Theorem 10 is proven as an aid in determining a characterization of when a tree-like space is wide.

THEOREM 10. *If (M, d) is wide tree-like with wide realization $\{C_n\}$ with respect to d, then $(M, d) = (M, d^*)$ where d^* is as defined above.*

Proof. Let $x \in M$, $\varepsilon > 0$, and $S_\varepsilon^*(x)$ denote a d^*-sphere with radius ε about x. Since $\{C_n\}$ is a wide realization of (M, d) with respect to d, there exists N_1 and $0 < \delta_1 < \varepsilon/4$ with the property that for $n \geqslant N_1$, if $b \in B \cap C_n$ and $d(x, \bar{b}) > \varepsilon/4$, then $d(x, A^*) \geqslant \delta_1$ where B is as in the proof of Theorem 8 and A is an arm of b in C_n such that $x \notin A^*$. Pick N_2 such that for each $n \geqslant N_2$, $\|C_n\| < \delta_1$ and then let $N = N_1 + N_2$. Let $l \in C_N$ such that $x \in l$ and assert that $x \in l \subset S_\varepsilon^*(x)$. Let $y \in l$.

If $n \geqslant N$ and $b \in B \cup C_n$ such that $d(x, \bar{b}) > \varepsilon/4$, then x and y belong to the same arm of b in C_n since $\{x, y\} \subset l$ and $|l| < \delta_1$. Therefore, $d_b(x, y) = d(x, y) < \delta_1 < \varepsilon/4$. If b is such that $d(x, \bar{b}) \leqslant \varepsilon/4$, then pick $z \in \bar{b}$ such that $d(x, z) \leqslant \varepsilon/4$. The definition of d_b infers that

$$d_b(x, y) \leqslant d(x, z) + d(z, y) \leqslant d(x, z) + d(z, x) + d(x, y) < \varepsilon/4 + \varepsilon/4 + \delta_1 < 3\varepsilon/4 .$$

For all cases when $n \geqslant N$ and $b \in B \cap C_n$ we can now conclude that $d_b(x, y) < 3\varepsilon/4$.

If $n < N$ and $b \in B \cap C_n$, then the definition of the realization $\{C_n\}$ yields a link $l_n \in C_n$ such that $\{x, y\} \subset l \subset l_n$. By definition of d_b, then $d_b(x, y) = d(x, y)$. Since $y \in l$ and $|l| < \varepsilon/4$, then $d_b(x, y) = d(x, y) < \varepsilon/4$. Thus, this paragraph and the above paragraph convinces us that for each $b \in B$, $d_b(x, y) < 3\varepsilon/4$ which assures that $d^*(x, y) < \varepsilon$. The conclusion is, $x \in l \subset S_\varepsilon^*(x)$.

If $x \in M$ and $\varepsilon > 0$, then $x \in S_\varepsilon^*(x) \subset S_\varepsilon(x)$ since $d^*(x, y) \geqslant d(x, y)$. From this fact and the above arguments we can now conclude that $(M, d) = (M, d^*)$.

Theorem 8 and Theorem 10 imply a characterization for the wide tree-like spaces. This characterization is revealed in Theorem 11.

THEOREM 11. *If (M, d) is a tree-like space and d^* is as defined in this section, then $(M, d) = (M, d^*)$ if and only if (M, d) is wide.*

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On the fundamental dimension of approximatively 1-connected compacta

by

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Abstract. The aim of the present paper is to give a homological characterization of the fundamental dimension for approximatively 1-connected compacta and to give some applications of this characterization.

The main result is the theorem which states that for every approximatively 1-connected compactum $X \neq \emptyset$ with $\mathrm{Fd}(X) < \infty$ the fundamental dimension of X is equal to the smallest integer number $n \geqslant 0$ such that X is acyclic (in the sense of Čech cohomology) in all dimensions $\geqslant n$.

We prove also that for every movable approximatively 1-connected continuum X with infinite fundamental dimension and for every natural number n there exists a natural number $m \geqslant n$ such that m-dimensional Čech cohomology group of X with coefficients in the group of integer numbers is not trivial.

From these theorems we deduce in particular that for every $n \geqslant 3$ there exists a sequence $\{Q_p^n\}_{p=2}^{\infty}$ of polyhedra such that $\dim Q_p^n = \mathrm{Fd}(Q_p^n) = n$ and $\mathrm{Fd}(Q_q^m \times Q_p^n) = \max(m, n)$ for all relatively prime natural numbers p and q.

Introduction. By K we denote the Hilbert cube. The *fundamental dimension* of a compactum X (denoted by $\mathrm{Fd}(X)$) is the minimum of the dimensions of compacta Y with $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ (see [4] p. 31). We say that a pointed compactum $(X, x_0) \subset (K, x_0)$ is *approximatively n-connected* (see [3], p. 266) if for every neighborhood V of X there exists a neighborhood V_0 of X such that every map of the pointed n-sphere (S^n, a) into (V_0, x_0) is null homotopic in (V, x_0). It is known that the approximative n-connectivity of a pointed compactum $(X, x_0) \subset (K, x_0)$ depends only on the pointed shape of (X, x_0) (see [3], p. 267). Thus a pointed compactum (Y, y_0) (not necessarily lying in K) is said to be *approximatively n-connected* if there is a pointed compactum $(X, x_0) \subset (K, x_0)$ which is approximatively n-connected and homeomorphic to (Y, y_0). We say also that a compactum Y is *approximatively n-connected* if (Y, y_0) is approximatively n-connected for every $y_0 \in Y$ (see [3], p. 266).

Let $H_n(X, A; G)$ (or $H^n(X, A; G)$) denote for every pair (X, A) of compacta and every Abelian group G the n-dimensional Čech homology (or cohomology) group of (X, A) with coefficients in G. By $H_c^n(X; G)$ we denote for every locally compact space X the n-dimensional Alexander cohomology group with compact supports and with coefficients in G (see [21], p. 320).

It is known ([21], p. 321) that if A is a compactum and B is a closed subset of A then for all q and all G the groups $H_c^q(A \setminus B; G)$ and $H^q(A, B; G)$ are isomorphic.

If X is a non-empty locally compact metric space then we say that a *coefficient of cyclicity* of X with respect to the Abelian group G (denoted by $c_G(X)$) is equal to n (where $n \geqslant 0$ is a integer number) if $H_c^m(X; G) = 0$ for all $m > n$ and $H_c^n(X; G) \neq 0$. Moreover, we set $c_G(\emptyset) = -1$ and if $X \neq \emptyset$ and for every $n = 1, 2, \dots$ there exists an integer number $m \geqslant n$ such that $H_c^m(X, G) \neq 0$, then we set $c_G(X) = \infty$.

In the sequel the coefficient of cyclicity of X with respect to the group of integer numbers Z is denoted by $c(X)$.

It is clear that if X is a compactum then we can replace in the above definition $H_c^n(X, G)$ by $H^n(X, G)$ and $H_c^m(X, G)$ by $H^m(X, G)$.

Since the Čech cohomology groups are shape invariants ([16], p. 54), the coefficient of cyclicity with respect to an arbitrary group G is also a shape invariant.

Let $\mathfrak{R}_1$ denote the group of real numbers modulo 1.

It is well known (see [12], p. 137 and p. 124) that for every compactum X the group $H_n(X; \mathfrak{R}_1)$ is the character group $H^n(X; Z)$. This implies that $\max(m\colon H^m(X; Z) \neq 0) = \max(m\colon H_m(X, \mathfrak{R}_1) \neq 0) = c(X)$.

N. Steenrod has proved ([23], p. 690) that for each Abelian group G and every compactum X the group $H_n(X; G)$ is the direct sum of two groups, one determined uniquely by G and $H_n(X; \mathfrak{R}_1)$, the other by G and $H_{n+1}(X; \mathfrak{R}_1)$. These groups are trivial if $H_{n+1}(X; \mathfrak{R}_1) = H_n(X; \mathfrak{R}_1) = 0$. Therefore for every compactum X and each abelian group G and every natural number $m > c(X)$ the group $H_m(X; G)$ is trivial.

From the universal-coefficient formula for the Čech cohomology ([21], p. 336) we infer also that for every compactum X and every $m > c(X)$ the group $H^m(X, G)$ is trivial for all G.

The *homological dimension* of a locally compact metric space X with respect to the group G (denoted by $\dim_G X$) is the maximum of the coefficients of cyclicity of open subsets of X with respect to the group G. This definition differs only formally from the classical definition (see [14], p. 7).

Fundamental dimension and the coefficient of cyclicity play in the theory of shape analogous roles to those played by the dimension and the homological dimension in topology.

The well-known theorem of P. S. Aleksandroff states that $\dim X = \dim_Z X$ for every finite-dimensional compactum X ([14], p. 8).

The aim of this paper is to give a homological characterization of fundamental dimension for approximatively 1-connected compacta. As the main results we obtain the theorems which state that if X is an approximatively 1-connected compactum with finite fundamental dimension (or X is a movable approximatively 1-connected continuum and the fundamental dimension of X is arbitrary) then $\operatorname{Fd}(X) = c(X)$.

From the last statements we deduce that for every natural number $n \geqslant 3$ there exists a sequence of polyhedra $Q_2^n, Q_3^n, \dots$ such that $\operatorname{Fd}(Q_p^n) = \dim Q_p^n = n$ and $\operatorname{Fd}(Q_p^n \times Q_q^m) = \max(m, n)$, where p and q are relatively prime natural numbers. In particular, there exist polyhedra X, Y such that $\operatorname{Fd}(X) = \operatorname{Fd}(Y) = \operatorname{Fd}(X \times Y) = \dim X = \dim Y = 3$.

Our example answers the following questions of K. Borsuk (see [4], p. 33):

(1) Is it true that $\operatorname{Fd}(X \times Y) > \operatorname{Fd}(X) + 1$ for all non-empty compacta X and Y such that $\operatorname{Fd}(Y) \geqslant 1$?

(2) Is it true that there exist polyhedra X and Y such that $\operatorname{Fd}(X \times Y) < \operatorname{Fd}(X) + \operatorname{Fd}(Y)$ and $\operatorname{Fd}(X) \neq 0 \neq \operatorname{Fd}(Y)$?

These questions are connected with the problem of the characterization of all compacta X for which there exists a compactum $Y \neq \emptyset$ such that $\operatorname{Fd}(X \times Y) < \operatorname{Fd}(X) + \operatorname{Fd}(Y)$. Applying the above mentioned homological characterization of the fundamental dimension we prove in the section five that for every approximatively 1-connected compactum with $\operatorname{Fd}(X) < \infty$ the following conditions are equivalent:

(a) There exists an approximatively 1-connected compactum Y such that $\operatorname{Fd}(X \times Y) < \operatorname{Fd}(X) + \operatorname{Fd}(Y)$.

(b) There exists an abelian group G such that $c_G(X) < \operatorname{Fd}(X)$.

The proof of this theorem is obtained by an easy modification of the proof of an analogous theorem for the homological dimension (see [14], p. 25).

The main theorems is also applied to compute the fundamental dimension of the suspension $\Sigma(X)$ of a compactum X. In the fourth section we show that if X is a compactum with $\operatorname{Fd}(X) < \infty$ (or X is a movable continuum and $\operatorname{Fd}(X)$ is arbitrary), then

$$\operatorname{Fd}(\Sigma(X)) = \begin{cases} c(X)+1 & \text{when } c(X) > 0, \\ 0 & \text{when } c(X) = 0 \text{ and } X \text{ is connected}, \\ 1 & \text{when } c(X) = 0 \text{ and } X \text{ is not a continuum}. \end{cases}$$

In [20] the author has shown that $\operatorname{Fd}(X \cup Y) \leqslant \max(\operatorname{Fd}(X), \operatorname{Fd}(Y), \operatorname{Fd}(X \cap Y) + 1)$ for all compacta X, Y.

In the last paragraph we study the problem of estimation from below of the fundamental dimension of the union of two compacta with a small fundamental dimension of their common part.

We assume that the reader is familiar with the theory of shape for compacta (see [1], [2], [3], [4], [5], [16], [17] and [18]).

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§ 1. Auxilliary notions and theorems. Let X, Y be compacta and let $f\colon X\to Y$ be a map and let n be a natural number or 0. We say (see [20]) that $\omega(f)\leqslant n$ iff there exists a map $g\colon X\to Y$ homotopic to f and such that $\dim g(X)\leqslant n$. If Y is a polyhedron, then $\omega(f)\leqslant n$ iff f is homotopic to a map $h\colon X\to Y$ such that $h(X)$ lies in the combinatorial n-skeleton of a triangulation of Y.

The author has proved (see [20], p. 214) the following

(1.1) THEOREM. *Let $X=\varprojlim\{X_k, p_k^{k+1}\}$, where X_k is a polyhedron for every $k=1,2,\dots$ Then the following conditions are equivalent:*

(i) $\mathrm{Fd}(X)\leqslant n$,

(ii) *$\omega(p_k)\leqslant n$ for every $k=1,2,\dots$, where $p_k\colon X\to X_k$ is the natural projection,*

(iii) *for every $k=1,2,\dots$ there exists a $k'>k$ such that $\omega(p_k^{k'})\leqslant n$.*

We need in the sequel the following theorem ([7], p. 376).

(1.2) THEOREM. *If X is a compactum and $\mathrm{Fd}(X)\leqslant n$ then $\omega(f)\leqslant n$ for every polyhedron Y and every map $f\colon X\to Y$.*

The proofs of the main results are based on some theorems and notions of the theory of obstructions obtained and introduced by Sze-Tsen Hu (see [9] and Chapter VII of [11]). We shall recall these notions and theorems in the compact and absolute case (Hu considered the general case of pairs of metric spaces).

Let X be a compactum and let $\mathfrak{U}$ be an open finite covering of X. By $N(\mathfrak{U})$ we denote the nerve of $\mathfrak{U}$. A map $\alpha_{\mathfrak{U}}\colon X\to N(\mathfrak{U})$ is called a *canonical map of* $\mathfrak{U}$ iff for each point $x\in X$, $\alpha_{\mathfrak{U}}(x)$ is contained in the closed simplex of $N(\mathfrak{U})$ whose vertices correspond to the members of containing x. It is known that for every finite open covering $\mathfrak{U}$ of a compactum X there exists a canonical map $\alpha_{\mathfrak{U}}\colon X\to N(\mathfrak{U})$.

Let Y be an arcwise connected space and let Y_0 be an arcwise connected closed subspace of Y and let $y_0\in Y_0$. For every $q=1,2,\dots$ the space Y is said to be *q-aspherical* relative to Y_0 if and only if $\pi_q(Y, Y_0, y_0)=0$ (see [11], p. 204 and [9], p. 110). The pair (Y, Y_0) is said to be *n-simple* (for $n\geqslant 1$) ([10], p. 138 and [21], p. 385) if $\pi_1(Y_0, y_0)$ acts trivially on $\pi_n(Y, Y_0, y_0)$. If Y_0 is simply connected, (Y, Y_0) is n-simple for every $n\geqslant 1$ ([21], p. 385).

Let $Y, Y_0\in\mathrm{ANR}$ be connected compacta and $y_0\in Y_0\subset Y$ and assume that X is a compactum. A map $f\colon X\to Y$ is said to be *n-deformable* into Y_0 ([11], p. 211) iff there exist a finite open covering $\mathfrak{U}$ of X and a map $\xi_{\mathfrak{U}}\colon N(\mathfrak{U})\to Y$ such that $\xi_{\mathfrak{U}}\big(N(\mathfrak{U})^{(n)}\big)\subset Y_0$ and $\xi_{\mathfrak{U}}\alpha_{\mathfrak{U}}$ is homotopic to f for every canonical map $\alpha_{\mathfrak{U}}$ of $\mathfrak{U}$, where $N(\mathfrak{U})^{(n)}$ is the n-skeleton of $N(\mathfrak{U})$.

S. T. Hu has shown ([11], p. 215) the following

(1.3) THEOREM. *Let $Y, Y_0\in\mathrm{ANR}$ be continua and $y_0\in Y_0\subset Y$. Let X be a compactum. Then every map $f\colon X\to Y$ is n-deformable into Y_0 if the following three condition are satisfied:*

(a) *Y is 1-aspherical relative to Y_0,*

(b) *(Y, Y_0) is q-simple for every q satisfying $1<q\leqslant n$,*

(c) *$H^q\big(X, \pi_q(Y, Y_0, y_0)\big)=0$ for every q satisfying $1<q\leqslant n$.*

Now let us prove the following lemmas:

(1.4) LEMMA. *Suppose that Z is a connected and simply connected polyhedron and let $Z^{(m)}$ be a combinatorial m-skeleton of a triangulation of Z, where $m\geqslant 2$. Then Z is q-aspherical relative to $Z^{(m)}$ for every $q\leqslant m$ and $(Z, Z^{(m)})$ is n-simple for every $n\geqslant 1$.*

Proof. Let $z_0\in Z^{(2)}$. Then $\pi_q(Z, Z^{(m)}, z_0)=0$ for every $q\leqslant m$ (see [21], p. 403) and it is evident ([21], p. 138) that $\pi_1(Z^{(2)}, z_0)=\pi_1(Z^{(m)}, z_0)=0$.

(1.5) LEMMA. *Suppose that $X\neq\emptyset$ is a continuum with $c(X)=n<\infty$ and $f\colon X\to Z$ is a map of X into a simply connected and connected polyhedron Z. Let $z_0\in Z$ and let Z_0 denote a subpolyhedron of Z which is the n-skeleton of a triangulation of Z when $n\geqslant 2$ and which is equel to $\{z_0\}$ for the case when $n<2$. Then f is m-deformable to Z_0 for every natural number m and if $\mathrm{Fd}(X)<\infty$ then $\omega(f)\leqslant n$.*

Proof. Let m be a natural number. We can assume that z_0 is a vertex of a triangulation of Z.

Suppose that $n\geqslant 2$. From Lemma (1.4) we infer that the pair $(Z, Z^{(m)})$ satisfies the conditions (a), (b) of Theorem (1.3) and moreover, $\pi_q(Z, Z^{(n)}, z_0)=0$ for every $1<q\leqslant n$. In Introduction we shown that if Y is a compactum then $H^k(Y; G)=0$ for every natural number $k>c(Y)$ and each abelian group G. Therefore $H^q\big(X; \pi_q(Z, Z^{(n)}, z_0)\big)=0$ for $q>n$. Thus the condition (c) is satisfied and f is m-deformable to $Z^{(m)}=Z_0$.

Suppose now that $c(X)=n<2$. Our assumption implies that $H^m(X; G)=0$ for every $m\geqslant 2$ and each abelian group G.

Using Theorem (1.3) for the case where $Y=Z$, $Y_0=Z_0=\{z_0\}$ and $X=X$ we infer that f is m-deformable to $\{z_0\}=Z_0$.

Let $\mathrm{Fd}(X)=n_0<\infty$. We know that f is n_0-deformable to Z_0 and that $\dim Z_0<c(X)=n$. This means that there are an open finite covering $\mathfrak{U}$ of X and a map $\xi_{\mathfrak{U}}\colon N(\mathfrak{U})\to Z$ such that $\xi_{\mathfrak{U}}\big(N(\mathfrak{U})^{(n_0)}\big)\subset Z_0$ (where $N(\mathfrak{U})^{(n_0)}$ is the n_0-skeleton of $N(\mathfrak{U})$) and $\xi_{\mathfrak{U}}\alpha_{\mathfrak{U}}$ is homotopic to f for an arbitrary canonical map $\alpha_{\mathfrak{U}}\colon X\to N(\mathfrak{U})$.

From (1.2) we infer that there exists for every canonical map $\alpha_{\mathfrak{U}}\colon X\to N(\mathfrak{U})$ a map $\beta\colon X\to N(\mathfrak{U})$ homotopic to $\alpha_{\mathfrak{U}}$ and such that $\beta(X)\subset N(\mathfrak{U})^{(n_0)}$.

Therefore $\xi_{\mathfrak{U}}\beta(X) \subset Z_0$, $\xi_{\mathfrak{U}}\beta$ and f are homotopic and $\omega(f) \leqslant n$. Thus the proof of Lemma (1.5) comes to an end.

If (X, x_0), (Y, y_0) are pointed topological spaces and $f\colon (X, x_0) \to (Y, y_0)$ is a map we denote by $f_{\#}\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ the induced homomorphism.

(1.6) Lemma. *Suppose that* (W, w_0), (Z, z_0) *are pointed connected polyhedra and let* $f\colon (Z, z_0) \to (W, w_0)$ *be a map such that* $f_{\#}\colon \pi_1(Z, z_0) \to \pi_1(W, w_0)$ *is a null homomorphism. Then there are a simply connected and connected polyhedron* $Z_1 \supset Z$ *and a continuous extension* $\tilde{f}$ *of* f *mapping* (Z_1, z_0) *into* (W, w_0).

Proof. We can assume that Z is a subpolyhedron of E^m (for some m) such that if $(x_1, x_2, \ldots, x_m) \in Z$ then $x_m = 0$. Let $b = (0, 0, \ldots, 0, 1) \in E^m$ and let $\mathcal{C}$ be a triangulation of Z.

Let us consider the polyhedron Z_1 which is the union of Z and a cone with a vertex b and a base $Z^{(1)}$, where $Z^{(1)}$ is the 1-skeleton of Z.

Let $\mathcal{C}_1$ be a triangulation of Z_1, which is a collection of all simplexes belonging to $\mathcal{C}$ and all simplexes which are cones with the vertex b and the base which is a 1-dimensional simplex $\sigma \in \mathcal{C}$ and all their faces.

Let $Z_1^{(1)}$ be a 1-skeleton of $\mathcal{C}_1$.

It is evident that there exists a retraction r of $Z_1^{(1)} \cup Z$ to Z such that $r(b) = z_0$.

If $\dot{\sigma}$ is a boundary of a 2-simplex $\sigma \in \mathcal{C}_1 \setminus \mathcal{C}$ then a map $\mathrm{fr}|\dot{\sigma}\colon (\dot{\sigma}, b) \to (W, w_0)$ is null homotopic. Hence there is a continuous extension $f_\sigma\colon \sigma \to W$ of $\mathrm{fr}|\dot{\sigma}$ for every 2-simplex $\sigma \in \mathcal{C}_1 \setminus \mathcal{C}$.

Setting

$$\tilde{f}(z) = \begin{cases} f(z) & \text{for every } z \in Z, \\ f_\sigma(z) & \text{for every } z \in \sigma, \text{ where } \sigma \in \mathcal{C}_1 \setminus \mathcal{C} \text{ is a 2-simplex}, \end{cases}$$

we get a continuous extension $\tilde{f}\colon Z_1 \to W$ of f. It is evident that $\pi_1(Z_1, z_0) = 0$. Thus (1.6) is proved.

Remark. Lemma (1.6) and its proof was communicated to the author by A. Trybulec.

Let (X_k, x_k^0) be a pointed compactum and let $p_k^{k+1}\colon (X_{k+1}, x_{k+1}^0) \to (X_k, x_k^0)$ be a map for every $k = 1, 2, \ldots$ Then $\{x_k^0\}_{k=1}^{\infty} = x_0 \in X = \varprojlim\{X_k, p_k^{k+1}\}$ and we write $(X, x_0) = \varprojlim\{(X_k, x_k^0), p_k^{k+1}\}$.

Let (X, Y) be topological spaces and $A \subset X$ and let $f_0, f_1\colon X \to Y$ be maps which agree on A. Then we say that f_0 *is homotopic to* f_1 *relative to* A (denoted by $f_0 \simeq f_1 \operatorname{rel} A$) if there exists a homotopy $\varphi\colon X \times [0, 1] \to Y$ such that $\varphi(x, 0) = f_0(x)$ and $\varphi(x, 1) = f_1(x)$ for $x \in X$ and $\varphi(x, t) = f_0(x)$ for $x \in A$ and $t \in [0, 1]$.

Let us observe that following proposition holds true:

(1.7) Proposition. *If* $(X, x_0) = \varprojlim\{(X_k, x_k^0), p_k^{k+1}\}$ *and* $(Y, y_0) = \varprojlim\{(X_k, x_k^0), q_k^{k+1}\}$, *where* X_k *is a polyhedron and* $p_k^{k+1}, q_k^{k+1}\colon (X_{k+1}, x_{k+1}^0) \to (X_k, x_k^0)$ *are maps such that* $p_k^{k+1} \simeq q_k^{k+1} \operatorname{rel}\{x_{k+1}^0\}$ *then* $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0)$.

In fact, the pair $(\mathrm{id}_N, \mathrm{id}_X)\colon \underline{(X, x_0)} = \{(X_k, x_k^0), p_k^{k+1}\} \to \{(X_k, x_k^0), q_k^{k+1}\} = \underline{(Y, y_0)}$ (N is the set of natural numbers) is a shape equivalence of ANR-sequences $\underline{(X, x_0)}$ and $\underline{(Y, y_0)}$ associated with (X, x_0) and (Y, y_0), respectively, (see [16], p. 45 and [17], p. 62).

§ 2. The main theorems. Let us prove the following

(2.1) Theorem. *If* (X, x_0) *is an approximatively* 1*-connected pointed continuum with* $\mathrm{Fd}(X) < \infty$ *then* $\mathrm{Fd}(X) = c(X)$.

Proof. Let $\{(X_k, x_k^0), p_k^{k+1}\}$ be an inverse sequence such that X_k is a connected polyhedron for $k = 1, 2, \ldots$ and $(X', x_0') = \varprojlim\{(X_k, x_k^0), p_k^{k+1}\}$ is homeomorphic to (X, x_0).

It is known (see Lemma (6.1) of [19]) that for every k there exists a $\gamma(k) > k$ such that

$$(p_k^{\gamma(k)})_{\#}\colon \pi_1(X_{\gamma(k)}, x_{\gamma(k)}^0) \to \pi_1(X_k, x_k^0)$$

is a null homomorphism.

Let $p_k = p_k^{\gamma(k)} p_{\gamma(k)}\colon (X', x_0') = \varprojlim\{(X_k, x_k^0), p_k^{k+1}\} \to (X_k, x_k^0)$ be a natural projection for every $k = 1, 2, \ldots$ From Lemma (1.6) we infer that there are a connected and simply connected polyhedron $\hat{X}_{\gamma(k)} \supset X_{\gamma(k)}$ and a continuous extension $\hat{p}_k^{\gamma(k)}\colon X_{\gamma(k)} \to X_k$ of $p_k^{\gamma(k)}$. We can assume that $p_k^{\gamma(k)}$ is a simplicial map (see [21], p. 126 and (1.7)). From Lemma (1.5) we infer that $\omega(ip_{\gamma(k)}) \leqslant c(X)$, where $i\colon X_{\gamma(k)} \to \hat{X}_{\gamma(k)}$ is the inclusion. This means that there exists a homotopy $\varphi\colon X' \times [0, 1] \to \hat{X}_{\gamma(k)}$ such that

$$\varphi(x, 0) = p_{\gamma(k)}(x) \quad \text{for} \quad x \in X$$

and

$$\dim \varphi(X' \times \{1\}) \leqslant c(X).$$

Let $\psi\colon X' \times [0, 1] \to X_k$ be a homotopy defined by the formula

$$\psi(x, t) = \hat{p}_k^{\gamma(k)} \varphi(x, t) \quad \text{for} \quad (x, t) \in X' \times [0, 1].$$

Then

$$\psi(x, 0) = p_k(x) \quad \text{for} \quad x \in X'$$

and

$$\dim \psi(X' \times \{1\}) \leqslant c(X)$$

and therefore

$$\omega(p_k) \leqslant c(X).$$

We conclude that

$$\mathrm{Fd}(X) \leqslant c(X).$$

It is clear that $c(X) \geqslant \mathrm{Fd}(X)$.

The proof of Theorem (2.1) is finished.

Remark. One can give an other proof of Theorem (2.1), in which we do not have to use Theorem (1.6) but we must use some results obtained by E. Spanier and J. H. C. Whitehead (see [22]).

K. Borsuk has proved (see [5]) that a compactum X is approximatively n-connected iff each of its components is approximatively n-connected. It is known also (see [20]) that $\mathrm{Fd}(X) \leqslant n$ iff for every component X_0 of X the inequality $\mathrm{Fd}(X_0) \leqslant n$ holds true.

From these facts and (2.1) we get

(2.2) THEOREM. *If X is an approximatively 1-connected compactum with $\mathrm{Fd}(X) < \infty$ then $\mathrm{Fd}(X) = c(X)$.*

(2.3) Remark. The assertion of (2.2) is false if one omits the assumption that X is approximatively 1-connected. Let $\mathcal{T}$ be a triangulation of a Poincaré sphere, i.e. of a 3-manifold P with the homology groups of the 3-sphere but with the first homotopy group non-trivial. Let G denote the interior of a 3-simplex of $\sigma \in \mathcal{T}$ and let $X = P \setminus G$. One can verify that $\mathrm{Fd}(X) = 2$ and $c(X) = 0$.

Considering the special case of movable continua, we get the following

(2.4) THEOREM. *Let (X, x_0) be an approximatively 1-connected pointed continuum and let X be a movable. Then $\mathrm{Fd}(X) = c(X)$.*

Proof. Let $c(X) = n < \infty$. We can assume that $(X, x_0) = \varprojlim\{(X_k, x_k^0), p_k^{k+1}\}$, where X_k is a connected polyhedron and $p_k^{k+1}\colon X_{k+1} \to X_k$ is a simplicial map (see (1.7)) for every $k = 1, 2, \ldots$

It is known ([15], p. 272) that

(2.5) For every polyhedron Y and every map $f\colon X \to Y$ there exist a natural number k and a map $f'\colon X_k \to Y$ such that $f' p_k \simeq f$, where $p_k\colon X \to X_k$ is the natural projection.

(2.6) For every polyhedron Y and all maps $f, g\colon X_k \to Y$ such that $f p_k \simeq g p_k$ there exists a natural number $k' > k$ such that $f p_k^{k'} \simeq g p_k^{k'}$.

Since X is movable and approximatively 1-connected, we can assume (see Lemma (6.1) of [19] and [18], p. 651) that

(2.7) $(p_k^{k+1})_{\#}\colon \pi_1(X_{k+1}, x_{k+1}^0) \to \pi_1(X_k, x_k^0)$ is a null homomorphism for every $k = 1, 2, \ldots$

and

(2.8) For all natural numbers n, n' such that $n' > n$ there is a map $r\colon X_{n+1} \to X_n$ satisfying the homotopy relation $p_n^{n'} r \simeq p_n^{n+1}$.

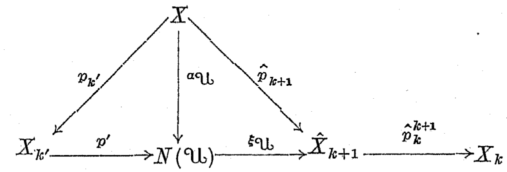
Let k be a fixed natural number. Lemmas (1.6) and (2.7) imply that there exist a connected and simply connected polyhedron $\hat{X}_{k+1} \supset X_{k+1}$ and a continuous extension $\hat{p}_k^{k+1}\colon \hat{X}_{k+1} \to X_k$ of p_k^{k+1}.

Let Y_{k+1} (Y_k) be a subpolyhedron of $\hat{X}_{k+1}$ (X_k) which is the n-skeleton of a triangulation of X_{k+1} when $n \geqslant 2$ and which is equal to $\{x_{k+1}^0\}$ $(\{x_k^0\})$ for the case when $n < 2$.

From Lemma (1.5) we conclude that $\hat{p}_{k+1} = i p_k^{k+1}\colon X \to \hat{X}_{k+1}$ is m-deformable to Y_{k+1}, where $m = \dim X_{k+1}$ and $i\colon X_{k+1} \to \hat{X}_{k+1}$ if the inclusion.

This means that there exists an open finite covering $\mathfrak{U}$ of X and a map $\xi_{\mathfrak{U}}\colon N(\mathfrak{U}) \to \hat{X}_{k+1}$ such that $\xi_{\mathfrak{U}}(N(\mathfrak{U})^{(m)}) \subset Y_{k+1}$ and $\xi_{\mathfrak{U}} a_{\mathfrak{U}}$ is homotopic to $\hat{p}_{k+1}$ for arbitrary canonical map $a_{\mathfrak{U}}\colon X \to N(\mathfrak{U})$.

From (2.5) we infer that there exist $k' > k+1$ and a map $p'\colon X_{k'} \to N(\mathfrak{U})$ such that $p' p_k \simeq a_{\mathfrak{U}}$, where $a_{\mathfrak{U}}\colon X \to N(\mathfrak{U})$ is a canonical map.



We can assume that

(2.9) $$p'(X_k^{(m)}) \subset N(\mathfrak{U})^{(m)}.$$

Let us observe that

(2.10) $$\xi_{\mathfrak{U}} p' p_{k'} \simeq \hat{p}_{k+1} = i p_k^{k'} p_{k'}$$

($i\colon X_{k+1} \to \hat{X}_{k+1}$ is the inclusion).

From (2.6) and (2.10) we infer that there exists $k'' > k'$ such that

$$\xi_{\mathfrak{U}} p' p_{k'}^{k''} \simeq i p_{k+1}^{k'} p_{k'}^{k''} = i p_{k+1}^{k} = p_{k+1}^{k''}.$$

Hence

(2.11) $$\hat{p}_k^{k+1} \xi_{\mathfrak{U}} p' p_{k'}^{k''} \simeq p_k^{k+1} i p_{k+1}^{k''} = p_k^{k''}$$

and by (2.9)

(2.12) $$\hat{p}_k^{k+1} \xi_{\mathfrak{U}} p' p_k^{k}(X_{k''}^{(m)}) \subset Y_k.$$

Let $r\colon X_{k+1} \to X_{k''}$ be a map (see (2.8)) such that

(2.13) $$p_k^{k''} r \simeq p_k^{k+1}.$$

Since $\dim X_{k+1} = m$, we can assume that

(2.14) $$r(X_{k+1}) \subset X_{k''}^{(m)}.$$

The statements (2.11) and (2.13) imply that

$$p_k^{k+1} \simeq p_k^{k''} r \simeq \hat{p}_k^{k+1} \xi_{\mathfrak{U}} p' p_{k'}^{k''} r.$$

By (2.12) and (2.14) we have

$$\hat{p}_k^{k+1}\xi_{\mathfrak{U}}p'p_{k'}^{k''}r(X_{k+1})\subset Y_k .$$

Therefore $\omega(p_k^{k+1})\leqslant \dim Y_k\leqslant c(X)=n$ and Theorem (1.1) implies that $\mathrm{Fd}(X)=n$.

(2.15) Remark. The assumption of movability in Theorem (2.1) is essential. Indeed, let X be an acyclic approximatively 1-connected continuum described by Kahn in [13]. Then $\mathrm{Fd}(X)=\infty$ (see [8], p. 172) and $c(X)=0$.

§ 3. Fundamental dimension of the Cartesian product of simply connected polyhedra. For every compactum X we denote by $\Sigma(X)$ the suspension of X. By C or Z_p we denote (respectively) the field of complex numbers and the cyclic group of finite order p.

Consider the set D consisting of all points $z\in C$ such that $|z|\leqslant 1$, its subset S consisting of all $z\in D$ such that $|z|=1$ and the map $a_p\colon S\to S$ given by the formula

$$a_p(z)=z^p \quad \text{for} \quad z\in S$$

and for every $p=2,3,\ldots$

Let Q_p be the decomposition space of the upper semicontinuous decomposition of D into the following sets:

(a) The single points of $D\setminus S$.

(b) The sets $a_p^{-1}(x)$ with $x\in S$.

Then Q_p is a polyhedron and it is known that $H^2(Q_p;Z)$ is isomorphic to Z_p and $H^1(Q_p;Z)$ is the trivial group for every $p\geqslant 2$.

Applying the universal-coefficient formula for cohomology (see [21], p. 336) we infer that if p and q are relatively prime natural numbers then

(3.1) $$H^i(Q_p;Z_q)=0 \quad \textit{for every } i>0 .$$

Let $Q_p^2=Q_p$ for every $p=2,3,\ldots$ and let $Q_p^m=\Sigma(Q_p^{m-1})$ for every $p=2,3,\ldots$ and every $m=3,4,\ldots$

From (3.1) we infer that the group $H^m(Q_p^m;Z)$ is isomorphic to Z_p and the groups $H^i(Q_p^m;Z)$ are trivial for every i satisfying the inequality $0<i<m$ and that $H^i(Q_p^m,Z_q)=0$ for every $i>0$, where p and q are relatively prime natural numbers.

It is evident that $\dim Q_p^m=m=\mathrm{Fd}(Q_p^m)$.

It is known (see [14], p. 6) that

(3.2) $$H^n(X\times Y;G)\approx \sum_{l+k=n} H^l\big(X;H^k(Y;G)\big) \textit{ for all compacta } X,Y \textit{ and each abelian group } G.$$

Computing $H^i(Q_p^m\times Q_q^n;Z)$ by the formula (3.2) we see that $H^i(Q_p^n\times Q_q^n;Z)=0$ for all relatively prime natural numbers p and q and every i satisfying the inequality $m+n\geqslant i>\max(m,n)$. Moreover, if $n\neq m$ then $H^m(Q_p^m\times Q_q^n;Z)\approx Z_p$ and $H^n(Q_p^m\times Q_q^n;Z)\approx Z_q$ and if $n=m$ then $H^m(Q_p^m\times Q_q^n,Z)\approx Z_q\oplus Z_p$.

Hence $c(Q_p^m\times Q_q^n)=\max(m,n)$ for all relatively prime natural numbers p and q and all $m\geqslant 2$.

Since Q_p^m is the suspension of some connected polyhedron for every $m=3,4,\ldots$ we infer that the first homotopy group of Q_p^m is trivial when $m\geqslant 3$.

Thus we obtain from (2.2) the following

(3.3) THEOREM. *For every $m\geqslant 3$ there exists a sequence $\{Q_p^m\}_{p=2}^{\infty}$ of polyhedra such that $\dim Q_p^m=m=\mathrm{Fd}(Q_p^m)$ and $\mathrm{Fd}(Q_p^m\times Q_q^n)=\max(m,n)$ for all relatively prime natural numbers p and q.*

Remark. In the case when p and q are not relatively prime we can prove in a similar way that $\mathrm{Fd}(Q_p^m\times Q_q^n)=m+n$ for all $m,n\geqslant 3$.

§ 4. Fundamental dimension of the suspension of a compactum. If X,Y are compacta and $f\colon X\to Y$ is a map then we denote by $\Sigma(f)$ the suspension of f.

Let X be a continuum. Then there is an inverse sequence $\{X_k,p_k^{k+1}\}$, where X_k is a connected polyhedron for every $k=1,2,\ldots$ such that X is homeomorphic to $\varprojlim\{X_k,p_k^{k+1}\}$. This implies that $\Sigma(X)$ is homeomorphic to $\varprojlim\{\Sigma(X_k),\Sigma(p_k^{k+1})\}$. It is clear that $\Sigma(X_k)$ is a simply connected polyhedron for every $k=1,2,\ldots$ Therefore (see Lemma (6.1) of [19]) X is an approximatively 1-connected continuum.

Hence we obtain

(4.1) LEMMA. *If X is a continuum then $\Sigma(X)$ is an approximatively 1-connected continuum.*

The author has proved ([20]) that if X is a compactum and A is a closed subset of X then

(4.2) $$\mathrm{Fd}(X/A)\leqslant\max\big(\mathrm{Fd}(X),\mathrm{Fd}(A)+1\big).$$

From (4.2) we infer (by easy induction) that the following lemma holds

(4.3) LEMMA. *Let $X\neq\emptyset$ be a compactum and let $A_1,A_2,\ldots,A_n$ be non-empty disjoint closed subsets of X. If Y denote the hyperspace of the upper semicontinuous decomposition of X into the sets $A_1,A_2,\ldots,A_n$ and the single points of $X\setminus\bigcup_{i=1}^{n}A_i$ then* $\mathrm{Fd}(Y)\leqslant\max\big(\mathrm{Fd}(X),\mathrm{Fd}(A_1)+1,\ldots,\mathrm{Fd}(A_n)+1\big)$.

Now let us prove the following

(4.4) THEOREM. *For every compactum $X \neq \emptyset$ with a finite fundamental dimension*

$$\mathrm{Fd}(\Sigma(X)) = \begin{cases} c(X)+1 & \text{when } c(X) > 0, \\ 0 & \text{when } c(X) = 0 \text{ and } X \text{ is connected}, \\ 1 & \text{when } c(X) = 0 \text{ and } X \text{ is not a continuum}. \end{cases}$$

Proof. It was proved (see [20]) by W. Holsztyński that for every compactum X there exists a compactum X' such that $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ and $\mathrm{Fd}(X) = \dim X'$. K. Borsuk has shown ([2], p. 250) that the shape of the suspension of a compactum X depends only on the shape of X. Therefore we can assume that $X \subset E^m$ for a certain m.

It is convenient for our purposes to regard E^m as the subset of E^{m+1} consisting of all points $x = (x_1, x_2, \ldots, x_m, x_{m+1})$ with $x_{m+1} = 0$.

For every natural number n there are a natural number l_n, a function a_n: $\{1, 2, \ldots, l_{n+1}\} \to \{1, 2, \ldots, l_n\}$ and disjoint non-empty subpolyhedra $X_n^1, X_n^2, \ldots, X_n^{l_n}$ of E^m such that

$$X_{n+1}^i \subset X_n^{a_n(i)} \quad \text{for every } i = 1, 2, \ldots, l_{n+1}, \tag{4.5}$$

$$X = \bigcap_{n=1}^{\infty} (X_n^1 \cup X_n^2 \cup \ldots \cup X_n^{l_n}). \tag{4.6}$$

Let $X_n = X_n^1 \cup X_n^2 \cup \ldots \cup X_n^{l_n}$. The conditions (4.5) and (4.6) imply that

$$X = \bigcap_{n=1}^{\infty} X_n \quad \text{and} \quad X_n \supset X_{n+1} \quad \text{for every } n = 1, 2, \ldots$$

Let $b_n^i = (b_1^{ni}, b_2^{ni}, \ldots, b_m^{ni}, 0) \in X_n^i \subset E^m$ and $a_{n,\varepsilon}^i = (b_1^{ni}, b_2^{ni}, \ldots, b_m^{ni}, \varepsilon) \in E^{m+1}$ for all natural numbers n, i such that $1 \leqslant i \leqslant l_n$ and for $\varepsilon = -1, 1$.

Consider now for every $k = 1, 2, \ldots$ and every $i \leqslant l_n$ a subpolyhedron V_n^i of E^m which is the union of the cone with the base X_n^i and the vertex $a_{n,-1}^i$ and the cone with the base X_n^i and the vertex a_{n1}^i.

It is clear that V_n^i is homeomorphic to $\Sigma(X_n^i)$ and that V_n^i is a simply connected polyhedron.

Let $A_n^1 = \bigcup_{i=1}^{l_n} \{a_{n,-1}^i\}$, $A_n^2 = \bigcup_{i=1}^{l_n} \{a_{n,1}^i\}$ and $V_n = \bigcup_{i=1}^{l_n} V_n^i$.

Consider now maps s_k^{k+1}: $A_{k+1}^1 \to A_k^1$, r_k^{k+1}: $A_{k+1}^2 \to A_k^2$ and p_k^{k+1}: $V_{k+1} \to V_k$ defined by the formulas

$$s_k^{k+1}(a_{k+1,-1}^i) = a_{k,-1}^{a_k(i)},$$

$$r_k^{k+1}(a_{k+1,1}^i) = a_{k,1}^{a_k(1)}$$

$$p_k^{k+1}(z) = (1-t)x + t a_{k,\varepsilon}^{a_k(1)}$$

for every $z = (1-t)x + t a_{k+1,\varepsilon}^i$ where $x \in X_{k+1}^i$ and $\varepsilon = -1, 1$.

Setting $Y = \varprojlim\{V_k, p_k^{k+1}\}$, $A_1 = \varprojlim\{A_k^1, s_k^{k+1}\}$ and $A_2 = \varprojlim\{A_k^2, r_k^{k+1}\}$ one easily sees that $A_1 \cap A_2 = \emptyset$, $\dim A_1 = \dim A_2 = 0$ and $A_1, A_2 \subset Y$.

Consider now the hyperspace W of the upper semicontinuous decomposition of Y into the sets A_1, A_2 and the single points of $Y \setminus (A_1 \cup A_2)$ and let W_n denote the quotient space V_n in which A_n^1 is identified to one point and A_n^2 is identified to another point. For every natural number k the map p_k^{k+1}: $V_{k+1} \to V_k$ induces a map u_k^{k+1}: $W_{k+1} \to W_k$. We note that W_k is homeomorphic to $\Sigma(X_k)$ and that $\varprojlim\{W_k, u_k^{k+1}\}$ is homeomorphic to W and $\Sigma(X)$.

From Lemma (4.3) we infer that

$$\mathrm{Fd}(\Sigma(X)) = \mathrm{Fd}(W) \leqslant \max(\mathrm{Fd}(Y), 1). \tag{4.7}$$

Let $\square(X)$ and $\square(Y)$ denote respectively the sets of all components of X and Y.

The conditions (4.5) and (4.6) imply that there exists a one-to-one correspondence Λ: $\square(X) \to \square(Y)$ such that for every $X_0 \in \square(X)$ continuum $\Lambda(X_0) \in \square(Y)$ is homeomorphic to the suspension of X_0. From Lemma (4.1) we infer that each component of Y is an approximatively 1-connected compactum. It follows (see [5]) that Y is an approximatively 1-connected compactum.

Since Y is an approximatively 1-connected compactum and $\mathrm{Fd}(Y) < \infty$, we conclude that $\mathrm{Fd}(Y) = c(Y)$.

The continuity property of the Čech cohomology implies that the groups $H^n(Y; G)$ and $H^n(W; G)$ are isomorphic for every Abelian group G and every $n \geqslant 2$: Therefore

$$\max(1, c(Y)) = \max(1, c(W)) = \max(1, c(\Sigma(X))).$$

The inequality (4.7) implies that

$$c(\Sigma(X)) \leqslant \mathrm{Fd}(\Sigma(X)) = \mathrm{Fd}(W) \leqslant \max(1, \mathrm{Fd}(Y)) = \max(1, c(Y)) = \max(1, c(\Sigma(X))). \tag{4.8}$$

It is known that

(4.9) *For every compactum X and each Abelian group G the groups $H^n(X; G)$ and $H^{n+1}(\Sigma(X); G)$ are isomorphic for $n \geqslant 1$.*

Suppose that $c(X) > 0$.

Then from (4.9) we infer that $c(\Sigma(X)) = c(X)+1 \geqslant 2$ and from (4.8) we infer that $\mathrm{Fd}(\Sigma(X)) = c(\Sigma(X)) = c(X)+1$.

Suppose that $c(X) = 0$.

One knows that $H^1(\Sigma(X); Z) = 0$ when X is a continuum and $H^1(\Sigma(X); Z) \neq 0$ when Y is not a connected compactum.

If X is connected then W is homeomorphic to Y and $\Sigma(X)$. Therefore $\mathrm{Fd}(\Sigma(X)) = c(Y) = 0$.

If X is not a continuum, then $c(\Sigma(X)) = 1$ and from (4.8) we infer that $\mathrm{Fd}(\Sigma(X)) = 1$.

This completes the proof of Theorem (4.4).

K. Borsuk has proved ([2], p. 251) that the suspension $\Sigma(X)$ of a movable compactum X is movable. This fact, Theorem (2.4) and Lemma (4.1) imply at once the following

(4.10) THEOREM. *Let $X \neq \emptyset$ be a movable continuum. Then*

$$\mathrm{Fd}(\Sigma(X)) = \begin{cases} c(X)+1 & \text{when } c(X) > 0, \\ 0 & \text{when } c(X) = 0. \end{cases}$$

Remark. The assumption of the movability in Theorem (4.10) is essential. Let X be the Kahn's continuum (see [13]). Then $\mathrm{Fd}(\Sigma(X)) = \infty$ and $c(X) = 0$ (see [8] and (2.15)).

§ 5. Fundamental dimension of the Cartesian product of approximatively 1-connected compacta. In this section we adopt the notations of [14]. By Q we denote the group of rational numbers and for every prime natural number p we denote by R_p the group of rational numbers which can be represented in the form $\frac{m}{n}$, where m and n are integer numbers and p does not divide n. Let us also put $Q_p = Q/R_p$ for every prime number p.

If G is an Abelian group, then we denote by $\sigma(G)$ the collection of Abelian groups defined by the following conditions:

(a) $Q \in \sigma(G)$ iff G contains an element of infinite order.

(b) If p is a prime number, then $Z_p \in \sigma(G)$ iff G contains an element g of order p^k (where k is a natural number) such that g is not divisible by p.

(c) $Q_p \in \sigma(G)$ iff G contains an element of order p.

(d) $R_p \in \sigma(G)$ iff there is an element a of G such that for every integer number $n \geqslant 0$ the number p^{n+1} does not divide $p^n a$.

(e) If $H \neq Q, Q_p, Z_p, R_p$ then H does not belong to $\sigma(G)$ (p is a prime number).

In the sequel we need the following three propositions:

(5.1) PROPOSITION. *For every locally compact metric space X and every Abelian group G the equality $c_\sigma(X) = \max_{H \in \sigma(G)} \{c_H(X)\}$ holds true.*

(5.2) PROPOSITION. *If X is a locally compact metric space and p is a prime number, then*

$$c_{Q_p}(X) \leqslant c_{Z_p}(X) \leqslant c_{Q_p}(X)+1,$$
$$c_Q(X) \leqslant c_{R_p}(X),$$
$$c_{Q_p}(X) \leqslant \max(c_Q(X), c_{R_p}(X)-1),$$
$$c_{R_p}(X) \leqslant \max(c_Q(X), c_{Q_p}(X)+1).$$

(5.3) PROPOSITION. *Let X, Y be locally compact metric spaces and let p be a prime number. Then*

$$c_{Z_p}(X \times Y) = c_{Z_p}(X) + c_{Z_p}(Y),$$
$$c_Q(X \times Y) = c_Q(X) + c_Q(Y),$$
$$c_{Q_p}(X \times Y) = \max(c_{Q_p}(X) + c_{Q_p}(Y), c_{Z_p}(X \times Y) - 1),$$
$$c_{R_p}(X \times Y) = \begin{cases} c_{R_p}(X) + c_{R_p}(Y) & \text{if } c_{Q_p}(X) = c_{R_p}(X) \text{ or } c_{Q_p}(Y) = c_{R_p}(Y), \\ \max(c_{Q_p}(X \times Y)+1, c_Q(X \times Y)) & \text{if } c_{Q_p}(X) < c_{R_p}(X) \text{ and } c_{Q_p}(Y) < c_{R_p}(Y). \end{cases}$$

It is known that if we replace in (5.1), (5.2) and (5.3) c_G by $\dim_G$, c_H by $\dim_H$, c_Q by $\dim_Q$, c_{Z_p} by $\dim_{Z_p}$, c_{R_p} by $\dim_{R_p}$ and c_{Q_p} by $\dim_{Q_p}$, then we get theorems which hold true (see [14], p. 12 and p. 14 and p. 15). Moreover, studying [14], one can observe that the proofs of those theorems contain the proofs of Propositions (5.1), (5.2) and (5.3). Therefore we omit them.

One knows ([14], p. 24) that for every fixed prime number p there is a compactum FQ_{p2} and a simple closed curve $B_p \subset FQ_{p2}$ (denoted in [14] by $\dot{X}_\infty$) such that

$$H^2(FQ_{p2}/B_p; R_p) \approx H^2_c(FQ_{p2} \setminus B_p; R_p) \neq 0$$

and

$$H^2(FQ_{p2}/B_p, G) = H^2_c(FQ_{p2} \setminus B_p; G) = 0$$

when $G = Q, Z, Z_p, Z_q, R_q, Q_p, Q_q$ and q is a prime number $\neq p$.

Let $A_p = \Sigma(FQ_{p2}/B_p)$.

It follows from the analysis of the construction FQ_{p2} that FQ_{p2} is a continuum. From Lemma (4.1) and (4.9) we infer that A_p is an aproximatively 1-connected compactum such that

$$c_{R_p}(A_p) = \mathrm{Fd}(A_p) = c(A_p) = 3$$

and

$$c_G(A_p) < 3 \quad \text{when} \quad G = Q, Z, Z_p, Z_q, R_q, Q_p, Q_q \text{ and } p \neq q.$$

Let us prove the following

(4.4) THEOREM. *For every approximatively 1-connected compactum X with $\mathrm{Fd}(X) < \infty$ the following conditions are equivalent:*

(i) *For every Abelian group G the equality $c_G(X) = \mathrm{Fd}(X)$ holds true.*

(ii) *If $Y \neq \emptyset$ is an approximatively 1-connected compactum then* $\mathrm{Fd}(X \times Y) = \mathrm{Fd}(X) + \mathrm{Fd}(Y)$.

(iii) $\mathrm{Fd}(X \times A_p) = \mathrm{Fd}(X) + 3$ *for every prime number p.*

Proof. (i) implies (ii). Let Y be an approximatively 1-connected compactum and $\mathrm{Fd}(Y) < \infty$. From Theorem (2.2) and Proposition (5.1) we infer that $c(X) = \max\{c_{R_R}(X)\}$, $c(Y) = \max\{c_{R_R}(Y)\}$, $c(X\times Y) = \mathrm{Fd}(X\times Y) = \max\{c_{R_R}(X\times Y)\}$. If $c_{Q_p}(X) = c_{R_p}(X) = c(X) = \mathrm{Fd}(X)$ for every prime number p, then $c_{R_p}(X\times Y) = c_{R_p}(X)+c_{R_p}(Y)$ and

$$\max\{c_{R_p}(X\times Y)\} = \max\{c_{R_p}(X)+c_{R_p}(Y)\} = \max\{\mathrm{Fd}(X)+\mathrm{Fd}(Y)\}$$
$$= \mathrm{Fd}(X)+\mathrm{Fd}(Y).$$

It is clear that (ii) implies (iii).

(iii) implies (i). From (5.3) we infer that $c_{R_p}(X\times A_p) = \mathrm{Fd}(X)+3$ for every $p\neq q$. Therefore $c_{R_p}(X\times A_p) = \mathrm{Fd}(X)+3$. This implies that $c_{Q_p}(X) = c_{R_p}(X) = \mathrm{Fd}(X) = c(X)$. Propositions (5.2) and (5.1) imply that $c_G(X) = c(X) = \mathrm{Fd}(X)$ for every Abelian group G.

§ 6. Some problems. Let us prove the following

(6.1) PROPOSITION. *If X, Y are compacta and*

$$c(X\cap Y) < \max(c(X), c(Y)),\quad \textit{then}\ c(X\cup Y) = \max(c(X), c(Y)).$$

Proof. It is known (see [14], p. 5) that for every integer number n there are homomorphisms

$$\Delta^n\colon H^{n-1}(X\cap Y; Z)\to H^n(X\cup Y; Z),$$
$$\varphi^n\colon H^n(X\cup Y; Z)\to H^n(X; Z)\oplus H^n(Y; Z),$$
$$\psi^n\colon H^n(X; Z)\oplus H^n(Y; Z)\to H^n(X\cup Y; Z)$$

such that the sequence

$$\ldots\to H^{n-1}(X\cap Y;\ Z)\xrightarrow{\Delta^n} H^n(X\cup Y; Z)\xrightarrow{\varphi^n} H^n(X, Z)\oplus H^n(Y; Z)$$
$$\to H^n(X\cap Y; Z)\xrightarrow{\Delta^{n+1}} H^{n+1}(X\cup Y; Z)\to\ldots$$

is exact.

From the definition of $c(X)$ we infer that there exists a natural number n such that $H^n(X; Z)\oplus H^n(Y; Z)\neq 0$ and $H^m(X\cap Y; Z) = 0$ for every $m\geqslant n$. This implies that $\varphi^n\colon H^n(X\cup Y; Z)\to H^n(X; Z)\oplus H^n(Y; Z)$ is an epimorphism and $H^n(X\cup Y; Z)\neq 0$ and for all $m>n$ the groups $H^m(X\cup Y; Z)$, $H^m(X; Z)\oplus H^m(Y; Z)$ are isomorphic. We infer that $c(X\cup Y) = \max(c(X), c(Y))$ and the proof of (6.1) is finished.

(6.2) COROLLARY. *Suppose that X and Y are compacta, $\mathrm{Fd}(X)<\infty$ and $\mathrm{Fd}(X\cap Y)<\mathrm{Fd}(X)$. If X is approximatively 1-connected compactum then $\mathrm{Fd}(X\cup Y)\geqslant\mathrm{Fd}(X)$.*

Proof. Corollary (6.2) is an immediate consequence of (6.1) and (2.2).

Corollary (6.2) gives a partial answer to the following

(6.3) PROBLEM. *Is it true that for all compacta X, Y such that $\mathrm{Fd}(X) > \mathrm{Fd}(X\cap Y)$ the inequality $\mathrm{Fd}(X\cup Y)\geqslant\mathrm{Fd}(X)$ holds?*

The author has proved (see [20]) that for all compacta X, Y we have

(6.4) $$\mathrm{Fd}(X\cup Y)\leqslant\max(\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X\cap Y)+1).$$

Problem (6.3) is connected with the problem of the homological characterization of the fundamental dimension for the class all finite-dimensional compacta.

Let us prove the following

(6.5) THEOREM. *For every fixed natural number $n_0\geqslant 2$ the following three propositions are equivalent:*

(a) *For every compactum X with a finite fundamental dimension* $\mathrm{Fd}(X)\leqslant\max(n_0, c(X))$.

(b) *For all compacta X, Y such that $n_0+1\leqslant\mathrm{Fd}(X)<\infty$ and $\mathrm{Fd}(X\cap Y)<\mathrm{Fd}(X)$ the inequality $\mathrm{Fd}(X\cup Y)\geqslant\mathrm{Fd}(X)$ holds true.*

(c) *If X is a compactum and Y is a continuum such that $n_0+1\leqslant\mathrm{Fd}(X)<\infty$, $\dim(X\cap Y) = 1$, $\dim Y = 2$ and $\mathrm{Fd}(Y) = 0$, then $\mathrm{Fd}(X\cup Y) = \mathrm{Fd}(X)$.*

Proof. From (6.1) we infer that (a) ⇒ (b) and from (6.4) we infer that (b) ⇒ (c).

In order to prove that (c) ⇒ (a), consider a compactum X such that $\mathrm{Fd}(X)<\infty$. There exists an inverse sequence of polyhedra (with fixed triangulations) $\{X_k, p_k^{k+1}\}$ such that $p_k^{k+1}\colon X_{k+1}\to X_k$ is a simplicial map for every $k = 1, 2, \ldots$ and

(6.6) $$\mathrm{Sh}(X) = \mathrm{Sh}(X'),$$

where $X' = \varprojlim\{X_k, p_k^{k+1}\}$. Let $X_k^{(1)}$ denote the 1-skeleton of X_k.

We can assume that X_k is a subpolyhedron of E^{m_k} such that $(x_1, x_2, \ldots, x_{m_k})\in X_k$ implies $x_{m_k} = 0$. Let $b_k = (0, 0, \ldots, 0, 1)\in E^{m_k}$ and let Z_k be a polyhedron which is the union of X_k and of a cone A_k with the vertex b_k and a base $X_k^{(1)}$.

Consider maps $q_k^{k+1}\colon A_{k+1}\to A_k$, $r_k^{k+1}\colon Z_{k+1}\to Z_k$ defined by the formulas

$$q_k^{k+1}(z) = (1-t)p_k^{k+1}(y)+tb_{m_k}\ \text{for every}\ z = (1-t)y+tb_{m_{k+1}}\ \text{where}\ y\in X_{k+1};$$

$$r_k^{k+1}(z) = \begin{cases} p_k^{k+1}(z) & \text{for every } z\in X_{k+1},\\ q_k^{k+1}(z) & \text{for every } z\in A_{k+1}.\end{cases}$$

Setting $A = \varprojlim\{A_k, q_k^{k+1}\}$ and $W = \varprojlim\{Z_k, r_k^{k+1}\}$, one easily sees that $\dim A = 2$, $\mathrm{Fd}(A) = 0$, $\dim A\cap X' = 1$, $W = X'\cup A$ and that W is an approximatively 1-connected compactum (see [19], Lemma (6.1)).

From (6.6) we infer that $\mathrm{Fd}(X') = \mathrm{Fd}(X)$ and $c(X') = c(X)$.

Proposition (6.1) implies that $c(W) = \max(c(X'), c(A)) = c(X')$. Hence

$$\mathrm{Fd}(X) = \mathrm{Fd}(X') = \mathrm{Fd}(W),$$

$$\max(n_0, c(W)) = \max(n_0, c(X')) = \max(n_0, c(X)).$$

Therefore (c) ⇒ (a) and the proof is finished.

(6.7) Problem. *Is it true that there exists a natural number* $n_0 \geqslant 2$ *such that propositions* (a), (b) *and* (c) *from* (6.5) *hold?*

The author knows only the example of a compactum X such that $c(X) = 0$ and $\mathrm{Fd}(X) = 2$ (see Remark (2.3)).

One also knows the example of an acyclic non-movable curve C (see [6] and [18], p. 652).

(6.8) Problem. *Is it true that* $\mathrm{Fd}(C^n) = n$?

A positive answer to the question (6.8) would give negative answers to the problems (6.7) and (6.3) (see (6.5)).

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