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Compacta weak shape equivalent to ANR's

by

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Abstract. We give necessary and sufficient conditions for a compactum to be weak shape equivalent to an ANR and for a tower of CW complexes to be weakly equivalent to a CW complex in pro-homotopy.

1. Introduction. There is a remarkable similarity between the theory of shape and the theory of etale homotopy as developed by Artin and Mazur in [1]. We will exploit this connection to obtain new information about shape. We feel that our methods are of importance in shape theory quite apart from the use we make of them. Therefore, when the paper is not self-contained, we try to introduce the reader to the supporting material. Shape only appears in § 6 but the reader who knows shape may be surprised to find that much of the earlier sections is familiar.

In our main theorem, Theorem 2 of Section 6, we give necessary and sufficient conditions for a movable compact metric space (movable compactum) to be weak shape equivalent to a metric absolute neighborhood retract (ANR). Since the ANR's need not be compact, our theorem is given in terms of Fox's theory of shape for metric spaces. The terms "movable" and "weak shape equivalent" will be defined in Section 6. The class of movable compacta, introduced by Borsuk in [3], is a large and interesting class of compacta. Weak shape equivalence is defined by analogy with weak homotopy equivalence. There is a related notion which we call "very weak shape equivalence." A very weak shape equivalence is a shape morphism which induces isomorphisms on the (inverse limit) shape groups. Theorem 2 states that *every* connected movable compactum is very weak shape equivalent to an ANR, and that the word "very" may be dropped if and only if the shape groups, when topologized with the inverse limit topology, turn out to be discrete. The latter criterion is useful as it can usually be checked. We give examples in Section 7.

We give an alternative version of our theorem (Theorem 1 of Section 5) in the language of etale homotopy. The notion of movability is useful there too. As a corollary to Theorem 1 we are able to give a partial answer to a question posed by Sullivan.

That there is a connection between shape and etale homotopy has been noticed independently by T. Porter [17] and S. Mardešić [13]. We are grateful for helpful conversations with Louis Mahony, Prabir Roy, Dennis Sullivan and John Walsh, and for correspondence with Tim Porter.

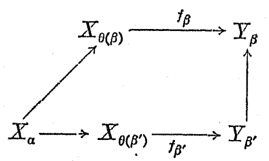
Notation. Throughout the paper, N denotes the natural numbers.

2. Pro-categories. The concept of pro-category was first introduced by Grothendieck in [7]. We will give the definition and then explain its connection with shape theory.

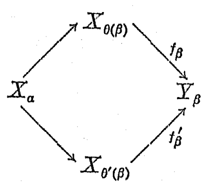
If C is any category one can form a new category pro(C) whose objects are inverse systems $\{X_a\}_{a\in A}$ of objects of C indexed by directed sets A. But the morphisms are not ordinary morphisms of inverse systems. In formal language the *set* of morphisms from $\{X_a\}_{a\in A}$ to $\{Y_\beta\}_{\beta\in B}$ is

$$\lim_{\overleftarrow{\beta}}\lim_{\overrightarrow{a}} C(X_a, Y_\beta)$$

where $C(X_a, Y_\beta)$ is the set of morphisms in C from X_a to Y_β. Since this definition gives no insight we will give another. To define a morphism from $\{X_a\}_{a\in A}$ to $\{Y_\beta\}_{\beta\in B}$ one prescribes a function $\theta\colon B\to A$ (which need not be order preserving) and morphisms $f\colon X_{\theta(\beta)}\to Y_\beta$ of C for each $\beta\in B$, subject to the condition that if $\beta\leqslant\beta'$ in B then for some $a\in A$ such that $a\geqslant\theta(\beta)$ and $a\geqslant\theta(\beta')$ the diagram



commutes (the unmarked maps are the structural or bonding maps of the inverse systems). But one identifies the morphisms $(\theta; f_\beta)$ and $(\theta'; f_{\beta'})$ if for each β there is an $a\in A$ such that $a\geqslant\theta(\beta)$, $a\geqslant\theta'(\beta)$ and the diagram



commutes. It is not hard to check (i) that the identification of morphisms in the above definition is an equivalence relation which respects composition; (ii) that C naturally embeds in pro(C), an object of C being regarded

as an inverse system over a one-element directed set; and (iii) that a (covariant) functor $T\colon C_1\to C_2$ induces a functor, also denoted by T, from pro(C_1) to pro(C_2).

An account of pro-categories can be found in the Appendix to the Artin–Mazur notes [1]. They allow A to be a "filtering category" (more general than a directed set) but we will never need the greater generality. The definition of pro(C) given here is taken from [2].

We shall wish to pass from an inverse system to a cofinal subsystem. We therefore observe:

PROPOSITION 1. *Let A be a directed set and B a cofinal directed subset. If $\{X_a\}_{a\in A}$ is an object of* pro(C), *it is equivalent in* pro(C) *to the object* $\{X_a\}_{a\in B}$.

A proof is given on page 150 of [1]: it is routine.

For the reader familiar with shape, here is an example. Let C be the category of compact ANR's and homotopy classes of maps. If one takes an "ANR system" in the sense of [13], and replaces each bonding map by its homotopy class, there results an object of pro(C). A "homotopy class of maps between ANR-systems" as defined in [11] is then precisely a morphism of pro(C).

The discussion in § 2 of [6] is also closely related.

3. Movability and the Mittag-Leffler condition. Let $\{X_a\}_{a\in A}$ be an object of pro(C), C being any category. This object is said to be *movable* if for each $a\in A$, there exists $\beta\geqslant a$ in A such that for every $\gamma\geqslant a$ there is a morphism $r_{\beta\gamma}\colon X_\beta\to X_\gamma$ making $p_{\gamma a}\circ r_{\beta\gamma}=p_{\beta a}$ where the p's are bonding morphisms. If C is the category of groups it is clear that a movable pro-group satisfies the *Mittag-Leffler condition*:

(ML) for each a, there exists $\beta\geqslant a$ such that for all $\gamma\geqslant\beta$ the bonding homomorphisms $p_{\gamma a}\colon X_\gamma\to X_a$ have the same image.

In [2], Atiyah and Segal observe that a pro-group satisfying (ML) may be identified with its topologized inverse limit in the following sense.

Let Q: (pro-groups) $\to$ (topological groups) be the functor which associates with each pro-group $\{X_a\}_{a\in A}$ its inverse limit $\varprojlim X_a$ topologized as a subgroup of the product ΠX_a where each X_a is given the discrete topology. Let P: (topological groups) $\to$ (pro-groups) be the functor which associates with each topological group X the pro-group $\{X/I_a\}$ where $\{I_a\}$ is the family of all open subgroups of X. Then one has (see [2])

PROPOSITION 2. *Let G be a group, $\{G_n\}_{n\in N}$ a pro-group satisfying* (ML) *and $f\colon G\to\{G_n\}$ a morphism of pro-groups. Then there is an equivalence $h\colon\{G_n\}\to PQ(\{G_n\})$ of pro-groups such that $h\circ f=PQ(f)$, where G is identified with $PQ(G)$.*

1*

Proof (sketch). Let $I_k = \{(x_i) \in \varprojlim\{G_n\} |\ x_i = 0,\ 1 \leqslant i \leqslant k\}$. The collection $\{I_k\}_{k \in N}$ is a basic system of open subgroups of the topological group $\varprojlim\{G_n\}$. The naturally defined inverse system $\{\varprojlim\{G_n\}/I_k\}_{k \in N}$ is cofinal in $PQ(\{G_n\})$. But the group $\varprojlim\{G_n\}/I_k$ is clearly isomorphic to the eventual image in G_k of the Mittag-Leffler system $\{G_n\}$. Thus the required equivalence h can be constructed easily.

For more information about the functors P and Q, see § 2 of [2].

The following properties of movability are obvious and will be used without comment in what follows. (i) Let A be a directed set and B a cofinal directed subset; $\{X_a\}_{a \in A}$ is a movable object of $\text{pro}(C)$ if and only if $\{X_a\}_{a \in B}$ is. (ii) If T: $C_1 \to C_2$ is a covariant functor, then T maps movable objects of $\text{pro}(C_1)$ to movable objects of $\text{pro}(C_2)$. (iii) If in (ii) C_2 is the category of groups, then T maps movable objects of $\text{pro}(C_1)$ to pro-groups satisfying (ML).

4. The simplicial inverse limit construction. The background for what follows can be found in Chapter VIII and §§ 2 and 3 of Chapter IX of the Bousfield-Kan notes [5]. The principal idea is to associate with an inverse system of ANR's an "inverse limit" which commutes with the homotopy functors π_i. The topological inverse limit does not have this property. But by passing to simplicial sets, such a construction is possible. We summarize.

Let H_0 be the category of connected pointed CW complexes and homotopy classes of maps. Let $\{X_n\}$ be an object of $\text{pro}(H_0)$ indexed by the natural numbers N. By the singular functor one may pass to the homotopy category of connected pointed simplicial sets: one obtains an inverse system of simplicial sets which up to isomorphism can be replaced by a system $\{\hat{X}_n\}$ of simplicial fibrations (to be precise, one uses Axiom CM 5 of [5], p. 242). Let X be the simplicial inverse limit. Denoting the projection $\hat{X} \to \{\hat{X}_n\}$ by φ, there is (by Theorem 3.1 of [5], p. 254) for each $i \geqslant 1$ a short exact sequence of groups and of pointed sets if $i = 0$!

$$0 \to \varprojlim{}^1\{\pi_{i+1}(\hat{X}_n)\} \to \pi_i(\hat{X}) \xrightarrow{\varphi_*} \varprojlim\{\pi_i(\hat{X}_n)\} \to 0$$

where $\varprojlim{}^1$ is a functor which vanishes on pro-groups satisfying (ML) (see 3.5, p. 256 of [5]). If we apply the geometric realization functor $|\cdot|$, the sequence will still be exact. But by Chapter VIII § 3 of [5], the system $\{|\hat{X}_n|\}$ is isomorphic in $\text{pro}\, H_0$ to $\{X_n\}$. If the original system $\{X_n\}$ is movable, the pro-groups $\{\pi_i(|\hat{X}_n|)\}$ will satisfy (ML) (see the end of Section 3). In summary we have

PROPOSITION 3. *Let $\{X_n\}_{n \in N}$ be a movable object of* $\text{pro}(H_0)$. *Then there is a connected (pointed) simplicial set $\hat{X}$ and a morphism f: $|\hat{X}| \to \{X_n\}$ of* $\text{pro}(H_0)$ *such that f_*: $\pi_i(|\hat{X}|) \to \varprojlim\{\pi_i(X_n)\}$ is an isomorphism of groups for all $i \geqslant 1$.*

Remark. The bonding morphisms of objects of $\text{pro}(H_0)$ are homotopy classes of maps. In general it is not possible to pass from such an object to an inverse system with topological (or simplicial) bonding maps, as we must do when we make the bonding maps fibrations. But when the indexing set is N this transition from homotopy inverse systems to topological or simplicial "representatives" is clearly possible.

5. Our main theorem in $\text{pro}(H_0)$. Again H_0 denotes the category of pointed, connected CW complexes and homotopy classes. A morphism f: $\{X_a\}_{a \in A} \to \{Y_\beta\}_{\beta \in B}$ of $\text{pro}(H_0)$ will be called a *weak equivalence* if for each $i \geqslant 1$ the induced morphism $\pi_i(f)$: $\{\pi_i(X_a)\} \to \{\pi_i(Y_\beta)\}$ is an equivalence of pro-groups. Artin and Mazur give an example (p. 35 of [1]) of a weak equivalence which is not an equivalence of $\text{pro}(H_0)$: see also Sec. 7, Example 2 below. However, weak equivalences induce equivalences of homology pro-groups and cohomology groups (pp. 36–37 of [1]) and are therefore almost as good as equivalences for algebraic purposes.

By contrast, let us call the morphism f a *very weak equivalence* if the induced morphism of groups from $\varprojlim\{\pi_i(X_a)\}$ to $\varprojlim\{\pi_i(Y_\beta)\}$ is an isomorphism for each i. Proposition 3 then says that for any movable object $\{X_n\}_{n \in N}$ of $\text{pro}(H_0)$ there is a CW complex $|\hat{X}|$ and a very weak equivalence from $|\hat{X}|$ to $\{X_n\}$. When can we remove the word "very" from the last statement? The answer is:

THEOREM 1. *Let $\{X_n\}_{n \in N}$ be a movable object of* $\text{pro}(H_0)$. *There exists a connected (pointed) simplicial set $\hat{X}$ and a weak equivalence f: $|\hat{X}| \to \{X_n\}$ if and only if for each $i \geqslant 1$ the topologized group $\varprojlim\{\pi_i(X_n)\}$ is discrete.*

Proof. The "only if" part follows from the fact that Q, defined in Section 3, is a functor (movability is not needed). The "if" part is proved as follows. Let $|\hat{X}|$ and f be as in Proposition 3. f induces $\pi_i(f)$: $\pi_i(|\hat{X}|) \to \{\pi_i(X_n)\}$, a morphism between pro-groups which satisfy (ML). We apply $P \circ Q$ as in Section 3. $Q\pi_i(f) = f_*$ is an isomorphism of groups by Proposition 3, and of topological groups by the discreteness hypothesis. Thus $PQ\pi_i(f)$ is an equivalence of pro-groups. Proposition 2 then implies that $\pi_i(f)$ is also an equivalence.

As a corollary we have a partial answer to a question put to us by D. Sullivan.

COROLLARY. *If $\{X_n\}_{n \in N}$ is a movable object of* $\text{pro}(H_0)$ *such that $\varprojlim\{\pi_i(X_n)\}$ is discrete for all $i \geqslant 1$, then there is a CW complex $|\hat{X}|$ and a morphism f: $|\hat{X}| \to \{X_n\}$ of* $\text{pro}(H_0)$ *which induces an isomorphism between the singular cohomology of $|\hat{X}|$ and the Čech cohomology of $\{X_n\}$ (i.e., the direct limit cohomology of $\{X_n\}$).*

An example is given in Section 7 to illustrate the difference between weak and very weak equivalences.

6. Our Main Theorem in shape. The theorem (Theorem 2 below) would be most easily stated in terms of a "Čech shape theory" in which one would associate with each pointed connected space the natural object of $\mathrm{pro}(H_0)$ generated by the (pointed) nerves of all open covers of the space. But since no such theory exists in the literature we formulate our theorem in terms of Fox's theory of shape [6]. We shall now give the necessary definitions.

Let M_0 be the pointed category of connected closed subsets of (metric) ANR's, and maps. Let ANR_0 be the pointed category of connected metric ANR's and homotopy classes of maps. A typical object of M_0 is $(X, x_0) \subset (Z, x_0)$ where Z is an ANR. The set of all (pointed) connected open neighborhoods of X in Z, $\{(X_a, x_0)\}$ is an inverse system of ANR's directed by inclusion. If each inclusion map is replaced by its (pointed) homotopy class, one obtains an object $\{(X_a, x_0)\}$ of $\mathrm{pro}(\mathrm{ANR}_0)$. The Fox shape functor $S\colon M_0 \to \mathrm{pro}(\mathrm{ANR}_0)$ takes $(X, x_0) \subset (Z, x_0)$ to this object. S is defined on morphisms in the obvious way. A *shape morphism* from (X, x_0) to (Y, y_0) is a morphism of $\mathrm{pro}(\mathrm{ANR}_0)$ from $S(X, x_0)$ to $S(Y, y_0)$ (the ANR superspaces being suppressed). Two objects of M_0 *have the same pointed shape* if their S-images are equivalent in $\mathrm{pro}(\mathrm{ANR}_0)$.

This is essentially Fox's idea of shape. In § 3 of [6] he shows that, up to equivalence in $\mathrm{pro}(\mathrm{ANR}_0)$, $S(X, x_0)$ is independent of Z and of the embedding of X in Z. We will therefore suppress Z from now on. In § 4 of [6] he shows that on compact metric spaces, his definition of "having the same shape" agrees with Borsuk's.

If X is an ANR, we adopt the convention that $S(X, x_0)$ is the trivial inverse system $\{(X, x_0)\}$.

A connected pointed compactum (X, x_0) is called *movable* if $S(X, x_0)$ is a movable object of $\mathrm{pro}(\mathrm{ANR}_0)$; see Sec. 3. It follows as in [14] that this definition is independent of the ANR in which X is embedded etc. and is equivalent to Borsuk's original definition [3], [4]. Mardešić has shown that if X is n-dimensional and locally $(n-1)$-connected then (X, x_0) is movable, see [16] for a simple proof. Solenoids are not movable. Borsuk [4] has shown that if X is movable then $S(X, x_0)$ is independent of x_0 up to equivalence in $\mathrm{pro}(\mathrm{ANR}_0)$: his proof is geometrical but our statement follows from the equivalence of definitions.

By our convention, $S(X, x_0)$ is movable in $\mathrm{pro}(\mathrm{ANR}_0)$ whenever X is an ANR.

If $S(X, x_0) = \{(X_a, x_0)\}$ the ith *shape group* of (X, x_0) is $\varprojlim_a \{\pi_i(X_a, x_0)\}$.

As in Section 3 we may give these groups the inverse limit topology. The notions of *weak equivalence* and *very weak equivalence* in $\mathrm{pro}(\mathrm{ANR}_0)$ are defined as in $\mathrm{pro}(H_0)$, see Sec. 5. A shape morphism from (X, x_0) to (Y, y_0) is a *weak shape equivalence* (respectively *very weak shape equivalence*) if it is a weak equivalence (respectively very weak equivalence) of $\mathrm{pro}(\mathrm{ANR}_0)$.

THEOREM 2. *Let (X, x_0) be a connected pointed movable compactum. There is a connected pointed* ANR, *(Y, y_0), and a very weak shape equivalence $f\colon (Y, y_0) \to (X, x_0)$. Moreover, Y and f may be chosen so as to make f a weak shape equivalence if and only if the topologized shape groups of X are discrete.*

Proof. Let X be embedded in some ANR. Since X is compact the inverse system $S(X, x_0)$ has a cofinal subsystem $\{(X_n, x_0)\}_{n \in N}$ to which $S(X, x_0)$ is equivalent in $\mathrm{pro}(\mathrm{ANR}_0)$ by Proposition 1. Since $S(X, x_0)$ is movable so also is $\{(X_n, x_0)\}$. From now on we suppress base points. Let $G\colon \mathrm{ANR}_0 \to H_0$ be the functor which assigns to each ANR the geometric realization of its singular complex, and to each homotopy class of maps the corresponding homotopy class of realized simplicial maps (see for example [11]). By Proposition 3, there is a simplicial set $\hat{X}$ and a very weak equivalence $f\colon |\hat{X}| \to \{G(X_n)\}$ in $\mathrm{pro}(H_0)$. The CW complexes $|\hat{X}|$ and $G(X_n)$ can be triangulated ([11], p. 100) and can therefore be given metrics which do not alter their homotopy types ([11], pp. 128, 131). With such metrics they become ANR's ([9], p. 106). f and the bonding morphisms come from simplicial maps and so need not be changed: f becomes a very weak equivalence in $\mathrm{pro}(\mathrm{ANR}_0)$. The system $\{G(X_n)\}$, with metric topology is equivalent in $\mathrm{pro}(\mathrm{ANR}_0)$ to $\{X_n\}$ ([11], pp. 126, 131). Thus the first part of the theorem is proved. For the second part, one proves "only if" as in Theorem 1; to prove "if" one observes that, by Theorem 1, f is a weak equivalence when regarded as a morphism of $\mathrm{pro}(H_0)$ and so, by the above discussion, f remains a weak equivalence in $\mathrm{pro}(\mathrm{ANR}_0)$.

7. Examples, questions and remarks. Clearly shape equivalence implies weak shape equivalence implies very weak shape equivalence. With Examples 1 and 2 we show that neither of the reverse implications holds even for connected movable compacta. We then show, by Example 3, that movability in $\mathrm{pro}(H_0)$ is not preserved under weak equivalence. We pose some questions, and conclude the section with a remark on the "shape Hurewicz theorem" of K. Kuperberg.

Throughout this section we suppress base points.

EXAMPLE 1. Let T^ω be the countably infinite product of circles. $S(T^\omega)$ has a cofinal sub-system of the form $\{T^n\}_{n \in N}$ where T^n is the n-fold product of open annuli in the plane, and the bonding map from $T^{n+1} \to T^n$ is the obvious projection map. This is a movable object of $\mathrm{pro}(\mathrm{ANR}_0)$, hence T^ω is movable; see Sec. 3. The first shape group of T is not discrete.

By Theorem 2, there is an ANR, Y and a very weak shape equivalence $f\colon Y \to T^\infty$, but Y and f cannot be chosen so as to make f a weak shape equivalence.

EXAMPLE 2. Let $X = \prod_{n>0} S^n$ where S^n is the n-dimensional sphere. By examining a suitable inverse system $\{X_n\}$ as in Example 1, one can see that X is movable. This time the shape groups are all discrete. Hence, by Theorem 2, there is an ANR Y and a weak shape equivalence $f\colon Y \to X$. But f cannot be a shape equivalence. To see this let $\pi_*\colon \mathrm{ANR}_0 \to$ (groups) be the functor which assigns to each object Z the group $\prod_{i>0} \pi_i(Z)$ which is the direct product of the homotopy groups. One easily shows that the induced functor on pro-categories takes $\{X_n\}$ to a pro-group satisfying (ML) whose topologized inverse limit group is not discrete. The discussion in Section 3 implies that $\{X_n\}$ cannot be equivalent in pro(ANR_0) to an object of ANR_0.

EXAMPLE 3. Let X_n be the wedge (= one-point union) of spheres $\bigvee_{k>n} S^k$. Let $\{X_n\}_{n\in N}$ be the object of pro(H_0) whose bonding maps are generated by inclusions. All the homotopy pro-groups vanish: hence there is a weak equivalence from a point to $\{X_n\}$. Let $\{X_n\}$ is not movable. To see this, let $H_*\colon H_0 \to$ (groups) be the functor which assigns to each object Z the group $\prod_{i>0} H_i(Z)$. The induced functor on pro-categories takes $\{X_n\}$ to a pro-group which does not satisfy (ML).

EXAMPLE 4. There is also a non-movable compactum which is weak equivalent to a point: it was constructed by D. Kahn; see [8].

QUESTION 1. What additional hypotheses are needed in order to reverse the implications with which we opened this section?

QUESTION 2. Is there a (non-movable) compactum which is not very weak shape equivalent to an ANR? In this connection, see Sullivan's example on page 3.4 of [18].

QUESTION 3. Examples 2 and 3 suggest a possible strengthening of the definition of weak equivalence: f is a weak equivalence if and only if $\pi_i(f)$ $(i \geq 1)$ and $\pi_*(f)$ are equivalences of pro-groups. With this definition is there a true pro(H_0) Whitehead theorem? (i.e., is every weak equivalence an equivalence?)

CONCLUDING REMARK. We point out again that weak equivalences induces isomorphisms on homology both at the pro-group and inverse limit levels, and on direct limit cohomology ([1], pp. 36–37). If HH is the homotopy category of CW pairs, the homology pro-groups of objects of pro(HH) always satisfy the Exactness Axiom. Not so the inverse limit (or Čech) homology groups: but Overton [15] has shown that on movable pairs of compacta Čech homology is exact. This is because movability implies (ML), and the inverse limit functor on groups is exact in the presence of (ML). Similar remarks apply to homotopy exact sequences.

K. Kuperberg [10] has proved a "shape Hurewicz theorem" for movable connected pointed compacta: the vanishing of the first $n-1$ Čech homology groups and shape groups implies the isomorphism of the nth. We should point out that the stronger "pro(H_0) Hurewicz theorem" proved by Artin and Mazur in [1] and the discussion in § 3 above together give an alternative proof of Kuperberg's theorem.

Added in proof, November, 1975. Since this paper was written, two and a half years ago, much progress has been made. See our papers *Shapes of complexes, ends of manifolds, homotopy limits and the wall obstruction*, Ann. Math. 101 (1975), pp. 521–535, *The stability problem in shape and a Whitehead theorem in pro-homotopy*, Trans. Amer. Math. Soc. (to appear). *Infinite dimensional Whitehead and Vietoris theorems in shape and pro-homotopy*, Trans. Amer. Math. Soc. (to appear), and *The stability problem in shape and pro-homotopy* (submitted).

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О нерастягивающих отображениях компактов

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Abstract. Two sufficient conditions on the compact K, $K \subset E^n$, for which a non-expansive mapping $f\colon K \to K'$ has a fixed point, are given.

В этой заметке рассматриваются так называемые нерастягивающие отображения. Отображение f метрического пространства X в метрическое пространство Y называется *нерастягивающим*, если $\varrho(x, y) \geqslant \varrho(f(x), f(y))$. Из этого условия сразу следует и непрерывность.

Известны примеры (см. напр. [1], [2], [3], [4]) ациклических компактов K, которые не обладают свойством неподвижной точки, т. е. существует отображение $f\colon K \to K$ такое, что $f(x) \neq x$ для всех $x \in K$. Вопрос о существование таких континуумов на плоскости, а также одномерных континуумов, остается открытым. С другой стороны хорошо известно, что для сжимающего отображения (т. е. $\varrho(f(x), f(y)) \leqslant \lambda\varrho(x, y)$, где $\lambda < 1$ и не зависит от x и y) полного метрического пространства существует единственная неподвижная точка. Интересно знать имеют ли неподвижные точки нерастягивающие отображения ациклических компактов. Оказывается, что при некоторых условиях, которые всегда выполнены для ациклических континуумов на плоскости, это так.

Можно пойти дальше в этом направлении и поставить следующий вопрос. Пусть K — компакт в евклидовом пространстве E^n, все группы гомологий (1) которого имеют конечное число образующих. Пусть далее f — нерастягивающее отображение K в себя, для которого число Лефшеца $\Lambda_f \neq 0$. Будет ли в этом случае отображение f иметь неподвижную точку, т. е. такую точку, что $f(x) = x$?

Теорема 2 показывает, что это так, если расположение компакта K в E^n удовлетворяет некоторым условиям.

Теорема 1 относится к ациклическим континуумам.

ТЕОРЕМА 1. *Пусть K — компакт в пространстве E^n для которого существуют такие сколь угодно малые ε и окрестности U_ε компакта K, что*

(1) Имеются ввиду гомологии Александрова.