(\{x\})\big)(y)\geqslant\big(\alpha(K)\big)(y)\quad\text{for all } y\in X\,.$$

The author's attempts to decide if (b) of this theorem could be removed led to the notions of continuous perfect normality, continuous normality,

[1] If X is a space and U is a finite collection of subsets of X, then RU will denote the set $\{F\in\mathcal{F}X\mid F\subset\bigcup U$ and F intersects each member of $U\}$. Then the collection $\{RU\mid U$ is a finite collection of open subsets of $X\}$ forms a basis for a topology on X. The topology so induced is often called the Vietoris topology, the finite topology, or the exponential topology. Good studies of the Vietoris topology can be found in [7] and in [8].