A new proof of Lusin's theorem.

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Lusin's theorem [1]) states, roughly, that a measurable function is continuous on a subset of its domain of definition which approximates in measure as near as one wishes to that of the domain. The proof will rest on the following criterion for continuity [2]):

A necessary and sufficient condition that $f(x)$ defined on a set A be continuous is that, for every c of a set everywhere-dense in the real numbers, the sets where $f(x) \geqq c$ and $f(x) \leqq c$ be closed relative to A.

A set M is closed relative to N if $M = NF$ where F is a closed set.

Lusin's Theorem: *If $f(x)$ is a measurable function defined on the interval I: $0 \leqslant x \leqslant 1$, then for every $\varepsilon > 0$ there is a set $A \subset I$ such that $f(x)$ is continuous on A and $m(I - A) < \varepsilon$.*

Proof: Let ε be any positive number.

Let $c_1, c_2, \ldots c_n, \ldots$ be the set of rational numbers. Since $f(x)$ is measurable, the two sequences of sets

$$M_i = \mathrm{E}[f \geqslant c_i],\ M'_i = \mathrm{E}[f \leqslant c_i],\ (i = 1, 2, \ldots)$$

are measurable. These sets have, therefore, the following decompositions:

$$M_i = F_i + R_i,\ M'_i = F'_i + R'_i$$

[1]) N. Lusin, *Comptes Rendus* 154 (1912), p. 1688; W. Sierpiński, *Fund. Math.* t. III (1922), p. 320.

[2]) H. Hahn, Theorie d. reelen Funktionen, Berlin (1921), p. 129.

where F_i, F'_i are closed and $m(R_i) < \frac{\varepsilon}{2^{i+1}}$, $m(R'_i) < \frac{\varepsilon}{2^{i+1}}$ where ε is the number chosen above.

Consider the following sets:

$$\begin{aligned} Q &= R_1 + R_2 + \ldots + R_n + \ldots, \\ Q' &= R'_1 + R'_2 + \ldots + R'_n + \ldots, \\ B &= Q + Q' \\ A &= I - B. \end{aligned}$$

The substraction is valid since B is a subset of I. Computing the measures of these sets, we obtain

$$m(Q) \leqq \sum_{n=1}^{\infty} m(R_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2},$$

$$m(Q') \leqq \sum_{n=1}^{\infty} m(R'_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2},$$

$$m(B) \leqq M(Q) + m(Q') < \varepsilon,$$

$$m(I - A) < \varepsilon.$$

I say that $f(x)$ is continuous on A. Let $\overline{M}_i, \overline{M}'_i$ be the subsets of A where $f \leqslant c_i$ and $f \leqq c_i$ respectively. Then

$$\overline{M}_i = AM_i = AF_i + AR_i = AF_i$$

$$\overline{M}'_i = AM'_i = AF'_i + AR'_i = AF'_i$$

since R_i and R'_i are in the complement of A. $\overline{M}_i$ and $\overline{M}'_i$, being the products of A and the closed sets F_i and F'_i respectively, are closed relative to A. Since this is the case for every rational number c_i, the conditions of the above stated criterion are fulfilled and $f(x)$ is continuous on A.

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