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Kurepa's hypothesis and the continuum

by

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Abstract. Silver [5] proved that Con(ZFC + "there is an inaccessible cardinal") implies Con(ZFC+ CH+ "there are no Kurepa trees"). In order to obtain this result, he generically collapses an inaccessible cardinal to ω_2. Hence CH necessarily holds in his final model. In this paper we sketch Silver's proof, and then show how it can be modified to obtain a model in which there are no Kurepa trees and the continuum is anything we wish.

Introduction. We work in ZFC and use the usual notation and conventions. For details concerning the forcing theory we require, see Jech [3] or Shoenfield [4]. A *tree* is a poset $\underset{\sim}{T} = \langle T, \leqslant_T\rangle$ such that $\hat{x} = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$ for any $x \in T$. The order-type of $\hat{x}$ is the *height* of x in $\underset{\sim}{T}$, $ht(x)$. The αth *level* of $\underset{\sim}{T}$ is the set $T_\alpha = \{x \in T \mid ht(x) = \alpha\}$. $\underset{\sim}{T}$ is an ω_1*-tree* iff:

(i) $(\forall\alpha < \omega_1)(T_\alpha \neq \emptyset)\ \&\ (T_{\omega_1} = \emptyset)$;
(ii) $(\forall\alpha < \beta < \omega_1)(\forall x \in T_\alpha)(\exists y_1, y_2 \in T_\beta)(x <_T y_1, y_2\ \&\ y_1 \neq y_2)$;
(iii) $(\forall\alpha < \omega_1)(\forall x, y \in T_\alpha)(\lim(\alpha) \to [x = y \leftrightarrow \hat{x} = \hat{y}])$;
(iv) $(\forall\alpha < \omega_1)(|T_\alpha| \leqslant \omega)\ \&\ |T_0| = 1$.

For further details of ω_1-trees, see Jech [2].

If $\underset{\sim}{T}$ is an ω_1-tree, a *branch* of $\underset{\sim}{T}$ is a maximal totally ordered subset of $\underset{\sim}{T}$. A branch b of $\underset{\sim}{T}$ is *cofinal* if $(\forall\alpha < \omega_1)(T_\alpha \cap b \neq \emptyset)$. $\underset{\sim}{T}$ is *Kurepa* if it has at least ω_2 cofinal branches. If $V = L$, then there is a Kurepa tree. This result is due to Solovay. For a proof, see Devlin [1] or Jech [2]. More generally, if $V = L[A]$, where $A \subseteq \omega_1$, then there is a Kurepa tree, from which it follows that if there are no Kurepa trees, then ω_2 is inaccessible in L. (All of this is still due to Solovay, and is proved in [1] and [2].). Hence, in order to establish Con(ZFC+K), where K denotes the statement "there are no Kurepa trees", one must at least assume Con(ZFC+I), where I denotes the statement "there is an inaccessible cardinal".

Now, if M is any cardinal absolute extension of L, and if $\underset{\sim}{T}$ is a Kurepa tree in L, then $\underset{\sim}{T}$ will clearly be a Kurepa tree in M. Hence, if $\varkappa$ is any cardinal of cofinality greater than ω, we can, by standard arguments, find a generic extension of L, with the same cardinals as L, such that, in the extension, there is a Kurepa tree and $2^{\omega} = \varkappa$. Johnsbråten has pointed out that the consistency of $K + 2^{\omega} = \varkappa$ (for such $\varkappa$) is not so easily obtained. Now, Silver [5] has shown that $\mathrm{Con}(\mathrm{ZFC}+I) \to \mathrm{Con}(\mathrm{ZFC} + 2^{\omega} = \omega_1 + K)$. (And by Solovay's result above, the hypothesis here is as weak as possible). However, the method Silver employs necessarily makes $2^{\omega} = \omega_1$ hold, so as it stands the only hope to obtain $K + 2^{\omega} = \varkappa$ would seem to be to take Silver's model and blow-up the continuum generically to $\varkappa$. In fact this procedure does work (i.e. K is preserved), but the proof that it does is fairly delicate, as opposed to the corresponding argument for $\neg K$. Since we shall need all of the tricks employed by Silver in his proof of $\mathrm{Con}(\mathrm{ZFC}+K)$, we may as well commence by describing his argument.

Silver's model. We shall use M to denote an arbitrary countable transitive model (c.t.m.) of ZFC throughout. By *poset*, we mean, as usual in forcing, a poset P, with a maximum element $\mathbf{1}$, such that every $p \in P$ has at least two incompatible extensions in P, where $p, q \in P$ are *compatible*, written $p \sim q$, if there is $r \in P$ such that $r \leqslant p, q$. We say P satisfies the $\varkappa$ *chain condition* ($\varkappa$-c.c.), for $\varkappa$ an uncountable cardinal, if there is no pairwise incompatible subset of P of cardinality $\varkappa$. P is σ*-closed* if whenever $\langle p_\alpha |\ \alpha < \lambda < \omega_1\rangle$ is a decreasing sequence from P there is $p \in P$ such that $p \leqslant p_\alpha$ for all $\alpha < \lambda$. The following lemmas are standard. (See Shoenfield [4] for example.)

LEMMA 1 (Cohen; Solovay). *Let P be a poset in M, $\varkappa$ an uncountable regular cardinal in M. Let G be M-generic for P.*

(i) *If $M \models$ "P satisfies the $\varkappa$-c.c." then $\lambda \geqslant \varkappa$ is a cardinal in $M[G]$ iff λ is a cardinal in M.*

(ii) *If $M \models$ "P is σ-closed", then for all $\lambda < \omega_1$, $(M^{\lambda})^M = (M^{\lambda})^{M[G]}$, so in particular, $\omega_1^M = \omega_1^{M[G]}$ and $\mathfrak{P}^M(\omega) = \mathfrak{P}^{M[G]}(\omega)$.*

LEMMA 2 (Lévy). *Let $\varkappa$ be an inaccessible cardinal in M, P a poset in M such that $M \models$ "$|P| < \varkappa$". If G is M-generic for P, then $\varkappa$ is still inaccessible in $M[G]$.*

LEMMA 3 (Solovay). *Let P_1, P_2 be posets in M. If G_1 is M-generic for P_1 and G_2 is $M[G_1]$-generic for P_2, then G_1 is $M[G_2]$-generic for P_1, G_2 is M-generic for P_2, $G_1 \times G_2$ is M-generic for $P_1 \times P_2$, and $M[G_1][G_2] = M[G_2][G_1] = M[G_1, G_2] = M[G_1 \times G_2]$, where $P_1 \times P_2$ is the cartesian product of P_1 and P_2 with the partial ordering $\langle p_1, p_2\rangle \leqslant \langle q_1, q_2\rangle \leftrightarrow p_1 \leqslant_1 q_1 \,\&\, p_2 \leqslant_2 q_2$. Conversely, if G is M-generic for $P_1 \times P_2$, then $G_1 = \{p |\ \langle p, \mathbf{1}\rangle \in G\}$ is M-generic for P_1, $G_2 = \{q |\ \langle \mathbf{1}, q\rangle \in G\}$ is $M[G_1]$-generic for P_2, and $G = G_1 \times G_2$.*

Let $\varkappa$ be an uncountable cardinal. The poset $P(\varkappa)$ is defined as follows. An element p of $P(\varkappa)$ is a countable function such that $\mathrm{dom}(p) \subseteq \omega_1 \times \varkappa$ and $\mathrm{ran}(p) \subseteq \varkappa$, and if $\langle \alpha, \delta\rangle \in \mathrm{dom}(p)$, then $p(\alpha, \delta) \in \delta$. The ordering on $P(\varkappa)$ is defined by $p \leqslant q \leftrightarrow p \supseteq q$. If $P = P(\varkappa)$ and $\lambda < \varkappa$, we set $P_\lambda = \{p \restriction (\omega_1 \times \lambda) |\ p \in P\}$, $P^\lambda = \{p - p \restriction (\omega_1 \times \lambda) |\ p \in P\}$, and regard P_λ, P^λ as posets in the obvious manner. Clearly, $P \cong P_\lambda \times P^\lambda$, by a canonical isomorphism.

LEMMA 4 (Levy). *Let $\varkappa$ be an inaccessible cardinal in M, and set $P = [P(\varkappa)]^M$. Then, $M \models$ "P is σ-closed and satisfies the $\varkappa$-c.c.". If G is M-generic for P, then $\omega_1^M = \omega_1^{M[G]}$ and $\varkappa = \omega_2^{M[G]}$. Furthermore, if $\lambda < \varkappa$ is an uncountable regular cardinal in M, then $M[G \cap P_\lambda] \models$ "P^λ is σ-closed and satisfies $\varkappa$-c.c.".*

Proof. See Jech [3] or Silver [5]. For the last part, notice that as P_λ is σ-closed in M, $M[G \cap P_\lambda]$ has no new countable sequences from P^λ, whence P^λ is still σ-closed in $M[G \cap P_\lambda]$. Also, as we clearly have $P^\lambda \cong [P(\varkappa)]^{M[G \cap P_\lambda]}$, Lemma 2 will ensure that P^λ has the $\varkappa$-c.c. in $M[G \cap P_\lambda]$.

For later use, we shall give the proof of the next lemma in full.

LEMMA 5 (Silver). *Let P be a poset in M such that $M \models$ "P is σ-closed". Let $\underset{\sim}{T}$ be an ω_1-tree in M. Let G be M-generic for P. If b is a cofinal branch of $\underset{\sim}{T}$ in $M[G]$, then in fact $b \in M$.*

Proof. We may assume $\underset{\sim}{T} = \langle \omega_1, \leqslant_T\rangle$. Suppose that, in fact $b \notin M$. Working in M, we define sequences $\langle p_s |\ s \in 2^{\underset{\smile}{\omega}}\rangle$, $\langle x_s |\ s \in 2^{\underset{\smile}{\omega}}\rangle$ so that $p_s \in P$; $t \subseteq s \to p_s \leqslant p_t$; $x_s \in T$; $t \subset s \to x_t <_T x_s$; $|s| = |t| \to ht(x_s) = ht(x_t)$; and $x_{s \frown \langle 0\rangle} \neq x_{s \frown \langle 1\rangle}$. The definition is by induction on $|s|$. Pick $p_\varnothing \in P$ so that $p_\varnothing \Vdash$ "$\mathring{b}$ is a cofinal branch of $\check{\underset{\sim}{T}}$ & $\mathring{b} \notin \check{M}$". Let $x_\varnothing$ be the minimal element of $\underset{\sim}{T}$. Suppose p_s, x_s are defined ofr all $s \in 2^n$, and that $p_s \Vdash$ "$\check{x}_s \in \mathring{b}$", where $p_s \leqslant p_\varnothing$ in particular. Since $p_\varnothing \Vdash$ "$\mathring{b} \notin \check{M}$", we can clearly find $p_{s \frown \langle 0\rangle}$, $p_{s \frown \langle 1\rangle} \leqslant p_s$ (each $s \in 2^n$) and points $x_{s \frown \langle 0\rangle}, x_{s \frown \langle 1\rangle} >_T x_s$ such that $ht(x_{s \frown \langle 0\rangle}) = ht(x_{s \frown \langle 1\rangle})$ and $x_{s \frown \langle 0\rangle} \neq x_{s \frown \langle 1\rangle}$, for which $p_{s \frown \langle i\rangle} \Vdash$ "$\check{x}_{s \frown \langle i\rangle} \in \mathring{b}$", $i = 0, 1$. Furthermore, we may clearly do this in such a way that for any $s, t \in 2^{n+1}$, $ht(x_s) = ht(x_t)$. Since P is σ-closed, for each $f \in 2^{\underset{\smile}{\omega}}$ we may pick $p_f \in P$ such that $p_f \leqslant p_{f \restriction n}$ for all $n < \omega$. Also, as $|2^{\underset{\smile}{\omega}}| = \omega$, we may pick $\alpha < \omega_1$ such that $ht(x_s) < \alpha$ for all $s \in 2^{\underset{\smile}{\omega}}$. Since $p_f \leqslant p_\varnothing$ (each $f \in 2^{\omega}$), we can find $p'_f \leqslant p_f$ such that for some $x_f \in T_\alpha$, $p'_f \Vdash$ "$\check{x}_f \in \mathring{b}$". But, clearly, $p'_f \Vdash$ "$\check{x}_{f \restriction n} <_T \check{x}_f$" for all $n < \omega$, so by our construction, $f \neq g \to x_f \neq x_g$. (There are just two remarks called for here. Firstly, since $\underset{\sim}{T} \in M$, if $p'_f \Vdash$ "$\check{x}_{f \restriction n} <_T \check{x}_f$" then in fact $x_f <_T x_{f \restriction n}$. Secondly, if $f \neq g$ then for some $n < \omega$, $f \restriction n \neq g \restriction n$.). Thus $\{x_f |\ f \in 2^{\omega}\}$ is an uncountable subset of T_α, which is absurd. ∎

THEOREM 6 (Silver). *Let $\varkappa$ be an inaccessible cardinal in M. Let $P = [P(\varkappa)]^M$. Let G be M-generic for P. Then $M[G] \models$ "$2^{\omega} = \omega_1 + K$".*

Proof. By Lemmas 4 and 1, $M[G] \models$ "$2^{\omega} = \omega_1$" and $\omega_2^{M[G]} = \varkappa$. Also, $\omega_1^{M[G]} = \omega_1^M$, so the notion of an "$\omega_1$-tree" is absolute here. Let $\underset{\sim}{T}$ be an ω_1-tree in $M[G]$. We may assume $\underset{\sim}{T} = \langle \omega_1, \leqslant_T\rangle$. By the truth lemma, we can find an uncountable regular cardinal $\lambda < \varkappa$ of M such that $\underset{\sim}{T} \in M[G \cap P_\lambda]$. By Lemma 2, $\underset{\sim}{T}$ has fewer than $\varkappa$ cofinal branches in $M[G \cap P_\lambda]$. But by Lemma 4, $\underset{\sim}{P^\lambda}$ is σ-closed in $M[G \cap P_\lambda]$, and by Lemma 3, $G \cap P^\lambda$ is $M[G \cap P_\lambda]$-generic for P^λ, so by Lemma 5, $\underset{\sim}{T}$ has no cofinal branches in $M[G \cap P_\lambda][G \cap P^\lambda]$ other than those in $M[G \cap P_\lambda]$. Again by Lemma 3, $M[G \cap P_\lambda][G \cap P^\lambda] = M[G]$, so we see that $\underset{\sim}{T}$ has fewer than $\varkappa$ cofinal branches in $M[G]$. ■

The new model. We shall require the following well known result, proved in Jech [3].

LEMMA 7 (Marczewski). *Let λ be a limit ordinal,* $\mathrm{cf}(\lambda) = \omega_1$. *Let J be a collection of ω_1 finite subsets of λ. There is a finite subset X of λ and an uncountable subfamily J' of J such that $Y, Z \in J' \to Y \cap Z = X$.*

Let $\varkappa$ be an ordinal. The poset $C(\varkappa)$ is defined as follows. An element of $C(\varkappa)$ is a finite function p such that $\mathrm{dom}(p) \subseteq \varkappa$ and $\mathrm{ran}(p) \subseteq 2$. The partial ordering on $C(\varkappa)$ is defined by $p \leqslant q \leftrightarrow p \supseteq q$. Thus, if $\varkappa$ is an uncountable regular cardinal in M, $[C(\varkappa)]^M$ is the usual poset for adding $\varkappa$ Cohen generic subsets of ω to M. Note that in this case, $[C(\varkappa)]^M = C(\varkappa)$, both of these being defined by the same, absolute formula of set theory.

It is well known that if $\varkappa$ is an uncountable regular cardinal in M and G is M-generic for $C = [C(\varkappa)]^M$, then M and $M[G]$ have the same cardinals, by virtue of the fact that $M \models$ "C satisfies the countable chain condition", and $M[G] \models 2^\omega \geqslant \varkappa$. For our purposes, however, it will be useful to regard the procedure of forcing with C over M here as an iteration of length $\varkappa$. Accordingly, we make the following definitions.

Let U be the poset consisting of all maps p such that $\mathrm{dom}(p) = n$ for some $n \in \omega$ and $\mathrm{ran}(p) \subseteq 2$, ordered by $p \leqslant q \leftrightarrow p \supseteq q$. Thus $U \in M$ and U is the usual poset for adding one Cohen generic subset of ω to M.

Let $\varkappa \in On$. Set $C^*(\varkappa) = \{\varphi | \varphi\colon \varkappa \to U$ & for some finite set $X \subseteq \varkappa$, $\varphi(\alpha) \neq \emptyset \leftrightarrow \alpha \in X$ (we call X the *support* of φ, $\mathrm{supp}(\varphi))\}$, and partially order $C^*(\varkappa)$ by $\varphi \leqslant \psi \leftrightarrow (\forall \alpha \in \varkappa)(\varphi(\alpha) \supseteq \psi(\alpha))$. It is easily seen that forcing with $C^*(\varkappa)$ is equivalent to forcing with $C(\varkappa)$. In fact, the complete boolean algebra associated with both of these posets is the Borel algebra on $2^\varkappa$ factored by the ideal of all meager Borel subsets of $2^\varkappa$, where $2^\varkappa$ is given the product topology for the discrete topology on 2. Note also that the definition of $C^*(\varkappa)$ is, like $C(\varkappa)$, absolute for transitive models of ZFC containing $\varkappa$. The point of all of this is that forcing with $C^*(\varkappa)$ can be regarded as a process of forcing with U $\varkappa$ times, successively, using Lemma 3.

LEMMA 8. *Let $\varkappa$ be an uncountable cardinal in M, $\mathrm{cf}^M(\varkappa) > \omega$. Let $C = [C(\varkappa)]^M$. If G is M-generic for C, then $M[G] \models 2^\omega \geqslant \varkappa$, M and $M[G]$ have the same cardinals and cofinality function, and if $M \models 2^\omega \leqslant \varkappa$, then $M[G] \models 2^\omega = \varkappa$. Furthermore, if $\underset{\sim}{T} = \langle \omega_1^M, \leqslant_T \rangle$ is an ω_1-tree in M, and b is a cofinal branch of $\underset{\sim}{T}$ in $M[G]$, then $b \in M$.*

Proof. The last part of the lemma is the only non-standard part. Let $C^* = [C^*(\varkappa)]^M$. We may assume, by virtue of our above remarks, that G is M-generic for C^* rather than C. Let $\underset{\sim}{T} = \langle \omega_1^M, \leqslant_T \rangle$ be an ω_1-tree in M. We may assume that $\nu <_T \tau \to \nu < \tau$. Note that as $\omega_1^{M[G]} = \omega_1^M$, $\underset{\sim}{T}$ is still an ω_1-tree in $M[G]$.

If $\gamma < \varkappa$, then clearly $C^*(\gamma) = \{\varphi \restriction \gamma |\ \varphi \in C^*\}$. Set $G_\gamma = \{\varphi \restriction \gamma |\ \varphi \in G\}$. By Lemma 3, G_γ is M-generic for $C^*(\gamma)$ and $M[G]$ is a generic extension of $M_\gamma = M[G_\gamma]$. Clearly, $M_\varkappa = M[G]$, so it suffices to prove, by induction on $\gamma \leqslant \varkappa$, that if b is a cofinal branch of $\underset{\sim}{T}$ in M_γ, then $b \in M$.

For $\gamma = 0$ there is nothing to prove. Suppose the result holds for $\gamma < \varkappa$. If $H = \{\varphi(\gamma) |\ \varphi \in G\}$, then by Lemma 3, H is M_γ-generic for U and $M_{\gamma+1} = M_\gamma[H]$. Let b be a cofinal branch of $\underset{\sim}{T}$ in $M_{\gamma+1}$. It suffices, by virtue of the induction hypothesis, to show that $b \in M_\gamma$. This will be so if, whenever $p \in U$ and $p \Vdash$ "$\mathring{b}$ is a cofinal branch of $\check{\underset{\sim}{T}}$", there is $q \leqslant p$ such that $q \Vdash$ "$\mathring{b} \in \check{V}$". We work in M_γ. Let such a p be given. For each $q \leqslant p$, let $\alpha(q)$ be the supremum of all ordinals $\xi < \omega_1$ such that $q \Vdash$ "$\check{\nu} \in \mathring{b}$" for some ν on level ξ of $\underset{\sim}{T}$. Set $\alpha = \sup\{\alpha(q) |\ q \leqslant p\}$. By the truth lemma for forcing with U over M_γ, $\alpha = \omega_1$. Hence, as $|U| = \omega$, $\alpha(q) = \omega_1$ for some $q \leqslant p$. Set $b' = \{\nu \in T |\ q \Vdash$ "$\check{\nu} \in \mathring{b}$"$\}$. Then $b' \in M_\gamma$, and clearly $q \Vdash$ "$\mathring{b} = \check{b}'$", so we are done. Finally, suppose $\gamma \leqslant \varkappa$, $\lim(\gamma)$, and the result holds for all $\delta < \gamma$. There are three cases to consider.

Case 1. $\mathrm{cf}^M(\gamma) = \omega$. Let b be a cofinal branch of $\underset{\sim}{T}$ in M_γ. In M, let $\langle \gamma_n |\ n < \omega \rangle$ be cofinal in γ. Work in M_γ. By the truth lemma for forcing with $C^*(\gamma)$ over M, for each $\nu \in b$ we can find $p_\nu \in G_\gamma$ such that $p_\nu \Vdash$ "$\check{\nu} \in \mathring{b}$". Let $X_\nu = \mathrm{supp}(p_\nu)$. Since each X_ν is finite, and $\mathrm{cf}(\omega_1) > \omega$, we can find an uncountable set $b' \subseteq b$ such that $\nu \in b' \to X_\nu \subseteq \gamma_n$ for some fixed $n < \omega$. But clearly, $b = \{\nu \in T |\ (\exists p \in G_{\gamma_n})[p \Vdash$ "$\check{\nu} \in \mathring{b}$"$]\} \in M_{\gamma_n}$. Hence, by induction hypothesis, $b \in M$.

Case 2. $\mathrm{cf}^M(\gamma) = \omega_1^M$. Let b be a cofinal branch of $\underset{\sim}{T}$ in M_γ. Suppose, by way of contradiction, that $b \notin M$. By induction hypothesis, therefore, $\delta < \gamma \to b \notin M_\delta$, also. Work in M_γ. For each $\nu \in b$, pick $p_\nu \in G_\gamma$ such that $p_\nu \Vdash_{C^*(\gamma)}$ "$\check{\nu} \in \mathring{b}$", and let $X_\nu = \mathrm{supp}(p_\nu)$. If $\sup\{\max(X_\nu) |\ \nu \in b\} < \gamma$, then arguing as in Case 1 we see that $b \in M_\delta$ for $\delta = \sup\{\max(X_\nu) |\ \nu \in b\}$, and we are done. Hence we may assume $\sup\{\max(X_\nu) |\ \nu \in b\} = \gamma$. It follows, by Lemma 6, that we can find an uncountable set $b' \subseteq b$ and a finite set $X \subseteq \gamma$ such that $\nu, \tau \in b'$ and $\nu < \tau$ implies $X_\nu \cap X_\tau = X$ and such that $\nu \in b'$ implies $X_\nu \neq X$. Since $|U| = \omega$, we can find an uncountable set $b'' \subseteq b'$ such that $\nu, \tau \in b''$ implies $p_\nu \restriction X = p_\tau \restriction X = p$, say. From now on $\Vdash$ refers to the forcing relation for $C^*(\gamma)$ over M.

CLAIM. *There is $q \in C^*(\gamma)$, $\mathrm{supp}(q) \cap X = \emptyset$, and $\nu < \omega_1$ such that*

$\nu \notin b$ but $p \cup q \Vdash$ "$\check{\nu} \in \mathring{b}$", where $p \cup q \in C^*(\gamma)$ is defined from p and q in the obvious manner.

Suppose the claim is false. In M, set $d = \{\nu \in T | (\exists q \in C^*(\gamma)) [\mathrm{supp}(q) \cap \cap X = \emptyset \,\&\, p \cup q \Vdash \text{"}\check{\nu} \in \mathring{b}\text{"}]\}$. Since the claim fails, $d \subseteq b$. But for each $\nu \in b''$, if $q = p_\nu \restriction (X_\nu - X)$, then $\mathrm{supp}(q) \cap X = \emptyset$ and $p \cup q = p_\nu$ and $p_\nu \Vdash$ "$\check{\nu} \in \mathring{b}$", so $b \subseteq d$. Hence $b = d \in M$, a contradiction. This proves the claim.

Pick $q \in C^*(\gamma)$ as in the claim and let $\nu < \omega_1$ be such that $\nu \notin b$ and $p \cup q \Vdash$ "$\check{\nu} \in \mathring{b}$". Pick $\tau \in b''$, $\tau > \nu$, such that $X_\tau \cap \mathrm{supp}(q) = \emptyset$. (This is clearly possible). Clearly, $p_\tau \cup q = p \cup q \cup [p_\tau \restriction (X_\tau - X)] \in C^*(\gamma)$. But look $p_\tau \cup q \leqslant p_\tau$, so $p_\tau \cup q \Vdash$ "$\check{\tau} \in \mathring{b}$", and $p_\tau \cup q \leqslant p \cup q$, so $p_\tau \cup q \Vdash$ "$\check{\nu} \in \mathring{b}$". Hence, as $\nu < \tau$, $p_\tau \cup q \Vdash$ "$\check{\nu} <_T \check{\tau}$", which means $\nu <_T \tau$, of course. Thus, as $\tau \in b''$, $\nu \in b$, a contradiction.

Case 3. $\mathrm{cf}^M(\gamma) > \omega_1^M$. This case is trivial by the truth lemma for forcing with $C^*(\gamma)$ over M.

The lemma is proved. ■

The following is an analogue of Lemma 5.

LEMMA 9. *Let C, P be posets in M such that $M \vDash$ "C satisfies* c.c.c. *and P is σ-closed". Let G be M-generic for $C \times P$. (Thus $\omega_1^M = \omega_1^{M[G]}$.) Let $G_C = \{p \in C | \langle p, \mathbf{1}\rangle \in G\}$, $G_P = \{q \in P | \langle \mathbf{1}, q\rangle \in G\}$. (Thus G_C is M-generic for C, G_P is $M[G_C]$-generic for P, and $M[G_C][G_P] = M[G]$.) Let $\underset{\sim}{T}$ be an ω_1-tree in $M[G_C]$. If b is a cofinal branch of $\underset{\sim}{T}$ in $M[G]$, then $b \in \widetilde{M}[G_C]$.*

Proof. Notice that as P is not necessarily σ-closed in the sense of $M[G_C]$, we cannot argue exactly as in Lemma 5. However, with a little extra work, we can carry through an argument parallel to that of Lemma 5. We shall assume that $\underset{\sim}{T} = \langle \omega_1, \leqslant_T\rangle$, as before, and that $T_0 = \{0\}$. Let b be a cofinal branch of $\underset{\sim}{T}$ in $M[G]$. We shall suppose that $b \notin M[G_C]$ and derive a contradiction. By $\Vdash_C$ we shall mean C-forcing over M, and by $\Vdash_P$ P-forcing over $M[G_C]$. For simplicity, we shall assume that $\mathbf{1}_P \Vdash_P$ "b is a cofinal branch of $\underset{\sim}{T}$ not in $M[G_C]$". (In the general case we pick some $p \in G_P$ which forces this statement and work below p in P.) Similarly, we shall assume that $\mathbf{1}_C \Vdash_C$ ["$T = \langle \omega_1, \leqslant_T\rangle$ is an ω_1-tree $\&\, T_0 = \{0\} \,\&\, \mathbf{1}_P \Vdash_P$ ["b is a cofinal branch of $\underset{\sim}{T}$ not in $M[G_C]$"]"]. (To avoid awkward clashes of notation, we shall write b, x, ω_1 instead of $\mathring{b}$, $\check{x}$, $\check{\omega}_1$, etc. and rely on the context to provide the precise meaning.

We shall construct, in M, a sequence $\langle q_s | s \in 2^{\underset{\sim}{\omega}}\rangle$ of members of P, with $s \subseteq t \rightarrow q_t \leqslant_P q_s$, and a sequence $\langle \alpha_n | n < \omega\rangle$ of countable ordinals, such that, for $s \in 2^n$:

(i) $\mathbf{1} \Vdash_C (\exists x \in T_{\alpha_n})(q_s \Vdash_P x \in b)$;

(ii) $\mathbf{1} \Vdash_C$ "if $x \in T_{\alpha_n} \,\&\, q_s \Vdash_P x \in b$, then there are $x_0, x_1 \in T_{\alpha_{n+1}}$, $x_0 \neq x_1$, $x <_T x_0, x_1$, such that $q_{s \frown \langle i\rangle} \Vdash_P x_i \in b$ $(i = 0, 1)$".

By analogy with Lemma 5, this will give the required result. For, let us place ourselves in $M[G_C]$, whence the statements in (i) and (ii) above will be true. For each $f \in 2^\omega \cap M$, we obtain a $q_f \in P$, $q_f \leqslant_P q_{f \restriction n}$ for all $n < \omega$. (Since P is σ-closed in the sense of M.) Since C satisfies the c.c.c. in M, $2^\omega \cap M$ is uncountable. Hence, as in Lemma 5, we obtain an uncountable family $\{x_f | f \in 2^\omega \cap M\}$ of distinct elements of T_α, where $\alpha = \sup_{n<\omega} \alpha_n$.

The construction of the sequences $\langle q_s | s \in 2^{\underset{\sim}{\omega}}\rangle$ and $\langle \alpha_n | n < \omega\rangle$ is by induction. From now on we work in M.

CLAIM 1. *Let $\alpha < \omega_1$, $q \in P$, $\mathbf{1} \Vdash_C (\exists x \in T_\alpha)(q \Vdash_P x \in b)$. Then there are $q^0, q^1 \leqslant_P q$ such that $\mathbf{1} \Vdash_C$ "if $x \in T_\alpha \,\&\, q \Vdash_P x \in b$, then there is $\beta > \alpha$ and $x^0, x^1 \in T_\beta$, $x^0 \neq x^1$, $x <_T x^0, x^1$, such that $q^i \Vdash_P x^i \in b$".*

In order to prove Claim 1, we define, by induction, a sequence $\langle\langle p_\nu, x_\nu, x_\nu^0, x_\nu^1, q_\nu^0, q_\nu^1, \beta_\nu\rangle | \nu < \delta\rangle$ for some $\delta < \omega_1$, so that:

(i) $\nu < \delta \rightarrow p_\nu \in C \,\&\, q_\nu^i \in P \,\&\, x_\nu, x_\nu^i \in \omega_1 \,\&\, \beta_\nu < \omega_1$;

(ii) $\nu < \tau < \delta \rightarrow p_\nu \nsim p_\tau \,\&\, q_\tau^i \leqslant_P q_\nu^i \leqslant_P q$;

(iii) $p_\nu \Vdash_C$ ["$x_\nu \in T_\alpha \,\&\, q \Vdash_P x_\nu \in b$"];

(iv) $p_\nu \Vdash_C$ ["$x_\nu^i \in T_{\beta_\nu} \,\&\, x_\nu^0 \neq x_\nu^1 \,\&\, x_\nu \leqslant_T x_\nu^i \,\&\, q_\nu^i \Vdash_P x_\nu^i \in b$"].

The ordinal δ will be determined by the construction breaking down. This will occur when $\{p_\nu | \nu < \delta\}$ is a maximal pairwise incompatible subset of C. Hence, by c.c.c. for C, $\delta < \omega_1$.

Suppose we are at stage ν and that $\{p_\tau | \tau < \nu\}$ is not maximal pairwise incompatible. As P is σ-closed, we can find $r_\nu^i \in P$ such that $(\forall \tau < \nu)(r_\nu^i \leqslant_P q_\tau^i)$. Pick $p_\nu \in C$ incompatible with all the p_τ, $\tau < \nu$. By extending p_ν if necessary, we may assume that for some x_ν,

$$p_\nu \Vdash_C (x_\nu \in T_\alpha \,\&\, q \Vdash_P x_\nu \in b).$$

We may likewise assume further that there is $\beta_\nu > \alpha$ and $r_\nu^{i0}, r_\nu^{i1} \leqslant_P r_\nu^i$ and x_ν^{i0}, x_ν^{i1}, such that $x_\nu^{i0} \neq x_\nu^{i1}$ and

$$p_\nu \Vdash_C \text{"}[x_\nu^{ij} \in T_{\beta_\nu} \,\&\, x_\nu \leqslant_T x_\nu^{ij} \,\&\, r_\nu^{ij} \Vdash_P x_\nu^{ij} \in b]\text{"}.$$

Since $x_\nu^{i0} \neq x_\nu^{i1}$ for each i, we may assume that $x_\nu^{00} \neq x_\nu^{11}$ (say). Set $x_\nu^i \neq x_\nu^{ii}$, $q_\nu^i = q_\nu^{ii}$. That completes the construction. Pick q^i now so that $(\forall \nu < \delta)(q^i \leqslant_P q_\nu^i)$. The q^i are now as required. For let $p \in C$ be given. For some $\nu < \delta$, there is $p' \leqslant_C p, p_\nu$. By conditions (iii) and (iv) above, together with the fact that $q^i \leqslant_P q_\nu^i$, p' forces the statement in the claim. Hence as p was arbitrary, the set of $p \in C$ which force this statement is dense in C, which proves the claim.

To construct the sequences $\langle q_s | s \in 2^{\underset{\sim}{\omega}} \rangle$ and $\langle \alpha_n | n < \omega \rangle$ is now easy. Let $\beta(\alpha, q, q^0, q^1) = \sup_{\nu<\delta} \beta_\nu$ in the above proof of Claim 1. If now q_s, $s \in 2^n$, and α_n are defined, extend q_s. $(s \in 2^n)$ to q_s^0, q_s^1 as in Claim 1, and set $\alpha_{n+1} = \sup\{\beta(\alpha_n, q_s, q_s^0, q_s^1) | s \in 2^n\}$. Then extend q_s^i to $q_{s \frown \langle i \rangle}$ as required by means of:

Claim 2. *If $\alpha < \omega_1$, $q \in P$, there is $q' \leqslant_P q$ such that $\mathbf{1} \Vdash_C (\exists y \in T_\alpha)(q \Vdash_P y \in b)$.*

The proof of Claim 2 is similar to, but much easier than, the proof of Claim 1, so we shall omit it. The lemma is thus proved. ∎

Theorem 10. *Let $\varkappa$ be an inaccessible cardinal in M, and let λ be an arbitrary cardinal in M such that $\lambda \geqslant \varkappa$ and $\mathrm{cf}^M(\lambda) > \omega$. Let $P = [P(\varkappa)]^M$, $C = [C(\lambda)]^M$. Let G be M-generic for $P \times C$. Then $\omega_1^M = \omega_1^{M[G]}$, $\varkappa = \omega_2^{M[G]}$, λ and all other cardinals of M above $\varkappa$ are cardinals in $M[G]$ (so if $\lambda = \omega_{\varkappa+\gamma}^M$ then $\lambda = \omega_{2+\gamma}^{M[G]}$), $\mathrm{cf}^{M[G]}(\lambda) > \omega$, $M[G] \models$ "$2^\omega = \lambda$", and $M[G] \models$ "K".*

Proof. Let G_P, G_C be as above. Let $\underset{\sim}{T} = \langle \omega_1, \leqslant_T \rangle$ be an ω_1-tree in $M[G]$. By the truth lemma, pick $\gamma < \varkappa$ an uncountable regular cardinal of M such that $\underset{\sim}{T} \in M[G_P \cap P_\gamma][G_C]$. Let $N = M[G_P \cap P_\gamma]$. Notice that by Lemma 4, $\underset{\sim}{P}^\gamma$ is σ-closed in the sense of M. Also, by absoluteness, $C = [C(\lambda)]^N$, so C satisfies c.c.c. in N. Now, by Lemma 3, G_C is N-generic for C, so by the truth lemma for C-forcing over N we can find, in N, a set $X \subseteq \lambda$, $|X| = \omega_1$, such that $\underset{\sim}{T} \in N[G_C \cap C_X]$, where $C_X = \{p \restriction X | p \in C\}$. Now, $X \in N$, so in N there is a canonical isomorphism $C \cong C_X \times C^X$, where $C^X = \{p - p \restriction X | p \in C\}$. Thus, by Lemma 3 (applied to N), $G_C \cap C_X$ is N-generic for C_X, $G_C \cap C^X$ is $N[G_C \cap C_X]$-generic for C^X, and $N[G_C \cap C_X][G_C \cap C^X] = N[G_C]$. By Lemma 2, $\varkappa$ is inaccessible in $N[G_C \cap C_X] = M[G_P \cap P_\gamma][G_C \cap C_X]$. Hence $\underset{\sim}{T}$ has fewer than $\varkappa$ cofinal branches in $N[G_C \cap C_X]$. In $N[G_C \cap C_X]$, there is a canonical isomorphism $C^X \cong [C(\lambda)]^{N[G_C \cap C_X]}$. Hence, by Lemma 8 applied to $N[G_C \cap C_X]$, $\underset{\sim}{T}$ has no extra cofinal branches in $N[G_C] = N[G_C \cap C_X][G_C \cap C^X]$. But by Lemma 3 again, $M[G] = M[G_P][G_C] = M[G_P \cap P_\gamma][G_P \cap P^\gamma][G_C] = N[G_C][G_P \cap P^\gamma]$ and $G_P \cap P^\gamma$ is $N[G_C]$-generic for P^γ. So, applying Lemma 9 to N and the posets C, P^γ, we see that $\underset{\sim}{T}$ has no extra cofinal branches in $M[G]$. Hence $\underset{\sim}{T}$ is not Kurepa in $M[G]$. ∎

Added in proof. Using similar techniques to the above, we have since obtained a model of the theory ZFC+$2^\omega = \omega_2$+K+Martin's Axiom. The proof will appear elsewhere.

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