Results and problems concerning compactifications, compact subtopologies, and mappings

by

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Abstract. Some relationships between compactifications and compact subtopologies are investigated.

1. Introduction. All spaces (compactifications, subtopologies, etc.) in this paper are assumed to be Hausdorff unless otherwise stated. Though some of our results are stated in this general setting, all of them are new for the setting of metric spaces.

A *compact subtopology* for a space (X, T) is a compact topology S for X such that $S \subset T$. It is well-known that a space (X, T) has a compact subtopology if and only if there exists a one-to-one continuous function f from (X, T) onto a compact space Z (if such an f exists, then $S = \{v: f(V)$ is an open subset of $Z\}$ is easily seen to be a compact subtopology for (X, T); in fact, (X, S) is homeomorphic to Z). Recently, there has been considerable interest in spaces with compact subtopologies (for example, see [4], [7], [9], [10], [11] and [13]). The question of which metric spaces have (metrizable) compact subtopologies was originally posed by Banach [1]. In 1949 M. Katetov [5] solved this problem in the case of countable spaces by showing that a countable regular space has a compact subtopology if and only if it is scattered (a space X is *scattered* [5] provided every nonempty subset has an isolated point (in the relative topology); for countable metric spaces this is equivalent to X not containing a topological copy of the rational numbers).

In this paper we investigate mapping relationships between compactifications and compact subtopologies. We obtain some results about countable metric spaces which show that there is a very strong relationship between their compactifications and compact subtopologies. Since these results about countable spaces, together with Example 1 below, form

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the motivation for the paper, we broadly describe them in the next few sentences. Assume (X, T) is a countable metric space which has a compact subtopology. It is known, then, that (X, T) has a countable compactification [4]. We show that each countable compactification of (X, T) is homeomorphic to a compact subtopology of (X, T) (see section 2) and that each compact subtopology of (X, T) is a continuous image of a countable compactification of (X, T) (Corollary 1 of section 3).

EXAMPLE 1. Let X be the subset $A \cup B$ of the plane R^2, where $A = \{z \in R^2\colon 1 < |z| \leqslant 2\}$ and $B = \{z \in R^2\colon |z-4| \leqslant 1\}$. The space X has the disc $D = \{z \in R^2\colon |z| \leqslant 2\}$ as a one-to-one continuous image. However, X has no connected compactification, so D is not homeomorphic with any compactification. Note that X is locally compact, and that the compact subtopology D is an open continuous image of a compactification of X, namely the one-point compactification.

It is well-known that the Stone-Čech compactification of a completely regular space (X, T) can be continuously mapped onto any compact space which is a continuous image of (X, T). Hence, every compact subtopology of a completely regular space (X, T) is a continuous image of a compactification of (X, T). In view of this, the results mentioned above, and Example 1, we consider (A) is every compact subtopology of a space (X, T) an open continuous image of some compactification of (X, T)? In section 4 we show in a strong way that the answer to (A) is no; more specifically, we show that *no* compact subtopology for the half-line $[0, +\infty)$ is an open continuous image of a compactification of $[0, +\infty)$. Further, we show that for the line $(-\infty, +\infty)$ there are compact subtopologies which are homeomorphic to compactifications of $(-\infty, +\infty)$. On the other hand, some compact subtopologies for $(-\infty, +\infty)$ are not open continuous images of any compactification of $(-\infty, +\infty)$. Using [10], we determine exactly which ones are; in fact, we show that an open continuous function from a compactification of $(-\infty, +\infty)$ onto a compact subtopology of $(-\infty, +\infty)$ is a homeomorphism. Preceeding section 4, we give in section 3 a special process (Theorem 2) for mapping certain compactifications of a space onto compact subtopologies. We refer the reader to the beginning of section 3 for further exposition about this.

Throughout this paper we let $(-\infty, +\infty)$ and $[0, +\infty)$ denote the real line and the nonnegative reals respectively, each with the usual topology (unless otherwise specified); R^n denotes usual Euclidean n-dimensional space, $n = 1, 2, \ldots$

2. A result about countable spaces. In this section we prove the following result.

THEOREM 1. *Let (X, T) be a countable space. If (X, T) has a countable compactification Y, then it has a compact subtopology S such that (X, S) is homeomorphic to Y.*

Proof. Let Y be a countable compactification of X and let d denote a metric for Y (that Y is metrizable is well-known; even more information is in [8]). Since it will be clear from what we will do how to handle the case when $Y-X$ is finite, we assume for the purpose of proof that $Y-X$ is infinite. Let $y_1, y_2, \ldots$ be a (one-to-one) complete enumeration of $Y-X$. For each $n = 1, 2, \ldots$, choose a sequence $\{x_m^n\}_{m=1}^{\infty}$ such that

(1) for each $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$, x_m^n is isolated in Y (equivalently, isolated in X);

(2) $d(x_m^n, y_n) < \dfrac{1}{m \cdot n}$ for each $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$;

(3) $x_i^n \neq x_j^n$ whenever $i \neq j$;

(4) $\{x_1^n, x_2^n, \ldots\} \cap \{x_1^k, x_2^k, \ldots\} = \emptyset$ whenever $n \neq k$. Let $f\colon X \to Y$ be defined by

$$f(x) = \begin{cases} y_n, & \text{if} \quad x = x_1^n, \\ x_{m-1}^n, & \text{if} \quad x = x_m^n \text{ and } m \geqslant 2, \\ x, & \text{otherwise}. \end{cases}$$

Clearly f is one-to-one and sends X onto Y. Now we show f is continuous. Let $\{z_i\}_{i=1}^{\infty}$ be a sequence of distinct points of X such that $\{z_i\}_{i=1}^{\infty}$ converges to a point $z \in X$. Then, by (1), $z \in [X - \bigcup \{x_m^n\colon n = 1, 2, \ldots \text{ and } m = 1, 2, \ldots\}]$. If some subsequence $\{z_{i_j}\}_{j=1}^{\infty}$ of $\{z_i\}_{i=1}^{\infty}$ has each of its terms in $X - \bigcup \{x_m^n\colon n = 1, 2, \ldots \text{ and } m = 1, 2, \ldots\}$, then, since f is the identity on $X - \bigcup \{x_m^n\colon n = 1, 2, \ldots \text{ and } m = 1, 2, \ldots\}$, $\{f(z_{i_j})\}_{j=1}^{\infty}$ converges to $f(z)$. Hence, it suffices to assume $z_i \in \bigcup \{x_m^n\colon n = 1, 2, \ldots \text{ and } m = 1, 2, \ldots\}$ for all $i = 1, 2, \ldots$ Since $z \in X$, $z \neq y_n$ for all $n = 1, 2, \ldots$ Thus, we may assume (by using a subsequence of $\{z_i\}_{i=1}^{\infty}$ if necessary) that, for each $i = 1, 2, \ldots$ $z_i = x_{m(i)}^{n(i)}$ where $n(i+1) > n(i)$. Now, by (2) above, the diameter of $f(\{x_m^{n(i)}\colon m = 1, 2, \ldots\}) = \{x_m^{n(i)}\colon m-1, 2, \ldots\} \cup \{y_{n(i)}\}$ is less than or equal to $2/n(i)$. Hence, $d(f(z_i), z_i) \leqslant 2/n(i)$. It follows easily from this that $\{f(z_i)\}_{i=1}^{\infty}$ converges to z. Therefore, since $z \in [X - \bigcup \{x_m^n\colon n = 1, 2, \ldots \text{ and } m = 1, 2, \ldots\}]$, this proves $\{f(z_i)\}_{i=1}^{\infty}$ converges to $f(z)$. This completes our proof of the continuity of f.

3. Continuous functions from compactifications to compact subtopologies. In [13] Smirnov gives an extrinsic characterization of when a completely regular T_1-space has a compact subtopology. The characterization is in terms of the existence of a certain type of function from a compactification of (X, T) onto (X, T). In one direction, he notes that each compact subtopology of (X, T) is a continuous image of the Stone-Čech compactification of (X, T); his certain type of function is produced from this.

Furthermore, he notes that a compactification of (X, T) can be chosen with the same weight and dimension as (X, T) to do the job. In the next theorem we establish a process. It enables us to choose compactifications which can be continuously mapped onto compact subtopologies and which are closely related to the original space. This relationship enables us to obtain more information in specific instances than is obtainable via the softer approach in [13] (cf. our Corollary 1, etc.).

THEOREM 2. *Let (X, T) be a topological space, not necessarily Hausdorff, with a possibly non-Hausdorff compact subtopology S. If $H^{\sim}$ is any (possibly non-Hausdorff) compactification of (X, T), then there exists a possibly non-Hausdorff compactification $X^* \subset [X^{\sim} \times (X, S)]$, where $\times$ denotes cartesian product, and a continuous function ϱ from X^* onto (X, S).*

Proof. Let $i\colon (X, T) \to (X, S)$ denote the identity map. Let $G(i) = \{(x, i(x))\colon x \in X\}$ denote the natural embedding of the graph of i in $X^{\sim} \times (X, S)$. Since i is continuous, $G(i)$ is homeomorphic with (X, T). Hence, $\mathrm{cl}[G(i)]$ can be viewed as a (possibly non-Hausdorff) compactification of (X, T), where cl denotes closure. Let $X^* = \mathrm{cl}[G(i)]$ and let ϱ denote the restriction to X^* of the projection of $X^{\sim} \times (X, S)$ onto (X, S). Clearly, X^* and ϱ satisfy the desired conclusion.

COROLLARY 1. *Let (X, T) be a countable metric space. If (X, T) has a compact subtopology S, then (X, T) has a countable compactification X^* such that (X, S) is a continuous image of X^*.*

Proof. Since (X, T) is countable and has a compact subtopology, we have from [5] that (X, T) is scattered. Hence, (X, T) has a countable compactification [4, Theorem 8]. The result now follows by letting this countable compactification play the role of $X^{\sim}$ in the statement of Theorem 2.

Remark 1. It would be tempting to conjecture, in view of Theorem 1 and Corollary 1, that if a countable metric space (X, T) has a compact subtopology S, then (X, T) has a countable compactification which is homeomorphic to (X, S). However, this is false. In fact, letting

$$X = \{0, 1/2, 1/3, \ldots, 1/n, \ldots\} \cup \{-1, -2, \ldots, -n, \ldots\}$$

with the relative topology T from the line, we see that (X, T) has a compact subtopology S homeomorphic with $\{0, 1/2, 1/3, \ldots, 1/n, \ldots\}$ and yet that no compactification of (X, T) is even a continuous image of (X, S) (because any compactification of (X, T) has at least two limit points).

EXAMPLE 2. It may appear that Corollary 1 would remain true without the hypothesis of "metric." However, this is not true as is easily seen by letting (X, T) be the subspace of the Stone-Čech compactification of the integers consisting of the integers together with one point of the remainder. The space (X, T) is not metric; this is a simple consequence of the fact that a sequence of integers converges in the Stone-Čech compactification if and only if it is eventually constant (to see this, use the well-known fact about extendability of bounded real-valued continuous functions). Clearly then, (X, T) does not have a countable compactification. However, (X, T) is a countable completely regular scattered space.

COROLLARY 2. *Let (X, T) be a separable metric space. If (X, T) has a compact subtopology S, then (X, T) has a metric compactification X^* such that (X, S) is a continuous image of X^*. Furthermore, if (X, T) and (X, S) are each zero-dimensional, then the compactification X^* can be chosen so as to be zero-dimensional as well as metric.*

Proof. We prove both parts of the corollary simultaneously. Under the assumption that (X, T) is separable metric (respectively, in addition, zero-dimensional), we have that (X, T) can be embedded in the Hilbert cube (respectively, the Cantor middle-thirds set). Hence, (X, T) has a metric (respectively, zero-dimensional and metric) compactification $X^{\sim}$. Now, let us note that (X, S) is a metric space (see the beginning of the proof of Theorem 4.1 in [11, p. 246]; the author of [11] acknowledges that this fact and its proof in [11] were communicated to him by H. H. Corson). The result now follows by letting $X^{\sim}$ be as in Theorem 2 and then applying Theorem 2.

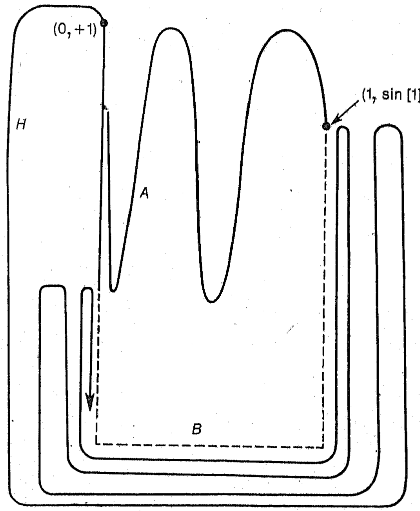
PROBLEM 1. (a) If (X, T) has a zero-dimensional compact subtopology, then must (X, T) be zero-dimensional (clearly such an (X, T) must be totally disconnected)? (b) Does every zero-dimensional space with a compact subtopology have a zero-dimensional compact subtopology?

Remark 2. Let (X, T) be a locally compact space which is not compact. Since (X, T) is locally compact, (X, T) has a compact subtopology (X, S) (to see this, simply take the one-point compactification and identify the "point at infinity" with some given point of X). Let X^* be a compactification of (X, T) constructed specifically by the procedure in the proof of Theorem 2. Let ϱ be the mapping obtained in the proof of Theorem 2. We now show that ϱ can not be an open mapping. Let j denote the (continuous) injection of (X, T) into X^*. Since (X, T) is locally compact, j takes open subsets of (X, T) to open subsets of X^*. Hence, if ϱ were open, the composition $\varrho \circ j$ would be an open mapping, a contradiction because $\varrho \circ j = i$, which implies $S = T$.

In relation to Theorem 2, it is natural to ask the following question: If (X, T) has a compact subtopology, does every compactification of (X, T) have some compact subtopology of (X, T) as a continuous image? Our next example shows that the answer to this question is no.

EXAMPLE 3. Let $A = \mathrm{cl}(\{x, \sin[1/x]\}\colon\ 0 < x \leqslant 1\}$, let B be the polygonal arc (dotted in Figure 1 below) defined by $B = \{(x, y) \in R^2\colon$

$x = 0$ and $-2 \leqslant y \leqslant -1\} \cup \{(x, y) \in R^2\colon\ 0 \leqslant x \leqslant 1$ and $y = -2\} \cup \cup \{(x, y) \in R^2\colon x = 1$ and $-2 \leqslant y \leqslant \sin[1]\}$, and let H be a topological copy of $[0, +\infty)$ beginning at $(0, +1)$ and compactifying on B as in Figure 1 $(H \cap A = \{(0, +1)\})$. The space (X, T) is $A \cup H$.



X = the solid lines
Fig. 1

Now, by pushing H up, it is easy to see that X has a compact subtopology homeomorphic to the object in Figure 2.

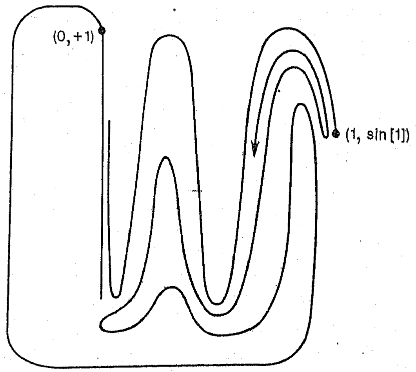


Fig. 2

Furthermore, let us observe that no compact subtopology for X can be arcwise connected. A rigorous argument for this can be obtained by using Theorem 7.2 of [10] (note that, under any one-to-one continuous function f from X onto a compact space Z, $f(H - \{(0, +1)\}) = f(X) - f(A)$ is an open subset of Z, hence locally compact). Thus, since $A \cup B \cup H$ is an arcwise connected compactification of X, no compact subtopology of X is a continuous image of $A \cup B \cup H$.

Note that, in Example 3 we found a space which had a compact subtopology and an arcwise connected compactification but no arcwise connected compact subtopology. This gives rise to the following question which we can not answer.

Problem 2. Is there a space (X, T) which has a compact subtopology and a locally connected compactification but no locally connected compact subtopology? We remark that if (X, T) is locally compact, then the answer is no. To see this, assume Y is a locally connected compactification of (X, T). Since (X, T) is locally compact, the one-point compactification $X \cup \{\infty\}$ of (X, T) is a continuous image of Y. Hence, $X \cup \{\infty\}$ is locally connected. Let $p \in X$. Then, the decomposition space obtained by identifying ∞ and p is a locally connected compact subtopology for (X, T).

4. Question (A). Before giving our answer to Question (A) stated in the Introduction, we prove the next two theorems which separately characterize all the open continuous images of $[0, +\infty)$ and of $(-\infty, +\infty)$.

Theorem 3. *If the space Z is an open continuous image of $[0, +\infty)$, then Z is one of the following three spaces:*
(1) *a single point,*
(2) *an arc,*
(3) *a half-line (homeomorphic to $[0, +\infty)$).*

Theorem 4. *If the space Z is an open continuous image of $(-\infty, +\infty)$, then Z is one of the following five spaces:*
(1) *a single point,*
(2) *an arc,*
(3) *a simple closed curve (homeomorphic to $\{z \in R^2\colon |z| = 1\}$),*
(4) *a half-line,*
(5) *a line (homeomorphic to $(-\infty, +\infty)$).*

To aid in the proof of Theorems 3 and 4, we first prove the following lemma.

Lemma 1. *Let X be a connected, locally compact, and locally connected separable metric space such that X does not contain a simple triod. Then, if X is not compact, the one-point compactification $X \cup \{\infty\} = Y$ of X is an arc or a simple closed curve.*

Proof. Assume X satisfies the given hypotheses. Since X is connected, ∞ is a noncut point of Y. Let a be another noncut point of Y (see 6.1 of [14, p. 54]) Now, assume Y is not an arc. Then, Y has a noncut point b with $b \notin \{a, \infty\}$ (see 6.2 of [14, p. 54]). Let $A \subset (Y-\{b\})$ be an arc with noncut points a and ∞, let $B \subset (Y-\{\infty\})$ be an arc with noncut points a and b, and let $C \subset (Y-\{a\})$ be an arc with noncut points b and ∞ (see 5.2 of [14, p. 38]). It follows (use that X does not contain a simple triod) that $A \cup B$ is an arc, with noncut points ∞ and b, and that $[A \cup B] \cap C = \{\infty, b\}$. Hence, $A \cup B \cup C$ is a simple closed curve. Now we show $A \cup B \cup C = Y$. Suppose there is a point

$$y_0 \in [Y-(A \cup B \cup C)] .$$

Again using 5.2 of [14, p. 38], there is an arc $D \subset X$ such that D has noncut points y_0 and a. Clearly (since $\infty \notin D$), $A \cup B \cup C \cup D$ contains a simple triod which is entirely contained in X. This contradiction establishes that $A \cup B \cup C = Y$ and completes the proof of the lemma.

Proof of Theorem 3 and Theorem 4. First note, from the openness and continuity that Z is a connected, locally compact, locally connected separable metric space. Also, from (iii) of 7.3 of [14, p. 147], Z does not contain a simple triod. We define a space X as follows: $X = Z$ if Z is not compact and $X = Z-\{p\}$ if Z is compact, where p is a noncut point of Z (see 6.1 of [14, p. 54]). Then, X satisfies all the conditions of Lemma 1 above (unless Z were a single point). Hence, the one-point compactification of X is an arc or a simple closed curve. This completes the proof of the theorems.

Remark 3. Another proof of Theorems 3 and 4 can be done using (iii) of 7.3 of [14, p. 147] and Theorem 1 of [2]. In particular, from the fact that Z does not contain a simple triod and the proof of Theorem 1 of [2], it follows that Z is a one-to-one continuous image of a point, $[0, 1]$, $[0, +\infty)$, or $(-\infty, +\infty)$. The rest of this proof can be completed using Theorem 7.1 of [10], the Structure Theorem in [9, p. 128], and a result in [7].

The next theorem gives the answer to question (A) promised in the Introduction.

THEOREM 5. *Let S be a compact subtopology for $[0, +\infty)$. Then, there is no compactification of $[0, +\infty)$, with the usual topology, which has $([0, +\infty), S)$ as an open continuous image.*

Proof. Let Y be a compactification of $[0, +\infty)$. Suppose there is an open continuous function f from Y onto $([0, +\infty), S)$. By the local compactness of $[0, +\infty)$, $[0, +\infty)$ is embedded as an open subset G of Y. Hence, the restriction of f to G is an open continuous function. Thus, by Theorem 3 above, it follows that $f(G)$ is an arc or is homeomorphic to $[0, +\infty)$. Furthermore, since G is a dense subset of Y, $f(G)$ is a dense subset of $([0, +\infty), S)$. However, by the Structure Theorem in [9, p. 128], no compact subtopology for $[0, +\infty)$ contains an arc or a space homeomorphic to $[0, +\infty)$ as a dense open subset. This contradiction establishes the theorem.

Now we devote our attention to compactifications and compact subtopologies of $(-\infty, +\infty)$. For the purpose of making the statements of our results fairly concise and self-contained, we recall a few preliminary notions from [10].

Following the terminology in [10, p. 2], a *half-ray curve* (respectively, *real curve*) is a compact connected space which is a one-to-one continuous image of $[0, +\infty)$ (respectively, $(-\infty, +\infty)$). Clearly, then, a half-ray curve is just a compact subtopology of $[0, +\infty)$ and a real curve is just a compact subtopology of $(-\infty, +\infty)$. In investigating real curves in [10] we made use of two special sets called singular sets. Specifically, let f denote a one-to-one continuous function from $(-\infty, +\infty)$ onto a real curve X. Then, the *singular sets of X with respect to f* [10, p. 6] are the following two sets:

(i) $K_+(X, f) = \{x \in X$: there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in R^1 such that $t_n \to +\infty$ as $n \to \infty$ and $\{f(t_n)\}_{n=1}^{\infty}$ converges to $x\}$.

(ii) $K_-(X, f) = \{x \in X$: there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in R^1 such that $t_n \to -\infty$ as $n \to \infty$ and $\{f(t_n)\}_{n=1}^{\infty}$ converges to $x\}$.

In [10, p. 9] we completely determine the topological types of singular sets that any real curve can have. There are uncountably many different such types but they can be divided, as in [10, p. 9], into five convenient types: (1) a point, (2) an arc, (3) a chainable continuum with exactly two arc components, (4) a half-ray triod, and (5) a half-ray curve. In what follows, when we say X has a "singular set of type (5)" we will mean that one of the singular sets is a half-ray curve. Finally, we recall from [10, pp. 13–15], that the singular sets of a real curve do not, in any crucial way, depend on the mapping f. Precisely, if g is another one-to-one continuous function from $(-\infty, +\infty)$ onto X, then

(1) $K_+(X, f) = K_+(X, g)$ and $K_-(X, f) = K_-(X, g)$ or

(2) $K_+(X, f) = K_-(X, g)$ and $K_-(X, f) = K_+(X, g)$.

THEOREM 6. *Suppose that g is an open continuous function from a compactification $L^\sim$ of the real line L onto a real curve X. Then g is a homeomorphism.*

Proof. Let X be a real curve, let $L^\sim$ be a compactification of $L = (-\infty, +\infty)$, and let g be an open continuous mapping of $L^\sim$ onto X. It follows from the openness of g and Theorem 4 that $g(L)$ is either an open and dense in X topological copy of $[0, +\infty)$ or of $[-\infty, +\infty)$ (note: no real curve can be a single point, an arc, or a simple closed curve). It can be readily verified by use of the Structure Theorem for Real Curves in [10] that no real curve contains an open topological copy of $[0, +\infty)$. Hence, $g(L)$ must be an open, dense topological copy of $(-\infty, +\infty)$. Note that the restriction of g to L must be a homeomorphism of L onto $g(L)$. Let $C = L^{\sim} - L$. The first thing we prove is that $g(L) \cap g(C) = \emptyset$. Suppose not; then there exists an $x \in C$ such that $g(x) = g(t)$ for some $t \in L$. But, then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \in L$ for all $n = 1, 2, \ldots$, $|t_n| \to +\infty$, and $\{g(t_n)\}_{n=1}^{\infty}$ converges to $g(x) = g(t)$. This contradicts that the restriction of g to L is a homeomorphism. This proves $g(L) \cap g(C) = \emptyset$. To see that g is a homeomorphism, all that remains is to show that g is one-to-one on C. To this end, suppose there exist distinct points x_1 and x_2 in C such that $g(x_1) = g(x_2)$. Let U_{x_1} and U_{x_2} be disjoint open sets in $L^{\sim}$ such that $x_1 \in U_{x_1}$ and $x_2 \in U_{x_2}$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $L \cap U_{x_1}$ such that $\{t_n\}_{n=1}^{\infty}$ converges to x_1. Then $\{g(t_n)\}_{n=1}^{\infty}$ converges to $g(x_1) = g(x_2)$. Since $g(U_{x_2})$ is an open subset of X, there exists a natural number n_0 such that $g(t_{n_0}) \in g(U_{x_2})$. But, since we have shown that $g(C) \cap g(L) = \emptyset$, $g(t_{n_0}) = g(s)$ for some $s \in (L \cap U_{x_2})$. This contradicts the one-to-oneness of g on L. This completes the proof of the theorem.

THEOREM 7. *Let X be a real curve. Then, X is a compactification of $(-\infty, +\infty)$ if and only if X has a singular set of type* (5).

Proof. First, a few preliminaries. Let f be a one-to-one continuous function from $(-\infty, +\infty)$ onto X and assume, by Lemma 2.5 of [10], that $K_+(X, f) = f([a, b])$ where $\infty < a \leqslant b < +\infty$. Let

$$Q = \{w \in (-\infty, a) \colon f(w) \in K_-(X, f)\}.$$

If $Q = \emptyset$, let $c = a$. If $Q \neq \emptyset$, let $c = \text{g.l.b.}(Q)$. That c is well defined is established on page 10 of [10]. It is easy to see that f restricted to $(-\infty, c)$ is a homeomorphism. Now with the preliminaries done, first assume X is a compactification of $(-\infty, +\infty)$. Then, by the local compactness of $(-\infty, +\infty)$, X must contain a dense open topological copy L of $(-\infty, +\infty)$. By the definition of c, $f((-\infty, c))$ is an open subset of X. Hence, $L \cap f((-\infty, c)) \neq \emptyset$. We now show $L \subset f((-\infty, c))$. Since X is not locally Euclidean at $f(c)$, $f(c) \notin L$. If $K_-(X, f)$ is not a single point, then $X - \{f(c)\}$ fails to be arcwise connected. Under these circumstances, since $L \cap f((-\infty, c)) \neq \emptyset$ and $f((-\infty, c))$ is an arc component of $X - \{f(c)\}$, $L \subset f((-\infty, c))$. On the other hand assume $K_-(X, f)$ consists only of a single point q. Since X can not be locally Euclidean at q, $q \notin L$. But $X - \{f(c), q\}$ is not arcwise connected and a repetition of an argument above gives that $L \subset f((-\infty, c))$. We have now shown that, in any case, $L \subset f((-\infty, c))$ (we mention that it is now simple to show that $f((-\infty, c)) = L$). From the denseness of L, we can now conclude that $\text{cl}[f((-\infty, c))] = X$. Thus, in particular, $\text{cl}[f((-\infty, c))] \supset f([c, +\infty))$. Hence, we must have $K_-(X, f) \supset f([c, +\infty))$ (note that, by the definition of c, this is actually an equality). Therefore, $K_-(X, f)$ is a singular set of type (5). Conversely, assume X has a singular set of type (5). Then, since $K_+(X, f)$ is an arc or a single point, $K_-(X, f)$ is this singular set. Thus, by definition of c, $K_-(X, f) = f([c, +\infty))$. Hence, $f((-\infty, c))$ is a dense (in X) topological copy of $(-\infty, +\infty)$. This proves X is a compactification of $(-\infty, +\infty)$.

COROLLARY 3. *Let X be a real curve. Then:*

(1) *X is an open continuous image of a compactification of $(-\infty, +\infty)$ if and only if X is homeomorphic to a compactification of $(-\infty, +\infty)$;*

(2) *X is an open continuous image of a compactification of $(-\infty, +\infty)$ if and only if X has a singular set of type* (5).

We state the following problem which is related to material in this paper.

PROBLEM 3. Is every compact subtopology of a countable space (X, T) an open continuous image of a countable (or any) compactification of (X, T)?

R. E. Chandler has pointed out to the authors that the technique used in the proof of Theorem 2 is well known to those working in the theory of compactifications.

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Infinite dimensional non-symmetric Borsuk-Ulam theorem

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Abstract. Let R^∞ be an infinite dimensional Banach space and $R^{\infty-1}$ a closed subspace of codimension one. If $X \subset R^\infty$, a mapping $f\colon X \to R^\infty$ is said to be a *compact vector field* or a *compact field* if the associated displacement mapping $F\colon X \to R^\infty$ defined by $F(x) = x - f(x)$ maps X into a compact subset of R^∞. It is proved that if X is closed and bounded and if the origin lies in a bounded component of the complement $R^\infty - X$ then for any compact field $f\colon X \to R^{\infty-1}$ there exist two points x and y in X, lying on opposite rays from the origin (i.e. $y = -\lambda x$ for some $\lambda > 0$), such that $f(x) = f(y)$. This is a generalization of a theorem of Granas which results by taking X to be the unit sphere in R^∞. The proof uses techniques analogous to those of Granas to reduce the problem to the finite dimensional case which was proved earlier by the author.

1. Introduction. The classic Borsuk-Ulam theorem states that if $f\colon S^n \to R^n$ is a map of the n-sphere into the Euclidean space R^n then there exists a pair of antipodal points $\{x, -x\}$ on S^n such that $f(x) = f(-x)$. Several generalizations of this theorem, preceeding in various directions, are known (see, for example, the references in [2]). In some of these generalizations the sphere is replaced by a more general space on which some suitable notaion of antipodality can be defined. In particular, the author [2] has proved the following theorem conjectured by Borsuk.

THEOREM A. *Let X be a compact subset of the Euclidean space R^{n+1} which disconnects it in such a way that the origin lies in a bounded component of $R^{n+1} - X$. Then for any map $f\colon X \to R^n$ there exist two points x and y in X, lying on opposite rays from the origin (that is, $y = -\lambda x$ for some $\lambda > 0$), such that $f(x) = f(y)$.*

On the other hand Granas [1] has extended the Borsuk-Ulam theorem from Euclidean spaces to infinite dimensional Banach spaces. Let R^∞ denote a fixed infinite dimensional Banach space and S^∞ the unit sphere in R^∞. By $R^{\infty-1}$ we mean a linear, closed subspace of R^∞ of co-dimension one. Because the unit sphere S^∞ is not compact, the Borsuk-Ulam theorem does not hold for an arbitrary map $f\colon S^\infty \to R^{\infty-1}$. However if the mapping f does not displace points of S^∞ too much (i.e. if the mapp-