On the number of generic models

by

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Abstract. Let T be a complete theory in a countable fragment $\mathfrak{L}_B$. Suppose A is a countable admissible set with $T \in A$. Then either (i) for each countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$ there is a unique $\mathfrak{L}_D$-generic model of T up to isomorphism, or (ii) for each countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$, there are continuum many $\mathfrak{L}_D$-generic models of T up to isomorphism.

Suppose M is a countable model of ZF and $P \in M$ is a partial ordering. It is well known, and easy to see, that there will be a subset of P in M generic over M, just in case there is a condition $p \in P$ such that all conditions greater than p are compatible. If there is no subset in M generic over M, then there will be continuum many different generic sets. We will show that a similar phenomenon occurs in model theoretic forcing. The obvious difficulty in going from the result about generic sets to generic models is that distinct generic sets will give rise to isomorphic generic models.

It is assumed that the reader is already familiar with the basic concepts involved in infinitary model theory, admissible sets, and forcing. Our treatment of these facts is almost self-contained with respect to forcing, is fairly suggestive in regard to the relevant notions of infinitary model theory, and is non-existent in the area of admissible sets and in particular the Lévy hierarchy. Any necessary background in these matters may be obtained through [1] or [4].

Our notation is quite standard, and is in keeping with [3]. We generally denote a structure by $\mathfrak{M}$, and its universe by M. The set of hereditarily countable sets, i.e. those sets whose transitive closure is countable, is denoted by HC. The set of all sets of rank less than some ordinal β is denoted by $R(\beta)$. Definitions of such notions as "fragment" can be found in [3].

Finally, we assume the reader is aware of how certain of what we do informally can be done formally within set theory, e.g. coding of formulas as sets, etc. since such an ability is implicit in the proof of our main result, Theorem 2.

§ 1. The concept of forcing was first utilized by Cohen to settle important questions of set theory. Subsequently, it was seen by Barwise [1] that this technique of "set-theoretic" forcing could be used in a weaker setting to construct new admissible sets from old ones. Latter, A. Robinson [5] considered applications of forcing in the model theory of finitary languages. This subject was pursued in [2] and elsewhere. Subsequently Barwise (unpublished) considered forcing in infinitary logic. Latter still, Keisler [3] employed forcing to re-examine some central questions in the model theory of the inifinitary language $\mathfrak{L}_{\omega_1\omega}$. In this paper we employ the "basic result on set-theoretic forcing on admissible sets" to obtain information on more model theoretic matters.

Much of what we do below was initially motivated by [3]. We restate some of the essential definitions and results of that paper, but for simplicity we consider a special case from the beginning.

We fix a countable alphabet $\mathfrak{L}$ and a countable fragment $\mathfrak{L}_B$ of $\mathfrak{L}_{\omega_1\omega}$. In addition, we introduce a countable set C of new constant symbols, and consider a new fragment $\mathfrak{K}_B$ whose formulas may be obtained from formulas of $\mathfrak{L}_B$ by replacing finitely many free variables by constant symbols in C. $\mathfrak{K}_{\infty\omega}$ is obtained from $\mathfrak{L}_{\infty\omega}$ in a similar way.

Now, given a consistent theory T in $\mathfrak{L}_B$, we define the set of forcing conditions, $\boldsymbol{P}$ to be the set of all finite sets S of sentences of $\mathfrak{K}_B$ such that $T \cup S$ is consistent.

For any condition $p \in \boldsymbol{P}$, and sentence $\varphi \in \mathfrak{K}_{\infty\omega}$, we define the forcing relation

$$p \Vdash \varphi,$$

by induction on the formation of φ:

$p \Vdash \varphi$ iff $\varphi \in p$ for φ atomic,

$p \Vdash \neg\varphi$ iff there is no $q \in \boldsymbol{P}$ such that $q \supseteq p$ and $q \Vdash \varphi$,

$p \Vdash \bigvee\Phi$ iff $p \Vdash \varphi$ for some $\varphi \in \Phi$,

$p \Vdash (\exists x)\varphi(x)$ iff $p \Vdash \varphi(c)$ for some $c \in C$.

As shorthand, we will always use p, q, r to denote elements of $\boldsymbol{P}$. Then we can state some of the basic facts about forcing familiar from set theory as

(i) if $p \Vdash \varphi$ and $q \supseteq p$, then $q \Vdash \varphi$,

(ii) $p \Vdash \bigwedge\Phi$ iff for each $\varphi \in \Phi$ and $q \supseteq p$, there is some $r \supseteq q$ such that $r \Vdash \varphi$,

(iii) $p \Vdash (\forall x)\varphi(x)$ iff for each $c \in C$ and $q \supseteq p$ there is some $r \supseteq q$ such that $r \Vdash \varphi(c)$.

The notion of weak forcing, $p \Vdash^{\omega} \varphi$ is defined by

$$p \Vdash^{\omega} \varphi \quad \text{iff} \quad p \Vdash \neg\neg\varphi.$$

Consequently,

(iv) $p \Vdash^{\omega} \varphi$ iff for each $q \supseteq p$ there is some $r \supseteq q$ such that $r \Vdash \varphi$, i.e. iff the set of conditions forcing φ is dense above p,

(v) if $p \Vdash \varphi$, then $p \Vdash^{\omega} \varphi$,

(vi) $p \Vdash^{\omega} \neg\varphi$ iff $p \Vdash \neg\varphi$,

(vii) $p \Vdash^{\omega} \bigwedge\Phi$ iff for all $\varphi \in \Phi$, $p \Vdash^{\omega} \varphi$,

(viii) $p \Vdash^{\omega} (\forall x)\varphi(x)$ iff for each $c \in C$, $p \Vdash^{\omega} \varphi(c)$,

(ix) $p \Vdash^{\omega} \bigvee\Phi$ iff for each $q \supseteq p$ there is some $r \supseteq q$ and some $\varphi \in \Phi$ such that $r \Vdash^{\omega} \varphi$,

(x) $p \Vdash^{\omega} (\exists x)\varphi(x)$ iff for each $q \supseteq p$ there is some $r \supseteq q$ and $c \in C$ such that $r \Vdash^{\omega} \varphi(c)$.

The reader will have noticed that in a certain sense $\bigvee$ and $\exists$ trade roles with $\bigwedge$ and $\forall$ upon switching from forcing to weak forcing. In particular, if for a fragment $\mathfrak{L}_D$ we define T_D^f to be the set of all sentences of $\mathfrak{L}_D$ weakly forced by 0, then for $\bigwedge\Phi \in \mathfrak{L}_D$,

$$\bigwedge\Phi \in T_D^f \quad \text{iff} \quad \varphi \in T_D^f \quad \text{for every } \varphi \in \Phi.$$

Given a fragment $\mathfrak{L}_A$, we say that a structure $\mathfrak{M}$ for $\mathfrak{L}$ is *A-generic* iff there is some mapping $c \to m_c$ of C onto M such that for each sentence $\varphi \in \mathfrak{K}_A$, $(\mathfrak{M}, m_c)_{c \in C} \Vdash \varphi$ iff for some finite set $p \subseteq \mathrm{th}_B((\mathfrak{M}, m_c)_{c \in C})$, $p \Vdash \varphi$.

Alternatively, a set $G \subseteq \boldsymbol{P}$ is said to be *A-generic* iff

(1) if $p \in G$ and $q \subseteq p$, then $q \in G$,

(2) if $p, q \in G$, then $p \cup q \in G$,

(3) for each sentence $\varphi \in \mathfrak{K}_A$,

there is some $p \in G$ such that either $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.

Given an A-generic set G, one may "effectively" construct, in the manner of Henkin's proof of the completeness theorem, an A-generic structure $\mathfrak{M}_G$, such that for the obvious canonical expansion $\mathfrak{M}_G'$ of $\mathfrak{M}_G$ to a structure for $\mathfrak{K}_A$, we have, for all $\varphi \in \mathfrak{K}_A$,

$$\mathfrak{M}_G' \Vdash \varphi \quad \text{iff} \quad p \Vdash \varphi \quad \text{for some } p \in G.$$

Conversely, every A-generic structure is $\mathfrak{M}_G$ for some A-generic set G, viz. $\mathrm{th}_B((\mathfrak{M}, m)_{m \in M})$.

Finally, using the fact that if $\mathfrak{L}_A$ is countable, every p is contained in an A-generic set, one can show

(†) for every countable fragment $\mathfrak{L}_A$, $\varphi \in \mathfrak{K}_A$, and $p \in \boldsymbol{P}$, $p \Vdash^{\omega} \varphi$ iff φ holds in $\mathfrak{M}_G'$ for every A-generic G with $p \in G$.

We wish to generalize the notion of generic set, and through it, the notion of generic structure. In doing so, we must be sure to retain the essential property expressed in (†). We make our generalization as concrete as possible.

Definition 1. By a *notion of genericity* we mean a function *, with domain the set of all countable fragments $\mathfrak{L}_D$, such that

(I) For each countable fragment $\mathfrak{L}_D$, $^*(\mathfrak{L}_D)$ is a set of D-generic sets.

(II) For each countable fragment $\mathfrak{L}_D$, and $p \in \boldsymbol{P}$, there is some $G \in {}^*(\mathfrak{L}_D)$ with $p \in G$.

(III) For each pair of countable fragment $\mathfrak{L}_{D'}$, $\mathfrak{L}_D$, such that

$$\mathfrak{L}_{D'} \supseteq \mathfrak{L}_D, \qquad {}^*(\mathfrak{L}_{D'}) \subseteq {}^*(\mathfrak{L}_D)\,.$$

We call sets $G \in {}^*(\mathfrak{L}_D)$, *$D$-* generic*, and structures isomorphic to the corresponding $\mathfrak{M}_G$, *D-* generic structures*. It again follows that $\mathfrak{M}'_G \Vdash \varphi$ iff $p \Vdash \varphi$ for some $p \in G$. Then, just as before, one can establish

(*†) for any countable fragment $\mathfrak{L}_D$, $\varphi \in \mathfrak{K}_D$, and $p \in \boldsymbol{P}$, $p \Vdash^\omega \varphi$ iff φ holds in $\mathfrak{M}'_G$, for every D-* generic G with $p \in G$.

§ 2. If we restrict our attention to sentences $\varphi \in \mathfrak{K}_B$, then since our forcing conditions p are also built up from $\mathfrak{K}_B$, the relation $p \Vdash^\omega \varphi$ is easily determined.

LEMMA 1. (i). *For $p \in \boldsymbol{P}$, $p \Vdash^\omega \bigwedge p$.*

(ii) $0 \Vdash^\omega \varphi$ *for $\varphi \in T$, whence for any notion of genericity *, every B-* generic structure is a model of T.*

(iii) $p \Vdash^\omega \varphi$ *iff $T \cup p \Vdash \varphi$, for $\varphi \in \mathfrak{K}_B$.*

Proof. (i) This is proved in [3].

(ii) Let $\varphi \in T$ and $p \in \boldsymbol{P}$. Then $p \cup \{\varphi\} \in \boldsymbol{P}$, and by (i), $p \cup \{\varphi\} \Vdash^\omega \varphi$. Thus there exists $q \supseteq p$ such that $q \Vdash^\omega \varphi$, and hence some $r \supseteq q \supseteq p$, $r \Vdash \varphi$. Therefore $0 \Vdash^\omega \varphi$.

(iii) First suppose $T \cup p \models \varphi$. Let G be any B-generic set with $p \in G$. Then, by (i) and (ii), $\mathfrak{M}'_G \models T \cup p$, whence $\mathfrak{M}'_G \models \varphi$. Therefore, $p \Vdash^\omega \varphi$.

Conversely, suppose $p \Vdash^\omega \varphi$. If not $T \cup p \models \varphi$, then $T \cup p \cup \{\neg\varphi\}$ is consistent, and so $p \cup \{\neg\varphi\} \in \boldsymbol{P}$ since $\varphi \in \mathfrak{K}_B$. Then, however, by the above, $p \cup \{\neg\varphi\} \Vdash^\omega \neg\varphi$, contradicting the fact that $p \Vdash^\omega \varphi$. ∎

From now on, we assume that T is a complete theory in $\mathfrak{L}_B$, i.e. for each sentence φ of $\mathfrak{L}_B$, either $T \models \varphi$ or $T \models \neg\varphi$, and T is consistent. In addition, we assume that A is admissible, and that $T \in A$. Our goal is to determine the number of non-isomorphic A-generic structures. As a first step we show that there is a unique A-generic structure up to isomorphism iff there is a unique D-generic structure up to isomorphism for some countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$. To do this we introduce the familiar notion of a prime model, and restate the basic facts concerning these models.

A model $\mathfrak{M} \models T$ is said to be a *prime model for T* iff $\mathfrak{M}$ can be $\mathfrak{L}_B$-elementary embedded in every model of T. A formula $\theta(x_1, \ldots, x_k)$ is said to be *complete for $\mathfrak{L}_B$ with respect to T* iff θ is consistent with T and for every formula $\varphi(x_1, \ldots, x_k)$ of $\mathfrak{L}_B$, either $T \vdash \theta \to \varphi$ or $T \vdash \theta \to \neg\varphi$. A model $\mathfrak{M} \vdash T$ is prime iff M is countable and every k-tuple of elements of M satisfies a complete formula for $\mathfrak{L}_B$ with respect to T. From this it follows, that any two prime models for T must be isomorphic. A theory T will have a prime model just in case for each formula $\varphi \in \mathfrak{L}_B$ consistent with T, there is a complete formula θ such that $T \models \theta \to \varphi$.

THEOREM 1. *Let * be a notion of genericity, T a complete theory in $\mathfrak{L}_B$, and A an admissible set with $T \in A$. Then the following are equivalent:*

(1) *T has a prime model.*

(2) *All A-* generic structures are isomorphic.*

(3) *There is some countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$ such that all D-* generic structures are isomorphic.*

Proof. The proof that (1) implies (2) follows easily from the obvious generalization of Theorem 2.1 (General form) of [3]. In fact, the unique A-*generic structure is a prime model. We only need to observe that since A is admissible, it can be shown [cf. 4] that for each k, the sentence $(\forall x_1) \ldots (\forall x_k) \bigvee \{\theta(x_1, \ldots, x_k)$: θ is a complete formula for $\mathfrak{L}_B$ with respect to $T\}$ is in $\mathfrak{L}_A$. In view of (III), (3) trivially follows from (2), and again, of course, the unique structure up to isomorphism is prime. This leaves the implication from (3) to (1), which can be proved in a manner similiar to a proof in [3]. We use the condition for the existence of a prime model stated above.

Let $\mathfrak{M}$ be the unique D-* generic structure up to isomorphism. Let $\varphi(x_1, \ldots, x_k)$ be consistent with T. Since T is complete,

$$T \models (\exists x_1) \ldots (\exists x_k)\varphi(x_1, \ldots, x_k)\,.$$

Now since $\mathfrak{M} \models T$, for some k-tuple $\langle m_1, \ldots, m_k\rangle$ of elements of M, $\mathfrak{M} \models \varphi[m_1, \ldots, m_k]$. Let Φ be the set of all formulas $\psi(x_1, \ldots, x_k)$ of $\mathfrak{L}_B$ such that $\mathfrak{M} \models \psi[m_1, \ldots, m_k]$. Consider the sentence $\tau = (\exists x_1) \ldots (\exists x_k) \bigwedge \Phi$. Choose some countable fragment $\mathfrak{L}_{D'} \supseteq \mathfrak{L}_D$ such that $\tau \in \mathfrak{L}_{D'}$. Then, since by (III) every D'-* generic structure is D-* generic, $\mathfrak{M}$ is the unique D'-generic structure up to isomorphism. Now, since $\tau \in \mathfrak{L}_{D'}$, and $\mathfrak{M} \models \tau$, there is some $p \in \boldsymbol{P}$ such that $p \Vdash \tau$. Then, for some $c_1, \ldots, c_k \in C$,

$$p \Vdash \bigwedge \{\psi(c_1, \ldots, c_k)\colon\ \psi \in \Phi\}\,,$$

whence,

$$p \Vdash^\omega \psi(c_1, \ldots, c_k) \quad \text{for each } \psi \in \Phi\,.$$

Consequently, by Lemma 1,

$$T \cup p \models \psi(c_1, \ldots, c_k); \qquad \text{or} \qquad T \models \bigwedge p \to \psi(c_1, \ldots, c_k)$$

for each $\psi \in \Phi$.

Now, suppose the constants appearing in p other than $c_1, \ldots, c_k$ are $d_1, \ldots, d_n$. Then, it is clear that

$$T \models (\exists y_1) \ldots (\exists y_n) \bigwedge p' \to \psi\,,$$

where the formulas of p' are obtained by systematically replacing d_i by y_i, a distinct variable not occuring in Φ, for each $i \leq n$, and c_j by x_j, for $j \leq k$.

We have thus succeeded in finding a complete formula θ, viz. $(\exists y_1) \dots (\exists y_n) \bigwedge p'$, such that $T \models \theta \to \varphi$, and so T has a prime model. ■

Our definition of notion of genericity is clearly stronger than we require to obtain the results to follow. For example, we could manage with * defined only on some set S of countable fragments such that for any $\varphi \in \mathfrak{L}_{\omega_1\omega}$, there is some $\mathfrak{L}_D \in S$ with $\varphi \in \mathfrak{L}_D$. Similarly, we could weaken (III) to the condition that for any $\varphi \in \mathfrak{L}_{\omega_1\omega}$ and $\mathfrak{L}_D \in S$, there is some $\mathfrak{L}_{D'} \supseteq \mathfrak{L}_D \cup \{\varphi\}$ such that D'-* genericity implies D-* genericity. In this case, part (3) of Theorem 1 would have to be altered to:

(3′) *For each $\varphi \in \mathfrak{L}_{\omega_1\omega}$ there is some $\mathfrak{L}_{D'}$ (in S) such that $\varphi \in \mathfrak{L}_{D'}$ and all D'-* generic structures are isomorphic.*

§ 3. We need to note a few facts about T^f_D. The first holds even if T is not complete.

Let ϱ be a permutation of C. For a sentence φ of $\mathfrak{K}_{\infty\omega}$, denote by φ^ϱ the sentence obtained from φ by replacing each occurrence of c in φ by $\varrho(c)$. Similarly, for $p \in \boldsymbol{P}$, define p^ϱ to be the set of all φ^ϱ such that $\varphi \in p$. Clearly p^ϱ will again be in $\boldsymbol{P}$. One can easily verify by induction on the formation of φ, that for any $p \in \boldsymbol{P}$ and sentence φ of $\mathfrak{K}_{\infty\omega}$,

$$p \Vdash \varphi \quad \text{iff} \quad p^\varrho \Vdash \varphi^\varrho .$$

In particular, if $\varphi \in \mathfrak{L}_{\infty\omega}$, then

$$p \Vdash \varphi \quad \text{iff} \quad p^\varrho \Vdash \varphi .$$

We now suppose that T is complete.

LEMMA 2. *Assume T is complete in $\mathfrak{L}_B$. Then*

(i) *If $p, q \in \boldsymbol{P}$ and p and q have no constants of C in common, then $p \cup q \in \boldsymbol{P}$.*

(ii) *If φ is a sentence of $\mathfrak{L}_{\infty\omega}$, and for some $p \in \boldsymbol{P}$, $p \Vdash \varphi$, then $0 \Vdash^\omega \varphi$.*

(iii) *For any countable fragment $\mathfrak{L}_D$, T^f_D is complete in $\mathfrak{L}_D$.*

(iv) *If $\bigvee \Phi \in T^f_D$, then $\varphi \in T^f_D$ for some $\varphi \in \Phi$.*

Proof. (i) Since $p \in \boldsymbol{P}$, and T is complete, $T \models (\exists x_1) \dots (\exists x_k) \bigwedge p'$, where the formulas of p' are obtained from those of p by substituting for constants of C distinct variables not occuring in sentences of p or q. Similarly, $T \models (\exists y_1) .. (\exists y_n) \bigwedge q'$ where q' is obtained in a similar way and the y_i's are distinct from the x_j's. Then, because of our choice of variables,

$$T \models (\exists x_1) \dots (\exists x_k)(\exists y_1) \dots (\exists y_n) \bigwedge p' \cup q' .$$

Consequently, $p \cup q$ is consistent with T, whence $p \cup q \in \boldsymbol{P}$.

(ii) Suppose $p \Vdash \varphi$ for some $p \in \boldsymbol{P}$ and $\varphi \in \mathfrak{L}_{\infty\omega}$. Let q be an arbitrary condition in $\boldsymbol{P}$. Let ϱ permute C so that all constant of C in p are mapped to constants not occuring in q. Then by our earlier observation, $p^\varrho \Vdash \varphi$. Moreover, by (i) above, $p^\varrho \cup q \in \boldsymbol{P}$. Since $p^\varrho \cup q \Vdash \varphi$, it follows that $0 \Vdash^\omega \varphi$.

(iii) Let $\mathfrak{L}_D$ be any countable fragment, and φ a sentence of $\mathfrak{L}_D$. Suppose $\varphi \notin T^f_D$, i.e. not $0 \Vdash^\omega \varphi$. Then, there is some $p \in \boldsymbol{P}$ such that $p \Vdash \neg\varphi$. Now, by (ii) above, $0 \Vdash \neg\varphi$, i.e. $\neg\varphi \in T^f_D$. Clearly T^f_D is consistent since every D-generic structure is a model of T^f_D. (Even if $\mathfrak{L}_D$ is uncountable, T^f_D is syntactically consistent.)

(iv) Suppose $\bigvee \Phi \in T^f_D$, i.e. $0 \Vdash^\omega \bigvee \Phi$. Then, there is some $p \in \boldsymbol{P}$ such that $p \Vdash \bigvee \Phi$. For some $\varphi \in \Phi$, we then have $p \Vdash \varphi$. Now, by (ii) above, $0 \Vdash^\omega \varphi$, i.e. $\varphi \in T^f_D$.

LEMMA 3. *If T is complete in $\mathfrak{L}_B$, and A is countable admissible with $T \in A$, then for any notion of genericity *, and fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$, either there is exactly one D-* generic model of T up to isomorphism, or there are uncountably many.*

Proof. For each structure it is possible to find a sentence of $\mathfrak{L}_{\infty\omega}$, called a Scott sentence, which characterizes the structure up to elementary equivalence in $\mathfrak{L}_{\infty\omega}$. In the case of a countable structure, a Scott sentence characterizes it up to isomorphism with respect to countable structures.

Assume that up to isomorphism, there are countably many D-* generic structures. Choose a Scott sentence σ for each isomorphism type of D-* generic model. Each σ can be chosen in $\mathfrak{L}_{\omega_1\omega}$. Let Σ be the set of all such σ. The sentence $\bigvee \Sigma$ is in $\mathfrak{L}_{\omega_1\omega}$, and so we may choose a countable fragment $\mathfrak{L}_{D'}$ with $\bigvee \Sigma$ in $\mathfrak{L}_{D'}$. Since every D'-* generic structure is a D-* generic structure, we may conclude that $\bigvee \Sigma \in T^f_{D'}$. Then by (iv) of the preceding lemma, for some $\sigma \in \Sigma$, $\sigma \in T^f_{D'}$. Hence all D'-* generic structures satisfy the Scott sentence σ and so are isomorphic. Finally, by Theorem 1, all D-* generic structures are isomorphic. ■

§ 4. We now set about the task of trying to replace "uncountably many" by "continuum many" in Lemma 3. We will consider a particular notion of genericity.

A set $S \subseteq \boldsymbol{P}$ is said to be *dense* iff for each $p \in \boldsymbol{P}$ there is some $q \in S$ such that $p \subseteq q$. For any sentence φ of $\mathfrak{K}_{\infty\omega}$, let $S_\varphi = \{p \in \boldsymbol{P}\colon p \Vdash \varphi$ or $p \Vdash \neg\varphi\}$. Then since given any $p \in \boldsymbol{P}$, either $p \Vdash \neg\varphi$ or there is some $q \supseteq p$ such that $q \Vdash \varphi$, S_φ is dense.

Now suppose that $\boldsymbol{P} \in A$, an admissible set. This will be the case if $T \in A$ and we choose $C \in A$. For $p \in \boldsymbol{P}$ and φ a sentence of $\mathfrak{K}_A$, the forcing relation "$p \Vdash \varphi$" is defined recursively by a $\underline{\Delta}$ relation. Therefore the relation "$p \Vdash \varphi$" is itself $\underline{\Delta}$-definable on A. In particular then, for each sentence φ of $\mathfrak{K}_A$, $S_\varphi \in A$ by $\underline{\Delta}$-separation. Therefore, any set $G \subseteq \boldsymbol{P}$ which satisfies the filter properties (1) and (2) of the definition of generic, and which intersects every dense subset of $\boldsymbol{P}$ in A, will be A-generic.

Now, we further assume that A is countable. We define a function S on countable fragments as follows:

$S(\mathfrak{L}_D) = \{G \subseteq \boldsymbol{P}$: G satisfies conditions (1) and (2) in the definition of generic and G intersects every dense set contained in every admissible set containing $\mathfrak{L}_D \cup \mathfrak{L}_A$ as a subset$\}$.

It then follows from our observation above, that S is a notion of genericity. The rather unaesthetic definition of S is caused by sets which are either not admissible or do not contain $\boldsymbol{P}$.

Our reason for considering S is expressed in the following result of [1].

LEMMA 4. *Suppose A is countable admissible, $P \in A$, and G is A-^{s} generic. Then G is contained in an admissible set with the same ordinals as A.*

Lemma 4 is very useful since we may "effectively" construct $\mathfrak{M}_G$ from G and so $\mathfrak{M}_G$ will also be in an admissible set with the same ordinals as A. Having placed $\mathfrak{M}_G$ in such an admissible set A', it follows [cf. 4] that css($\mathfrak{M}_G$), the canonical Scott sentence of $\mathfrak{M}_G$ has quantifier rank at most $o(A')+\omega$, where $o(A')$ is the smallest ordinal not in A', and "set theoretical" rank less than some fixed ordinal of $(A')^+$, the smallest admissible set containing A', depending upon how one codes formulas as sets.

By using canonical Scott sentences, we get only one Scott sentence for each isomorphism type, and so we may count these instead of the isomorphism types of the structures themselves. We should like to make use of the following generalization of a familiar result on analytic sets of reals.

LEMMA 5. *For any countable ordinal β, every $\underset{\sim}{\Sigma}$-definable subset of $HC \cap R(\beta)$ has the cardinality of the continuum, or is countable.*

We now state the main result and begin the final stages of the argument.

THEOREM 2. *Let T be a complete theory in $\mathfrak{L}_B$, $T \in A$, a countable admissible set. Then either*

(i) *there is a unique D-generic model of T up to isomorphism for each countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$, or*

(ii) *there are continuum many non-isomorphic D-generic models of T for each countable fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$.*

Proof. In view of Theorem 1, it is sufficient to "correctly" compute the number of non-isomorphic A-generic models.

By Lemma 3, if there are only countably many non-isomorphic A-generic models of T, there is a unique one.

Let us suppose, to the contrary, that there are uncountably many A-generic models up to isomorphism. Then in view of Lemma 3 and Theorem 1, there will be uncountably many A-^{s} generic models up to isomorphism.

Let Γ be the set of all canonical Scott sentences of A-^{s} generic structures. We can easily have $\boldsymbol{P} \in A$, by choosing $C \in A$. Hence we already know that there is some countable ordinal β such that $\Gamma \subseteq HC \cap R(\beta)$. In order to apply Lemma 5, we need to verify that Γ is actually $\underset{\sim}{\Sigma}$-definable. Choose $\beta > o(A)$ so that $\Gamma \in HC \cap R(\beta)$. The property of being the canonical Scott sentence of a structure is easily seen to be $\underset{\sim}{\Sigma}$, and $\mathfrak{M}_G$ is $\underset{\sim}{\Sigma}$-definable from G. All we really need to check is that the property of being an A-^{s} generic set is $\underset{\sim}{\Sigma}$. This in turn follows easily since,

"G is A-^{s} generic iff $G \subseteq \boldsymbol{P}$, G satisfies conditions (1) and (2) in the definition of generic, and $(\forall S \in A)$ [If $S \subseteq \boldsymbol{P}$ and S dense, then $G \cap S \neq 0$]", and this is only $\underset{\sim}{\Delta}_0$ over $HC \cap R(\beta)$.

Now, by Lemma 5, Γ has the cardinality of the continuum, i.e. there are continuum many non-isomorphic A-^{s} generic structures. Since every A-^{s} generic structure is also A-generic, we have obtained the desired result.

In addition, we easily obtain the following

COROLLARY. *Let T be a complete theory in $\mathfrak{L}_B$, $T \in A$, countable admissible. Then either*

(i) *there is an A-generic model of T whose canonical Scott sentence has quantifier rank less than $o(A)$, or*

(ii) *there are continuum many non-isomorphic A-generic models of T with canonical Scott sentences of quantifier rank at most $o(A)+\omega$.*

One may also observe that, in case (ii) of Theorem 2, for each fragment $\mathfrak{L}_D \supseteq \mathfrak{L}_A$, no D-generic model can have a Scott sentence in $\mathfrak{L}_D$. In general then, there will be $\mathfrak{L}_A$-generic models of T which are not $\mathfrak{L}_D$-generic.

It is clear that Theorem 2 and its corollary can be immediately extended to certain other notions of genericity *.

The reader familiar with Martin's Axiom will easily see that (keeping $\mathfrak{L}_B$ countable) it would allow us to extend these results to all fragments $\mathfrak{L}_D$ of cardinality less than the continuum. In this case, Theorem 2 could be obtained immediately in Lemma 3.

References

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