Compact metric spaces have binary bases

by

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Abstract. O'Connor has shown that every compact metric space is supercompact. However, his proof is valid only for spaces dense-in-itself. This result is strengthened here, namely by proving that every compact metric space has a binary base.

A family \( \mathcal{K} \) of subsets of a topological space is said to be binary (see [7]) if every subfamily \( \mathcal{K}' \) of \( \mathcal{K} \) such that \( \bigcap \{ \text{cl} A : A \in \mathcal{K}' \} = \emptyset \) contains two elements with disjoint closures.

A space \( X \) is said to be supercompact (see de Groot [3]) if there exists an open subbase \( \mathcal{S} \), called superbase, such that in every cover of \( X \) by means of elements of \( \mathcal{S} \) there exist two elements which cover \( X \).

It is clear that if \( X \) is a compact Hausdorff space and \( \mathcal{S} \) is a binary base of open sets in \( X \), then the family \( \mathcal{S} \) consisting of sets of the form \( X \setminus dU \), where \( U \in \mathcal{S} \), is a supersubbase on \( X \).

In the paper [4] O'Connor proved that every compact metric space is supercompact; his proof consists on construction of a special embedding of a given compact metric space into Hilbert cube. However, his proof is valid only in the case when the space has no isolated points. In fact, the assertion ([4], p. 32) that the points \( T_aD_a ... T_bD_b(a) \) and \( T_aD_a ... T_bD_b(b) \) lie on opposite sides of \( z_a \) for \( a, b \in M \), whenever \( D_a ... T_bD_b(a) \) and \( D_b ... T_bD_b(b) \) lie, does not follow from Lemma 2, because that lemma assures this only for \( a, b \in K \), where \( K \) is a dense-in-itself subset of a given uncountable compact metric space \( M \).

In this paper we prove a theorem (Theorem 2) which asserts that every compact metric space has a binary base. Clearly, this result contains O'Connor's one. Our proof is based on the Freudenthal's theorem on inverse expansions [1].

The question of the existence of supersubbases or binary bases for arbitrary compact Hausdorff spaces are still open. The first of these questions was raised by de Groot [3].

§ 1. Natural projections, pseudopolyhedra and non-tangent sets. By a polyhedron we mean a compact Euclidean polyhedron. The symbol \([P]\)
denotes a polyhedron with a triangulation $P$. If we say about a triangulation of a simplex, then we mean on the standard triangulation consisting of all faces of that simplex.

Let $x \in [P]$. Then we define the carrier and the star of $x$:
\[
\text{car}_{P}x = \bigcap \{ s \in P : x \in s \}, \\
\text{st}_{P}x = \bigcup \{ s \in P : x \in s \} \setminus \bigcup \{ s \in P : x \notin s \}.
\]

Clearly, the carrier of $x$ is a simplex and $x$ belongs to the geometrical interior of it.

The following definition and lemma are taken from Rogers's paper [6].

If $|P|$ is a polyhedron, a simple subdivision of $P$ is a complex $P'$ whose vertices consist of just one point $p_i$ from the geometrical interior of each simplex $s$ of $P$, such that the simplex determined by a set $V$ of vertices of $P'$ belongs to $P'$ if and only if there is a sequence $s_0, \ldots, s_k$ of simplexes of $P$, each except the last is a face of the next, such that $V = \{ p_{s_0}, \ldots, p_{s_k} \}$. If $P$ is of dimension $n$ and $k$ is a positive integer, then $P'$ is said to be of order $k$ if the barycentric coordinate of $p_i$ on each vertex of $s$ is not smaller than $(n+1)^{-k}$ for each $s$ of $P$.

**Lemma 1.** If $P'$ is a simple subdivision of the $n$-dimensional complex $P$ of order $k$, then $\text{mesh} P' \leq (1-\frac{1}{n+1})^k \text{mesh} P$.

Let $S$ be a simplex. If $p$ is a vertex of $S$, then $[S(p)]$ denotes the opposite to $p$ face of $S$. Let $q$ be a vertex from $S(p)$. Then by a natural projection $\rho_{S} : [S(p)] \rightarrow [S(p)] = \{ q \}$ we mean a linear map which identifies the vertex $p$ with the vertex $q$ and is identity on $[S(p)]$.

**Remark 1.** Each linear onto map between two simplexes can be represented as a composition of natural projections.

Let $|P|$ be a polyhedron and let $Q$ be contained in $P$. Then $\bigcup \{ s \in Q : s \cap P \}$ is called a pseudopolyhedron.

Using the Lefschetz's construction [3], Ch. 8, § 1, (5.2)) we obtain following two lemmas.

**Lemma 2.** If $|P_0|, \ldots, |P_k|$ are polyhedra contained in a polyhedron $|P|$ (there is no dependence between triangulations $P_0, \ldots, P_k$ and the triangulation $P$), then there exists a subdivision $P'$ of $P$ which induces subdivisions on each $P_i$.

**Lemma 3.** The union and the intersection of two pseudopolyhedra (polyhedra) is also a pseudopolyhedron (polyhedron). The prism over pseudopolyhedron (polyhedron) is a pseudopolyhedron (polyhedron).

The following corollary is a consequence of Remark 1 and Lemma 3.

**Corollary 1.** The counterimage of pseudopolyhedron (polyhedron) under a simplicial map is a pseudopolyhedron (polyhedron).

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A subset $M$ of a polyhedron $|P|$ is said to be non-tangent in $|P|$ if $\text{cl}(s \cap M) = s \cap \text{cl} M$ for each $s$ of $P$, or equivalently, if $x \in \text{cl} M$, then $x \notin \text{cl}(s \cap M)$, where $s$ is the carrier of $x$. A family consisting of non-tangent in $P$ sets will be called non-tangent family.

**Lemma 4.** Let $\pi$ be a simplicial map of a polyhedron $|P|$ onto a polyhedron $|P'|$ and let $M$ be non-tangent in $P'$. Then $\pi^{-1}(M)$ is non-tangent in $P$ and $\pi^{-1}(\text{cl} M) = \text{cl} \pi^{-1}(M)$.

**Proof.** Note, that the both conclusions follow from the following implication:
\[
x \in \pi^{-1}(\text{cl} M) \Rightarrow x \in \text{cl}(\pi^{-1}(M) \cap s) \quad \text{where} \quad s = \text{car}_{P}x.
\]

To prove this implication let $x$ belongs to $\pi^{-1}(\text{cl} M)$, let $s$ be the carrier of $x$ and let $s'$ be the carrier of $\pi(x)$. Since $M$ is non-tangent in $P'$, hence $\pi(s)$ belongs to $\text{cl}(M \cap s')$. Then there exists a sequence $(y_n : n = 1, 2, \ldots)$ of points of $M \cap s'$ converging to $\pi(x)$. We claim that there exists a sequence $(z_n : n = 1, 2, \ldots)$ of points of $\pi^{-1}(M) \cap s$ converging to $x$. Consider $\pi$ as a map from $s$ onto $s'$, which is sufficient for further considerations; so we can assume that $s'$ is a face of $s$.

1. If $\text{dim} s = \text{dim} s'$, then let $z_n = y_n$.

2. If $\text{dim} s = \text{dim} s' + 1$, then we can assume that $\pi$ is a natural projection which identifies vertices $p$ and $q$, where $q$ belongs to $s'$. Let $H$ be the (dim)-plane which contains the point $x$ and the face of $s'$ opposite to $q$. Then let $z_n$ be a (unique) point of $H$ such that $\pi(z_n) = y_n$.

Passing to the general situation, the proof reduces in view of Remark 1 to the cases 1 and 2.

Now we infer that $x \in \text{cl}(\pi^{-1}(M) \cap s)$.

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§ 2. Construction of some special binary bases on simplexes. Let $|S|$ be a simplex and let $p$ and $q$ be different vertices of $S$. A symbol $e^*_{p,q}$ denotes the one face of $S$ which contains $p$ and $q$. If a point $x$ belongs to $e^*_{p,q}$, then $H^*_x$ denotes the minimal hyperplane in $|S|$ passing through $x$ and $|S(p)|q$. If a point $x$ belongs to $e^*_{p,q}$ and $p \neq x \neq q$, then $H^{1}_{x}$ denotes the intersection of $|S|$ with the open half-space determined by $H^*_x$ to which $p$ belongs (do the same with $q$). If $x$ and $y$ are different points in $e^*_{p,q}$ such that $p \neq x \neq y$ and $p \neq q \neq y$, then $H^{2}_{x,y}$ denotes the intersection of $H^*_p$ and $H^*_q$. The sets $H^*_x$, $H^{1}_{x}$ will be called strata of $|S|$ with respect to $p$ and $q$. We shall omit indices in the symbols $H^*_x$ if misunderstanding is excluded.

Let $p^*_x : |S| \rightarrow |S(p)|$ be the natural projection and let $S$ be an arbitrary family of subsets of $|S(p)|$.

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By a lift of the family \( \mathcal{X} \) to \([S]\) by means of \( \varphi \), we mean the family of subsets of \([S]\) of the form:

1. \( \varphi^{-1}(\mathcal{A}) \cap H \), if \( \mathcal{A} \) is disjoint with \([S(p)](q)\),
2. \( \varphi^{-1}(\mathcal{A}) \), if \( \mathcal{A} \cap [S(p)](q) \neq \emptyset \),

where \( \mathcal{A} \) is a member of \( \mathcal{X} \) and \( H \) being a stratum.

**Lemma 5.** If \( \mathcal{A} \) is a non-tangent in \([S(p)] \) of \([S(p)](q)\), then \( \text{cl} \varphi^{-1}(\mathcal{A} \cap H) = \text{cl} \varphi^{-1}(\mathcal{A}) \cap \text{cl} H \), for every stratum \( H \).

**Proof.** Note that \( \text{cl} \varphi^{-1}(\mathcal{A} \cap H) = \varphi^{-1}(\text{cl} \mathcal{A} \cap \text{cl} H) \), \( H \) being open.

Therefore, it suffices to prove only the following inclusion:

\[ \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \subseteq \text{cl} \varphi^{-1}(\mathcal{A}) \cap \text{cl} H. \]

Since the map \( \varphi : H \to [S(p)] \) is a homeomorphism and, by Lemma 4, \( \text{cl} \varphi^{-1}(\mathcal{A}) = \varphi^{-1}(\text{cl} \mathcal{A}) \), we have

\[ \text{cl} \varphi^{-1}(\mathcal{A} \cap H) = \varphi^{-1}(\text{cl} \varphi^{-1}(\mathcal{A}) \cap H) = \varphi^{-1}(\text{cl} \mathcal{A} \cap H). \]

The desired inclusion follows now from the observation that \( \text{cl} H \) is the union of some sets \( H_i \).

**Lemma 6.** If \( \mathcal{A} \) is a non-tangent in \([S(p)] \) of \([S(p)](q)\), then

\[ \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \subseteq \text{cl} \mathcal{A} \cap H \cap s = \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \cap s, \]

for each \( s \in S \) and every stratum \( H \).

**Proof.** Let \( s \) be a face of \( S \). First let us consider the following two cases:

1. \( p \neq s \) and \( q \neq s \); if \( H \) is \( H_+ \) or \( H_- \), then \( \text{cl} H \cap s \subset [S(p)](q) \) and therefore \( \text{cl} \varphi^{-1}(\mathcal{A}) \cap \text{cl} H \cap s = \emptyset \); if \( H \) is \( H^0 \), then \( s \subset \text{cl} H^0 \) and therefore \( \text{cl} \varphi^{-1}(\mathcal{A}) \cap \text{cl} H \cap s = \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \cap s = \text{cl} (\varphi^{-1}(\mathcal{A} \cap H) \cap s) \) (the second equality follows from Lemma 4).
2. \( p \neq s \) and \( q \neq s \); then observe that \( \varphi^{-1}(\varphi^{-1}(s)) = s \) and then Lemmas 4 and 5 imply equalities:

\[ \text{cl} \varphi^{-1}(\mathcal{A}) \cap \text{cl} H \cap s = \varphi^{-1}(\text{cl} \mathcal{A} \cap s) \cap \text{cl} H = \varphi^{-1}(\text{cl} \mathcal{A} \cap s) \cap \text{cl} H = \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \cap \text{cl} H \cap s = \text{cl} \varphi^{-1}(\mathcal{A} \cap H) \cap s, \]

The proof of the required equality in the remaining two cases, is analogous to that of the case 1, or obvious.

**Corollary 2.** The property to be non-tangent is preserved by the lift operation.

**Theorem 1.** In every simplex \([S]\) there exists a binary base \( S \) non-tangent in \( S \), consisting of open pseudopolyhedra the intersections of which with each face of \([S]\) form a binary family and such that

1. if \( U, V \subseteq S \) and the sets \( U \cap \text{cl} V, U \cap \text{cl} s \) and \( \text{cl} U \cap s \) are non-empty, then the set \( U \cap \text{cl} V \cap s \) is so.

**Proof.** The construction of such a base will be given by the induction on the dimension of \([S]\).

If \( \dim(S) = 1 \), then the family of all non-tangent in \( S \) open in \([S]\) intervals is the desired one.

Let us assume that there exists a binary base \( S \) consisting of open pseudopolyhedra on an \( (n-1) \)-simplex \([S]\) of \( S \) being non-tangent in \( S \), which induces a binary family on each face of \([S]\) and fulfills the condition (1).

Let \( p \) and \( q \) be vertices of \( S \) such that \( p \neq q \) and \( q \neq q \). Let \( \varphi : [S] \to [S(p)](q) \) be the corresponding natural projection. Now we prove that the lift \( S \) of the base \( S \) to \([S]\) by means of \( \varphi \) is the family in question.

It follows from Corollary 2 that \( S \) is non-tangent in \( S \).

Clearly \( S \) is a base, having sets of arbitrarily small diameters.

To prove that \( S \) is binary and induces a binary family on each face of \([S]\) let \( s \) be a simplex of \( S \) and let \( U_1, \ldots, U_n \in S \) be such that \( U_i = \varphi^{-1}(V_i) \cap H, i = 1, \ldots, n, H \subseteq S \) and \( U_i = \varphi^{-1}(V_i), j = 1, \ldots, k, \) where \( V_i \) belong to \( H_i \) and \( H_i \) are strata. Non-trivial is only the case when \( p, q \neq s \) and then \( H \subseteq S \) such that \( H \cap \text{cl} H_1 \cap \cdots \cap \text{cl} H_k \).

By Lemma 6 we get

\[ \text{cl} \varphi^{-1}(V_i) \cap \cdots \cap \text{cl} \varphi^{-1}(V_k) \subseteq \text{cl} H_1 \cap \cdots \cap \text{cl} H_k \cap s = O. \]

Using the equality \( \varphi^{-1}(\varphi^{-1}(s)) = s \) we have

\[ \varphi^{-1}(\text{cl} V_i \cap \cdots \cap \text{cl} V_k \cap \varphi^{-1}(s)) \subseteq \text{cl} H_1 \cap \cdots \cap \text{cl} H_k \cap s = O. \]

The assumption \( H \subseteq \text{cl} H_1 \cap \cdots \cap \text{cl} H_k \) implies that \( \text{cl} V_1 \cap \cdots \cap \text{cl} V_k \cap \varphi^{-1}(s) = O \). Since \( \varphi^{-1}(s) \) is a face of \([S(p)]\) and \( S \) induces a binary family on each face of \([S(p)] \) hence there exist \( V_i \) and \( V_j \) from \( \{V_1, \ldots, V_k\} \) such that \( \text{cl} V_i \cap \text{cl} V_j \supseteq \varphi^{-1}(s) = O. \) Consequently, \( \text{cl} U_i \cap \text{cl} U_j \cap s = O, \) and all the more \( \text{cl} (U_i \cap s \cap \text{cl} U_j \cap s = O. \)

In order to prove the condition (1) let \( U, V \) and \( s \) be such that \( U, V \subseteq S \), \( s \subseteq S \) and the sets \( U \cap \text{cl} V, U \cap \text{cl} s \) and \( \text{cl} U \cap s \) are non-empty. We can assume that \( p \) and \( q \) belong to \( s \) (other cases are trivial). Since \( U \) and \( V \) are in \( S \), hence \( U = \varphi^{-1}(U \cap s), V = \varphi^{-1}(V \cap s) \), where \( U \) and \( V \) belong to \( S \) and \( E_1, E_2 \) are strata of \([S]\). Let, on the contrary, \( \text{cl} U \cap \text{cl} V \cap s = \emptyset \). Using Lemma 6 and the formula \( \varphi^{-1}(\varphi^{-1}(s)) = s \) we get

\[ \varphi^{-1}(\text{cl} U \cap \text{cl} V \cap \varphi^{-1}(s)) \subseteq \text{cl} E_1 \cap \text{cl} E_2 = O. \]
Since $\text{cl}U \cap \text{cl}V \neq \emptyset$ hence there exists a hyperplane $H'$ which is contained in $\text{cl}U \cap \text{cl}E_i$. But in this case $\text{cl}E' \cap \text{cl}\mathcal{G} \cap \mathcal{P}(\mathcal{P}) = \emptyset$; a contradiction with the fact that $P', \mathcal{G} \subseteq \mathcal{S}^\prime$ which fulfills (6).

The fact that $S$ consists of pseudopolyhedra follows immediately from Lemma 3.

§ 3. Further lemmas.

Lemma 7. For every simplex $[S]$ and a positive $\delta$ there exists a finite number of points of $[S]$, say $x_1, \ldots, x_n$, and open (in $[S]$) neighbourhoods of that points, say $U_1, \ldots, U_n$, which cover $[S]$ and such that

1. $U_i$ is a pseudopolyhedron non-tangent in $S$ and $\text{diam} U_i < \delta$,
2. $\text{cl}U_i \cap \text{cl}U_j \neq \emptyset$ implies that $\text{car}_{\delta}x_i$ is a face of $\text{car}_{\delta}x_j$ or conversely,
3. $\{U_1, \ldots, U_n\}$ is a binary family which induces a binary family on each face of $S$,
4. $s \in S$ and all the sets $\text{cl}U_i \cap \text{cl}U_j$, $\text{cl}U_i \cap s$ and $\text{cl}U_j \cap s$ are non-empty, then the set $\text{cl}U_i \cap \text{cl}U_j \cap s$ is so.

Proof. Let $S$ be a base constructed in Theorem 1. Let $\{x_1, \ldots, x_n\}$ be the $0$-skeleton of $S$. Since $S$ is a base hence there exist elements $U_1, \ldots, U_n$, of $\mathcal{S}$ with disjoint closures satisfying the conditions (1)–(5) (the conditions (3)–(5) in vacuum).

Let $S^{(k)}$ be the $k$-skeleton of $S$. Let us assume that we have points $x_1, \ldots, x_{n_k}$ in $[S^{(k)}]$ and sets $U_1, \ldots, U_{n_k}$ from $S$ which satisfy conditions (1)–(5) and such that $[S^{(k)}] \subset U_1 \cup \ldots \cup U_{n_k}$.

Now let $s^{k+1}$ be a $(k+1)$-simplex of $S^{(k+1)}$. The compactness of the set $D = s^{k+1} \cup \{U_1 \cup \ldots \cup U_{n_k} \}$ implies that there exist points $x_{n_k+1}, \ldots, x_p$ in $D$ and sets $U_{n_k+1}, \ldots, U_p$ in $S$ which cover $D$, satisfy (1) and such that

$x_i \in U_i \cap \text{cl}U_i \subset \text{st}_{\delta}x_i$ and $\text{diam} U_i < \delta$ for all $i$

and for a set $V$ from $\{U_1, \ldots, U_p\}$ such that $\text{cl}V \cap s^{k+1} = \emptyset$, then $\text{cl}V \cap \text{cl}(U_{n_k+1} \cup \ldots \cup U_p) = \emptyset$.

It is easy to see that the points $x_1, \ldots, x_{n_k}, \ldots, x_p$ and the sets $U_1, \ldots, U_{n_k}, \ldots, U_p$ satisfy conditions (1)–(5). Applying this construction successively to remaining $(k+1)$-simplexes we obtain, in finitely many steps, the points $x_1, \ldots, x_{n_{k+1}}$, in $[S^{(k+1)}]$ and the sets $U_1, \ldots, U_{n_{k+1}}$, in $S$ which satisfy conditions (1)–(5) and such that $[S^{(k+1)}] \subset U_1 \cup \ldots \cup U_{n_{k+1}}$.

Now the lemma follows by the induction.

Lemma 8. Let $[S]$ be a simplex and let $P'$ be a polyhedron such that $P' \subset S$. Let $\mathcal{G}$ be a finite binary family of subpolyhedra of $[S]$ such that

1. $[P'] \subset \mathcal{G}$,
2. if $[V] \not\in \mathcal{G}$, then $V \not\subset P'$.

Let $Q$ be a binary family on $[S]$ constructed as in Lemma 7.

Then the intersections of elements of $\mathcal{G} \cup Q$ with $[P']$ form a binary family on $[P']$ non-tangent in $P'$.

Proof. The fact that the family in question is non-tangent in $P'$ holds, because $\mathcal{G}$ is non-tangent in $P'$ and elements of $Q$ are closed.

In order to prove the binarity let $U_1, \ldots, U_n \in \mathcal{G}$ and $[V_1], \ldots, [V_n] \in \mathcal{Q}$ be such that the closures of each two members of $\{U_1, \ldots, U_n, [V_1], \ldots, [V_n]\}$ have non-empty intersection and each of $U_i$ has non-empty intersection with $[P']$.

Let $x_1, \ldots, x_n$ be points corresponding to sets $U_1, \ldots, U_n$ as in Lemma 7.

It follows from the fact that $\text{cl}U_i \cap [V_s] \neq \emptyset$ and from (3) that $x_s \not\in [V_s]$, for each $p$ and $q$. Hence

$x_1, \ldots, x_n \not\in [V_1] \cap \cdots \cap [V_n] \cap [P']$.

It is easy to prove using (3), by the induction on $j$, that all the points $x_i, i = 1, \ldots, n$, lie in one simplex $s$ from $V_1 \cap \ldots \cap V_n \cap P'$, $s$ being the carrier of one of them. This, together with (5), imply that

$\text{cl}U_i \cap \ldots \cap \text{cl}U_j \cap \text{cl}[V_1] \cap \ldots \cap \text{cl}[V_n] \cap [P'] = \emptyset$.

In the case of the lack of $V-s$ the proof holds with obvious simplifications. In the case of the lack of $V-s$ conclusion follows immediately from the hypotheses.

Lemma 9. Let $[P]$ be a polyhedron, let $\mathcal{G}$ be a finite binary family consisting of pseudopolyhedra non-tangent in $P$ and let $Q$ be a positive number.

Then there exists a finite open covering $\mathcal{S}$ of $[P]$ consisting of open pseudopolyhedra such that $\text{mesh}\mathcal{S} < \delta$ and $\mathcal{S} \cup \mathcal{G}$ is a binary family non-tangent in $P$.

Proof. It follows from Lemma 2 that there exists a subdivision $P'$ of $P$ which induces a triangulation on each nonempty intersection of closure of elements of each subfamily of $\mathcal{S}$, the elements of $\mathcal{S}$ are pseudopolyhedra. We can assume that $P' \subset S$, where $[S]$ is a simplex. Let $\mathcal{G} = \{\text{cl}A : A \in \mathcal{G}\}$. Define $\mathcal{S}$ to be the family of all intersections of elements of $Q$ with $[P']$, where $Q$ is a $\delta$-covering taken for $S$ according to Lemma 7. Lemma 8 assures that $\mathcal{S} \cup \mathcal{Q}$ is binary. Thus $\mathcal{S} \cup \mathcal{Q}$ is binary. The same Lemma 8 assures that $\mathcal{S} \cup \mathcal{Q}$ is non-tangent in $P$.

Lemma 10. Let $[P]$ be a polyhedron and let $W_1, \ldots, W_n$ be open in $[P]$ pseudopolyhedra non-tangent in $P$. Then there exists a simple subdivision $P^{(0)}$ of $P$ of order 2 such that all $W_1, \ldots, W_n$ are non-tangent in $P^{(0)}$. 
Proof. For each \( s \in P \) we denote by \( U(s) \) the set of all points of \( s \) having all the barycentric coordinates in \( s \) not smaller than \((\dim P + 1)^{-3}\). Now the thesis of our lemma may be stated as follows.

For each \( s \in P \) there exists \( p_s \in U(s) \) such that for each sequence

\[
(7)\quad s_0 \subseteq s_1 \subseteq \ldots \subseteq s_k \subseteq s
\]

and for each face \( A \) of \( \{p_{n_1}, \ldots, p_{n_k}, p_s\} \), the simplex determined by vertices \( p_{n_1}, \ldots, p_{n_k}, p_s \) there is for each \( i, j \leq m \)

\[
(8)\quad \text{cl}(A \cap W_l) = A \cap \text{cl}W_i.
\]

The proof of our lemma will be done by the induction on \( \dim s \).

If \( \dim s = 0 \), then \( p_s \) equals \( s \) and (8) follows easily from the assumption that \( W_i \) is non-tangent in \( P \).

Let \( p_s \) be already defined for simplexes of the \( n \)-skeleton of \( P \) and let \( s \in P \) be a \((n+1)\)-simplex. Now we are going to define \( p_s \) such that (8) holds for each \( A \) of each simplex \( \{p_{n_1}, \ldots, p_{n_k}, p_s\} \) where \( s_0, \ldots, s_k \) satisfy (7).

Take on each polyhedron \( \text{cl}(s \cap W_l) \) a triangulation \( T_l \) such that \( T_l \) induces on \( s \cap \text{bd}W_l \) (bd stands for the boundary in \([P]\)) a triangulation of \( T_l \) (the existence of such \( T_l \) follows from Lemma 2). Note that \( s \cap \text{bd}W_l \) is equal, in virtue of the non-tangence of \( W_l \) in \( P \), to \( \text{bd}(s \cap W_l) \) (bd stands for the boundary in \( s \)).

Consider all the hyperplanes in \( s \times \text{dimension} \) not greater than \( n \) determined by arbitrary families of simplexes of \( A' \), \( A'' \) and of points \( p_{n_1}, \ldots, p_{n_k} \) already defined which satisfy (7). Let \( A \) be the union of all such hyperplanes. It follows from the fact that \( A \) is nowhere dense in \( s \) that there exists a point \( p_s \) such that \( p_s \in U(s) \) and \( A \).

Let \( A \) be a face of \( \{p_{n_1}, \ldots, p_{n_k}, p_s\} \) where \( s_0, \ldots, s_k \) be such that (7) holds. To prove (8), let \( p \in A \cap \text{cl}W_i \). To prove that \( p \in \text{cl}(A \cap W_l) \) it suffices, in virtue of the inductive hypothesis, to consider only the case when \( p \in \text{int} s \), i.e. when \( s = \text{car}_s \). In consequence \( p_s \in \text{car}_s \). Clearly we can assume that \( p \in \text{bd}W_i \cap \text{bdgeo}A, W_l \) being non-tangent in \( P \).

If \( M \) is a subset of \([P]\), then by \( \text{cl}(M) \) we denote the hyperplane determined by \( M \).

Let \( t = \text{car}_s \) and let \( t' \in T_l \) be such that \( t' \subseteq t \) and \( t' \cap W_l \cap s \neq \emptyset \). Let \( t' = \text{intgeo} t' \). Clearly, \( t \subseteq W_l \cap s \), \( t' \subseteq \text{cl} t' \) and \( \dim H(t) > \dim H(t') \).

Let \( p_{n_1}, \ldots, p_{n_k} \) be all points from \( \{p_{n_1}, \ldots, p_{n_k}\} \) which lie in \( \text{car}_s \). We have \( \text{cl}(s' \cap t') = \text{cl}(s' \cap \{p_{n_1}, \ldots, p_{n_k}\}) \), because \( p_s \) belongs to the set on the right side. Then, by the definition of \( p_s \) we infer that

\[
\dim H((\text{car}_s \cap t') \cap H(t')) = n - 1. \tag{11}
\]

This implies, in virtue of \( p \in H((\text{car}_s \cap t') \cap H(t')) \),

\[
\dim H((\text{car}_s \cap t') \cap H(t')) = \dim H((\text{car}_s \cap t') \cap H(t)) + n - 1,
\]

\[
\dim H((\text{car}_s \cap t') \cap H(t)) + \dim H(t) = \dim H((\text{car}_s \cap t') \cap H(t)) + n - 1.
\]

But \( \dim H(t) > \dim H(t') \). So

\[
\dim H((\text{car}_s \cap t') \cap H(t)) > \dim H((\text{car}_s \cap t') \cap H(t')).
\]

Let

\[
g \in H((\text{car}_s \cap t') \cap H(t')).
\]

Then the open interval \( \langle p, q \rangle \) is contained in \( H((\text{car}_s \cap t') \cap H(t')) \). So there exists an \( r \in (p, q) \) such that \( (p, r) \subseteq \text{intgeo} t \subseteq (p, q) \) and hence \( p \in H(t) \cap A \).

Lemma 11. Let \( \mathcal{R} \) be a finite family of open in \([P]\) pseudopolyhedra in a polyhedron \([P]\) non-tangent in \( P \) and let \( \epsilon \) be a positive number. Then there exists a subdivision \( P' \) of \( P \) such that mesh \( P' < \epsilon \) and each element of \( \mathcal{R} \) is non-tangent in \( P' \).

Proof. By Lemma 10, there exists a simple subdivision \( P^0 \) of \( P \) of order 2 such that all elements of \( \mathcal{R} \) are non-tangent in \( P^0 \). By Lemma 1, mesh \( \{--(\dim P + 1)^{-3}\} \) mesh \( P^0 \). This implies that if we shall iterate the procedure described above, then we can find an \( r \) such that \( P^0 \) is a subdivision in question.

§ 4. Construction of a binary base on a compact metric space.

Theorem 2. Every compact metric space \( X \) has a binary base.

Proof. We shall construct an inverse sequence of polyhedra \([P_n]\) (the metric \( d_s \) on \([P]\) let be such that \( d_s([P_n]) < 1 \)) with simplicial onto bonding maps \( s_n \subseteq s_{n+1} \), whose inverse limit \( \chi \) with the standard metric

\[
d((x, y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n([\pi_n(x), \pi_n(y)]),
\]

where \( \pi_n \) denotes the standard projection of \( X \) onto \( P_n \), is homeomorphic to \( X \), and has a binary base \( \mathcal{B} = \{\pi_n: A \subseteq \mathcal{R}, n = 1, 2, \ldots\} \), where \( \mathcal{R} \) is a certain finite open covering of \([P]\). To do this it suffices in the procedure of the proof of Freudenthal's theorem, [1] (on the existence of such an expansion without the existence of a binary base on \( Y \) in the form of Pasykov [5]) (proof of Prop. 2, pp. 97–98; for \( \mu \) being the class of all polyhedra; we assume that the technique of maps of \( X \) into nerves of open finite coverings of \( X \) is known to the reader), which
consists on the inductive construction of \([P_n]\) and \(n_{n-1}\), to take into consideration the following two observations:

(1) if we have been already defined sequences \([P_i] \supseteq [P_{i-1}] \supseteq \cdots \supseteq [P_1] \supseteq [P_0]\)

... \(\vdash [P_n] \supseteq \cdots \supseteq [P_1] \supseteq [P_0]\), \(\rho_i\), \(\delta_i\) and \(\rho_i\), \(\delta_i\), such that, for \(1 < i < n\),

(a) \([P_i]\) is a polyhedron such that \(\text{diam}(P_i) < 1\) and \(\rho_i\) is a simplicial map of \([P_i]\) onto \([P_{i-1}]\), where \([P_{i-1}]\) is a certain subdivision of \([P_{i-1}]\),

(b) \(\rho_i\) is a finite family consisting of pseudopolyhedra open in \([P_i]\), non-tangent in \([P_i]\); for \(i < n\) and in \([P_n]\) for \(i = n\), such that \(\text{mesh}\, \rho_i < \delta_i\), \(\rho_i\) covers \([P_i]\) and \(\rho_i \cup \{x_i\}\) is \((\rho_i)^{-1}(A_2)\); \(A_2 \in \rho_i\), \(k < i\) is a binary family,

(c) \(\delta_i\) is a positive number such that if \(A \subset [P_i]\) and \(\text{diam}(A) < \delta_i\),

then for each \(j < i, \text{diam}(\rho_j(A)) < 1/3^i\), then for every positive number \(\varepsilon\) there exists a subdivision \(P'_i\) of \([P_i]\) such that \(\text{mesh}\, P'_i < \varepsilon\) and each element of \(\rho_i \cup \{x_i\}\) is \((\rho_i)^{-1}(A_2)\); \(A_2 \in \rho_i\), \(k < n\) is non-tangent in \([P_n]\), and

(2) if we have been already defined a simplicial map \(n_{n+1}\) of \([P_{n+1}]\) onto \([P_n]\), where \(P_n\) satisfies conditions from (1), then we can find

(c) a positive number \(n_{n+1}\) such that if \(A \subset [P_{n+1}]\) and \(\text{diam}(A) < n_{n+1}\),

then for each \(j < n+1, \text{diam}(\rho_{n+j}(A)) < 1/2^{n+1}\), and

(f) an open (in \([P_{n+1}]\), covering \(\rho_{n+1}\) of \([P_{n+1}]\), with \(\text{mesh} \rho_{n+1} < n_{n+1}\), consisting of pseudopolyhedra such that family

\[ \rho_{n+1} \cup \{x_{n+1}\}\] 

is binary and each element of that family is non-tangent in \([P_n]\).

In fact, (1) assures (see the note of Pasykno loco cit.) the existence of polyhedron \([P_{n+1}]\) and a simplicial map \(n_{n+1}: [P_{n+1}] \to [P_n]\), where \(P_n\) is as in (1) (for sufficiently small \(\varepsilon\)), such that they satisfy the conclusions of the Pasynkov's construction (in order to get a homeomorphism of \(X\) with the inverse limit of the inverse sequence \(\{P_n; n_{n-1}\}\)). In virtue of Lemma 4, the family \(\{n_{n+1}\}\) is \((\rho_i)^{-1}(A_2)\); \(A_2 \in \rho_i\), \(k < n+1\) is non-tangent in \([P_n]\). Now take \(\rho_{n+1}\) as in (2).

By the induction we have constructed an inverse sequence \(\{P_n; n_{n-1}\}\) with inverse limit \(Y\), a homeomorphism of \(X\) onto \(Y\) (by the procedure of Pasynkov loco cit.) and a sequence \(\rho_i\) satisfying the conditions of (1) and (2).

Now, easy calculation with projections \(n_i\) and maps \(n_i\) leads, in virtue of (b), to the formula

\[ n_i^{-1}(\text{cl}\, A_2) = \text{cl}(\rho_i^{-1}(A_2)) \quad \text{for} \quad A_2 \in \rho_i, \quad k < n. \]

Using this formula and applying (b) once again we infer that \(\rho_i\) is a binary family on \(Y\).

The fact that \(\rho_i\) is a base follows easily from (b) and (c).

To complete the proof observe that the assertions (1) and (2) follows immediately from Lemma 11 and Lemma 8, respectively.