On Δ-spaces and fundamental dimension in the sense of Borsuk

by

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Dedicated to Professor Kiiti Morita for his 60th birthday

Abstract. A class Δ of compacta which contains all triangulable spaces and 0-dimensional compacta is introduced and it is shown that for every compactum X there is a compactum Y belonging to Δ such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\mathrm{Fd}(X) = \dim Y$, where $\mathrm{Fd}(X)$ is the fundamental dimension of X in the sense of Borsuk. Let X be a finite dimensional compactum which is approximatively k-connected for $k = 0, 1, \ldots, n+1$. It is proved that if A is a closed subset of X with $\dim A \leqslant n$ then $\mathrm{Fd}(X) \leqslant \mathrm{Fd}(X/A)$.

§ 1. Introduction. In this paper we introduce a class Δ of compacta which is called *Δ-spaces*. The class Δ contains all triangulable spaces and 0-dimensional compacta. Every Δ-space is dimensionally fullvalued for paracompact spaces (see [7, p. 357]). In [2] and [3] K. Borsuk has defined shapes of compacta. The shape of a compactum X is denoted by $\mathrm{Sh}(X)$. By K. Borsuk [3, p. 31] the fundamental dimensional $\mathrm{Fd}(X)$ of X is defined as $\mathrm{Min}\{\dim Y\colon Y$ is a compactum such that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)\}$. We shall show that if $\mathrm{Fd}(X) = n$ then there is a Δ-space Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\dim Y = n$. In particular, for each compactum X we can find a compactum Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\dim Y = \mathrm{Fd}(X)$. Let A be a closed subset of X which is contractible in X and let X/A be the compactum obtained from X by contracting A to a point. Then it is shown that $\mathrm{Fd}(X/A) \geqslant \mathrm{Fd}(X)$, i.e. the quotient map: $X \to X/A$ raises the fundamental dimension.

Throughout the paper we assume that all spaces are metric and all maps are continuous.

§ 2. Δ-spaces.

DEFINITION 1. A compactum X is called a *Δ-space* if there is an inverse sequence $\{P_i, \pi_i^{i+1}\}$ whose limit space $\varprojlim P_i$ is X such that

(2.1) each P_i is a simplicial complex,

(2.2) each bonding map $\pi_i^{i+1}\colon P_{i+1} \to P_i$ is a simplicial map.

By Δ we mean the class of all Δ-spaces.

EXAMPLE 1. Let S_i, $i = 1, 2, \ldots$, be a 1-sphere considered as the set of all complex numbers z with $|z| = 1$ and let $\pi_i^{i+1}\colon S_{i+1} \to S_i$ be a map defined by the formula $\pi_i^{i+1}(z) = z^{p_i}$, where p_i is a positive integer. The inverse limit $S(p) = \varprojlim\{S_i, \pi_i^{i+1}\}$ is called the *solenoid generated by* $p = (p_1, p_2, \ldots)$. Obviously we can subdivide each S_i such that π_i^{i+1} is simplicial. Thus $S(p)$ is a Δ-space.

EXAMPLE 2. Let $p = (p_1, p_2, \ldots)$ be a sequence of positive integers such that $p_i > 1$ for each i. In [6; II, p. 106] we constructed a 2-dimensional continuum $R(p)$. Also, in case p_i is a divisor of p_{i+1} for each i, a 2-dimensional continuum $Q(p)$ was defined in [6; I, p. 390]. If each p_i is equal to a fixed prime p, then $Q(p)$ is a Pontrjagin's surface mod$p$. It is easy show that no continua $R(p)$ and $Q(p)$ are in Δ (see Theorem 2).

THEOREM 1. (1) *All polyhedra and 0-dimensional compacta belong to* Δ.

(2) *There is a 1-dimensional compact* AR X *such that* X *is not a* Δ*-space and* X *does not have any singularities of Peano, Alexandroff, Mazurkiewicz and Brouwer of type* (n, k) *in the sense of Borsuk* [1, Chap. VI].

Proof. Since (1) is obvious, it is enough to find an AR X satisfying (2). Let M be a set in the plane defined as follows: $M = \{(x, y)\colon 0 \leqslant x \leqslant 1$ and $y = 0$; $x = \frac{1}{2}$ and $0 \leqslant y \leqslant \frac{1}{2}\}$. We consider M as a 1-dimensional simplicial complex with 4-vertices: $(0, 0)$, $(\frac{1}{2}, 0)$, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. A subset $N = \{(x, y)\colon 0 \leqslant x \leqslant 1$ and $y = 0\}$ which is a subcomplex of M is a *base* of M, $(\frac{1}{2}, 0)$ is the *middle vertex*, the 1-simplex $\{(x, y)\colon x = \frac{1}{2}$ and $0 \leqslant y \leqslant \frac{1}{2}\}$ is the *middle simplex*. By $\pi\colon M \to N$ denote the simplicial map such that $\pi|N$ is the identity and π maps the middle simplex to the middle vertex. Let X_1 be a simplicial complex consisting of only one 1-simplex and its vertices. For each j, $j \leqslant k$, suppose that a 1-dimensional simplicial complex X_j and a bonding map $\pi_{j-1}^j\colon X_j \to X_{j-1}$ are constructed. In order to construct X_{k+1}, let us replace each 1-simplex s_i of X_k by a copy M_i of M such that the base N_i of M_i coincides s_i. The projection $\pi_k^{k+1}\colon X_{k+1} \to X_k$ is defined by putting $\pi_k^{k+1}|M_i = \pi\colon M_i \to N_i$ $(= s_i)$ on each M_i. We obtain an inverse sequence $\{X_j, \pi_j^{j+1}\}$. Set $X = \varprojlim X_j$. It is easy to show that X is a 1-dimensional compact AR and does not have any singularities in the theorem. Let us prove that X is not a Δ-space. Consider the subset X_0 of X consisting of all points (x_j), $x_j \in X_j$, of X such that for some k each x_j, $j \geqslant k$, is a vertex of X_j. Then X_0 is dense in X. Suppose that there is an inverse sequence $\{Y_i, \mu_i^{i+1}\}$ such that $X = \varprojlim Y_i$, Y_i is a simplicial complex and $\mu_i^{i+1}\colon Y_{i+1} \to Y_i$ is simplicial. We may assume that each μ_i^{i+1} is onto. Let $\mu_i\colon X \to Y_i$ be the projection. Then $\mu_i(X_0)$ is dense in Y_i for each i. For each point y of X_0, there is $Y_j \in \{Y_i\}$ such that $\mu_j(y)$ is a vertex of Y_j, because $\dim Y_i \leqslant 1$ for each i and y has a sufficiently small neighborhood in X whose boundary consists of more than 2-points. Hence $\mu_i(y)$, $i = 1, 2, \ldots$, has to be a vertex of X_i. This contradicts that $\mu_i(X_0)$ is dense in X_i.

For an abelian group G we define the homological dimension $d(X\colon G)$, the local homological dimension $\operatorname{loc} d(X\colon G)$, the cohomological dimension $D(X\colon G)$ and the local cohomological dimension $\operatorname{loc} D(X\colon G)$ of a space X as follows (see [6] and [7]):

$$d(X\colon G) = \operatorname{Max}\{n\colon \check{H}_n(X, A\colon G) \neq 0 \text{ for some closed set } A \text{ of } X\},$$

$$\operatorname{loc} d(X\colon G) = \operatorname{Min}\{n\colon \text{for every point } x \text{ and every neighborhood } U \text{ of } x \text{ in } X \text{ there is a closed neighborhood } V \text{ contained in } U \text{ such that } d(V\colon G) \leqslant n\}.$$

Here $\check{H}_*$ is the Čech homology group. $D(X\colon G)$ and $\operatorname{loc} D(X\colon G)$ are defined by using the Čech cohomology group $\check{H}^*$ in place of $\check{H}_*$ in the definition of $d(X\colon G)$ and $\operatorname{loc} d(X\colon G)$ respectively.

Let X be a Δ-space. There is an inverse sequence $\{P_i, \pi_i^{i+1}\}$ whose limit space is X such that P_i is a simplicial complex and π_i^{i+1} is simplicial for each i. We can assume that each π_i^{i+1} is onto. Let k be a non-negative integer such that $k \leqslant \dim X$. Since $\varprojlim P_i = X$, there is an integer m such that $\dim P_j \geqslant k$ for $j \geqslant m$. For every closed k-simplex σ_m of P_m, there is a k-simplex σ_j of P_j such that $\pi_m^j|\sigma_j\colon \sigma_j \to \sigma_m$ is a homeomorphism, where $\pi_m^j = \pi_m^{m+1} \ldots \pi_{j-1}^j$. Hence X contains a homeomorph of a closed k-simplex for every k, $k \leqslant \dim X$. Thus the following is obvious and a consequence of [7, Cor. 5].

THEOREM 2. *Let* X *be a finite or an infinite dimensional compactum belonging to* Δ. *For every abelian group* G, $d(X\colon G) = \operatorname{loc} d(X\colon G) = D(X\colon G) = \operatorname{loc} D(X\colon G) = \dim X$. *If* Y *is a paracompact Hausdorff space with* $\dim Y < \infty$, *then* $D(X \times Y\colon G) = \dim X + D(Y\colon G)$. *In particular,* X *is dimensionally full valued for paracompact spaces.*

It is not known that the cohomological dimension of an infinite dimensional compactum is infinite. However, if X is an infinite dimensional Δ-space, then $D(X\colon G) = \infty$ for each abelian group G.

§ 3. Fundamental dimension $\operatorname{Fd}(X)$. Let X be a compactum. K. Borsuk [3, p. 31] gave the following definition.

DEFINITION 2. $\operatorname{Fd}(X) = \operatorname{Min}\{\dim Y\colon Y$ is a compactum such that $\operatorname{Sh}(Y) \geqslant \operatorname{Sh}(X)\}$. $\operatorname{Fd}(X)$ is called the *fundamental dimension* of X.

THEOREM 3. *Let* X *be a compactum. Then* $\operatorname{Fd}(X) \leqslant n$ *if and only if there is a* Δ*-space* Y *such that* $\dim Y \leqslant n$ *and* $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$.

Proof. The "if" part follows from Definition 2. Let us prove the "only if" part. Let $\operatorname{Fd}(X) \leqslant n$. There is a compactum Z such that $\dim Z \leqslant n$ and $\operatorname{Sh}(X) \leqslant \operatorname{Sh}(Z)$. Let $\{P_i, \pi_i^{i+1}\}$ and $\{T_i, \mu_i^{i+1}\}$ be inverse sequences such that

(3.1) $$X = \underleftarrow{\lim} P_i \quad \text{and} \quad Z = \underleftarrow{\lim} T_i ,$$

(3.2) P_i and T_i are polyhedra and π_i^{i+1} and μ_i^{i+1} are onto, $i = 1, 2, \ldots,$

(3.3) $$\dim T_i \leqslant n , \quad i = 1, 2, \ldots$$

Since $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Z)$, by Mardešić and Segal [10] there exist sequences of maps $\underline{f} = \{f_i\}$: $\{P_i\} \to \{T_i\}$ and $\underline{g} = \{g_i\}$: $\{T_i\} \to \{P_i\}$ such that f_i: $P_{f(i)} \to T_i$ and g_i: $T_{g(i)} \to P_i$,

(3.4) for $i \leqslant j$ $f_i \pi_{f(i)}^{f(j)} \simeq \mu_i^j f_j$: $P_{f(j)} \to T_i$ and $g_i \mu_{g(i)}^{g(j)} \simeq \pi_i^j g_j$: $T_{g(j)} \to P_i$,

(3.5) $$g_i f_{g(i)} \simeq \pi_i^{fg(i)}\colon P_{fg(i)} \to P_i , \quad i = 1, 2, \ldots$$

(See [9] and [10] for notations.) We may assume that f: $N \to N$ and g: $N \to N$ are increasing, where N is the set of positive integers (see [9, Lemma 5]). Put $a_1 = 1$. Inductively, define $a_{i+1} = fg(a_i)$ for $i \geqslant 1$. The sequence $\{a_i\}$ is increasing. Consider the map $g_1 f_{g(1)}$: $P_{a_2} \to P_{a_1}$. Let K_{a_1} be a triangulation of P_{a_1}. By (3.3) there is a triangulation K_{a_2} of P_{a_2} and a simplicial map η_1: $K_{a_2} \to K_{a_1}$ such that $\eta_1 \simeq g_1 f_{g(1)}$ and $\eta(K_{a_2}) \subset K_{a_1}^n$, where K^n means the n-skeleton of K. For $j \leqslant k$, suppose that there exist triangulations K_{a_j} of P_{a_j} and simplicial maps η_{j-1}: $K_{a_j} \to K_{a_{j-1}}$ such that

(3.6) $$\eta_{j-1} \simeq g_{a_{j-1}} f_{g(a_{j-1})}\colon K_{a_j} \to K_{a_{j-1}} \quad \text{and} \quad \eta_{j-1}(K_{a_j}) \subset K_{a_{j-1}}^n .$$

By (3.3) there is a triangulation $K_{a_{k+1}}$ of $P_{a_{k+1}}$ and a simplicial map η_k: $K_{a_{k+1}} \to K_{a_k}$ such that (3.6) holds for $j = k+1$. Thus we have constructed triangulations K_{a_j} and simplicial maps η_j satisfying (3.6) for each $j = 1, 2, \ldots$ Set $M_j = K_{a_j}^n$ and $\mu_j^{j+1} = \eta_j | M_{j+1}$, $j = 1, 2, \ldots$ By (3.6) we have $\mu_j^{j+1}(M_{j+1}) \subset M_j$. Hence $\{M_j, \mu_j^{j+1}\}$ is an inverse sequence consisting of simplicial complexes with dimension $\leqslant n$. Let $Y = \underleftarrow{\lim} M_j$. Then Y is a Δ-space with $\dim Y \leqslant n$. To complete the proof, let us show that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. Without loss of generality we may suppose that $a_j = j$ for $j = 1, 2, \ldots$ For each j, put $i(j) = j$ and let i_j: $M_j \to P_j$ be the inclusion map, and put $h(j) = j+1$ and define h_j: $P_{h(j)} \to M_j$ by $h_j = \eta_j$. We have to show that

(3.7) $\underline{i} = \{i_j\}$: $\{M_j\} \to \{P_j\}$ and $\underline{h} = \{h_j\}$: $\{P_j\} \to \{M_j\}$ are maps between ANR sequences (see [9, p. 42]),

(3.8) $\underline{hi} \simeq \underline{1}_Y$ and $\underline{ih} \simeq \underline{1}_X$, where $\underline{Y} = \{M_j\}$ and $X = \{P_j\}$ and $\underline{1}_X$ and $\underline{1}_Y$ are the identity maps (see [9, p. 43]).

From the definition

(3.9) $$\mu_j^{j+1} = h_j i_{j+1} .$$

By (3.6) and (3.5) $i_j h_j = \eta_j \simeq g_j f_{g(j)} \simeq \pi_j^{j+1}$, that is,

(3.10) $$i_j h_j \simeq \pi_j^{j+1}.$$

Hence $i_j \mu_j^{j+1} = i_j h_j i_{j+1} \simeq \pi_j^{j+1}$ and $\mu_j^{j+1} h_{j+1} = h_j i_{j+1} h_j \simeq h_j \pi_{j+1}^{j+2}$. Thus (3.7) holds. Also (3.9) and (3.10) show that (3.8) holds. This completes the proof.

The following corollary is a consequence of Theorem 3.

Corollary 1. *For every finite or infinite dimensional compactum X there is a Δ-space Y such that* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *and* $\mathrm{Fd}(X) = \dim Y$.

For the proof it is enough to note that if Y is a Δ-space and $\{K_i, \pi_i^{i+1}\}$ is an inverse sequence of simplicial complexes such that $Y = \underleftarrow{\lim} K_i$ and each bonding map π_i^{i+1} is simplicial and onto, then the equality $\dim Y = \mathrm{Max}\{\dim K_i,\ i = 1, 2, \ldots\}$ holds.

Let us define the fundamental dimension $\mathrm{Fd}(X, A)$ for a pair of compacta (X, A) as follows: $\mathrm{Fd}(X, A) = \mathrm{Min}\{\dim Y\colon Y$ is a compactum and $\mathrm{Sh}(X, A) \leqslant \mathrm{Sh}(Y, B)$ for some closed set B of $Y\}$. (For the definition of the shape $\mathrm{Sh}(X, A)$ of a pair (X, A), see [2] and [3].) If A is an empty set, then $\mathrm{Fd}(X, A) = \mathrm{Fd}(X)$. Since $\mathrm{Sh}(X, A) \leqslant \mathrm{Sh}(Y, B)$ implies $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and $\mathrm{Sh}(A) \leqslant \mathrm{Sh}(B)$, we know that

$$\mathrm{Max}\{\mathrm{Fd}(X), \mathrm{Fd}(A)\} \leqslant \mathrm{Fd}(X, A) .$$

It is not known that $\mathrm{Fd}(X, A)$ is determined by $\mathrm{Fd}(X)$ and $\mathrm{Fd}(A)$. The following problem raises:

Problem. *For a pair of compacta* (X, A), *does it hold that* $\mathrm{Fd}(X, A) \leqslant \mathrm{Max}\{\mathrm{Fd}(X), F(A)+1\}$?

The following corollary gives a partial answer to the problem.

Corollary 2. *Let X be a compact* AR *and A its closed set. Then the relation* $\mathrm{Fd}(A) \leqslant \mathrm{Fd}(X, A) \leqslant \mathrm{Fd}(A)+1$ *holds.*

Proof. Since $\mathrm{Fd}(X) = 0$, it is obvious that $\mathrm{Fd}(A) \leqslant \mathrm{Fd}(X, A)$. To prove the second relation, let us assume that $\mathrm{Fd}(A) < \infty$. By Theorem 3 there is a compact Δ-space B such that $\dim B = \mathrm{Fd}(A)$ and $\mathrm{Sh}(B) = \mathrm{Sh}(A)$. To complete the proof, it is enough to construct a compact AR $M(B)$ such that $M(B) = \dim B + 1$ and $M(B)$ contains B as a closed set. Because, from $\mathrm{Sh}(A) = \mathrm{Sh}(B)$ follows $\mathrm{Sh}(X, A) = \mathrm{Sh}(M(B), B)$ and hence $\mathrm{Fd}(X, A) \leqslant \dim M(B) = \dim A + 1$. To construct $M(B)$, let $\{K_i, \pi_i^{i+1}\}$ be an inverse sequence consisting of simplicial complexes such that K_1 consists of only one vertex, $\dim K_i \leqslant \dim B$, π_i^{i+1}: $K_{i+1} \to K_i$ is simplicial for $i = 1, 2, \ldots$, and $\underleftarrow{\lim}\{K_i\} = B$. By $M(K_{i+1}, K_i, \pi_i^{i+1})$ denote the mapping cylinder constructed for the map π_i^{i+1}. $M(K_{i+1}, K_i, \pi_i^{i+1})$ is a simplicial complex and a union of the sets $K_{i+1} \times [0, 1)$ and K_i.

Consider a topological sum $N = \bigoplus_{i=1}^{\infty} M(K_{i+1}, K_i, \pi_i^{i+1})$. For each i, by identifying $K_{i+1}\times\{0\}$ of $M(K_{i+1}, K_i, \pi_i^{i+1})$ and K_{i+1} of $M(K_{i+2}, K_{i+1}, \pi_{i+1}^{i+2})$ in N we obtain a metrizable space M. Put $M(B) = M \cup B$. Give $M(B)$ the following topology: M is open in $M(B)$ and has its proper topology. Take $x \in B$. For $i = 1, 2, \ldots$, let V be an open neighborhood of $\pi_i(x)$ in K_i, where π_i is the projection of B to K_i. For $j > i$, consider an open set $(\pi_i^j)^{-1}V\times[0,1)$ of $M(K_j, K_{j-1}, \pi_{j-1}^j)$, where $\pi_i^j = \pi_i^{i+1}\ldots\pi_{j-1}^j$. The collection of the sets of the form $(\pi_i^{-1}(V)\cap B)\cup\bigcup_{j=i+1}^{\infty}(\pi_i^j)^{-1}V\times[0,1)$, where V ranges over open neighborhoods of $\pi_i(x)$ in K_i, $i = 1, 2, \ldots$, forms a neighborhood base of x in $M(B)$. Obviously $M(B)$ is compact and metrizable, and $\dim M(B) = \dim B+1$. Since $M(B)$ is contractible and locally contractible, $M(B)$ is an AR. This completes the proof.

Remark. Theorem 3 is given in the relative form as follows:

(3.11) *For a pair of compacta* (X, A), $\mathrm{Fd}(X, A) \leqslant n$ *if and only if there is a Δ-pair* (Y, B) *such that* $\dim Y \leqslant n$ *and* $\mathrm{Sh}(X, A) = \mathrm{Sh}(Y, B)$.

Here (Y, B) is called a *Δ-pair* if there is an inverse sequence $\{(P_i, S_i), \pi_i^{i+1}\}$ such that $(Y, B) = \varprojlim(P_i, S_i)$, (P_i, S_i) is a pair of simplicial complexes and $\pi_i^{i+1}\colon (P_{i+1}, S_{i+1})\to(P_i, S_i)$ is simplicial. To do it, the theorems of Mardešić and Segal ([9, Theorem 10] and [10, Theorem]) which we used in the proof of Theorem 3 have to be given in the form of sequences of ANR pairs. This is done by modifying slightly the definition of sequences of ANR pairs and maps between sequences of ANR pairs which were given in [9, p. 42]. The original definition by Mardešić and Segal is not suitable to prove the equivalence with Borsuk's shape theory for pairs of compacta (see [10]). The example given by Borsuk [5, p. 479] shows that their approach differs from Borsuk's theory for a pair of compacta.

Let X be a compactum in the Hilbert cube Q. By K. Borsuk [4, p. 266], X is said to be *approximatively k-connected* if for every neighborhood U of X in Q there is a neighborhood V contained in U such that every map of a k-sphere S into V is null homotopic in U. As proved by Borsuk [4, Theorem (2.1)], the approximative k-connectedness of X does not depend on imbeddings of X into Q and it is a shape invariant. In [8] we proved the following theorem.

Theorem 4. *Let* (X, A) *and* (Y, B) *be pairs of metric spaces and subsets and let* $f\colon (X, A)\to(Y, B)$ *be a perfect map such that* $f(X-A) = Y-B$ *and* $f(A) = B$. *If* $\dim Y \leqslant n$ *and* $f^{-1}(y)$ *is approximatively k-connected for each* $y \in Y$ *and* $k = 0, 1, \ldots, n$, *then* $\mathrm{Sh}_W(X) \geqslant \mathrm{Sh}_W(Y)$ *and* $\mathrm{Pos}(X, A) \geqslant \mathrm{Pos}(Y, B)$. *In addition, if* $\dim X \leqslant n$, *then* $\mathrm{Sh}_W(X) = \mathrm{Sh}_W(Y)$ *and* $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$.

Here we mean by $\mathrm{Pos}(X, A)$ and $\mathrm{Sh}_W(X)$ the position of (X, A) and the weak shape of X defined by Borsuk, *On positions of sets in spaces*, Fund. Math. 79 (1973), pp. 141–158, respectively. The relation $\mathrm{Pos}(X, A) \geqslant \mathrm{Pos}(Y, B)$ is defined by a similar way to the relation $\mathrm{Sh}_W(X) \geqslant \mathrm{Sh}_W(Y)$. Since $\mathrm{Sh}(X) = \mathrm{Sh}_W(X)$ for every compactum X, we know the following fact.

(3.12) Let f be a map of a compactum X into an n-dimensional compactum Y such that $f^{-1}(y)$ is approximatively k-connected for each $y \in Y$ and $k = 0, 1, \ldots, n$. Then $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ and $\mathrm{Fd}(X) \geqslant \mathrm{Fd}(Y)$. In addition, if $\dim X \leqslant n$, then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and $\mathrm{Fd}(X) = \mathrm{Fd}(Y)$.

By (3.12) we know that if $f\colon X\to Y$ is a map satisfying the condition of (3.12), then f does not raise the fundamental dimension. The following theorem concerns a map which raises the fundamental dimension.

Theorem 5. *Let X be a finite dimensional compactum. Suppose that one of the following conditions holds*: (1) *A is contractible in X*. (2) $\dim A \leqslant n$ *and X is approximatively k-connected for* $k = 0, 1, \ldots, n+1$. *Then the relation* $\mathrm{Fd}(X) \leqslant \mathrm{Fd}(X/A)$ *holds, where X/A is the quotient space obtained from X by contracting A to a point.*

Proof. Let $\hat{A}$ be a cone over A and let $\hat{X}$ be the compactum obtained from the disjoint union of X and $\hat{A}$ by identifying points of X and $\hat{A}$ corresponding to a point of A. We consider X and $\hat{A}$ as subsets of $\hat{X}$. We can prove that X is a fundamental retract of $\hat{X}$ (see [3, § 18]) if (1) or (2) holds. This is obvious if A is contractible in X, because X is a retract of $\hat{X}$. In case (2) is satisfied, we have to construct a fundamental sequence $\underline{h} = \{h_k\}\colon \hat{X}\to X$ such that $\underline{h}\underline{i}_X \simeq \underline{1}_X$, where $\underline{i}_X\colon X\to\hat{X}$ is the fundamental sequence generated by the inclusion map of X into $\hat{X}$ and $\underline{1}_X\colon X\to X$ is the fundamental identity sequence for X. To construct $\underline{h}$ we use the same argument as in the proof of Theorem 4 given in [8].

We consider $\hat{X}$ as a subset of the Hilbert cube Q. Let d be a fixed metric in Q. Since X is approximatively k-connected for $k = 0, \ldots, n+1$, by using the definition of the approximative k-connectedness repeatedly, it is known that there exists a complete system $\{W_k\colon k = 1, 2, \ldots\}$ of open neighborhoods of X in Q satisfying the following conditions for each $k = 1, 2, \ldots$:

(3.13) $$\overline{W}_{k+1}\subset W_k \quad\text{and}\quad d(X, Q-W_k) < 2^{-k}.$$

(3.14) If K is an $(n+2)$-dimensional simplicial complex and f is a partial realization of K into W_{k+1}, then f has an extension $f\colon K\to W_k$. Here by a *partial realization* of K into W we mean a map from a subcomplex of K containing the set of all vertices of K into W.

(3.15) Let K be an $(n+1)$-dimensional simplicial complex and L a subcomplex of K, and let f and g be maps of K into W_{k+1}. If there is a homotopy $H\colon L\times I\to W_{k+1}$ connecting $f|L$ and $g|L$, then there is an extension $h\colon K\times I\to W_k$ of H connecting f and g.

Since $\dim\hat{A}\leqslant n+1$, we can find finite open collections $\mathfrak{V}_k$, $k=1,2,\ldots$, in Q satisfying the following conditions:

(3.16) (i) $\hat{X}\subset\bigcup\{V\colon V\in\mathfrak{V}_k\}$ and $V\cap X\neq\emptyset$ for each $V\in\mathfrak{V}_k$.
(ii) $\mathfrak{V}_k$ and $\mathfrak{V}_k\cap\hat{X}=\{V\cap\hat{X}\colon V\in\mathfrak{V}_k\}$ are similar.
(iii) $\mathfrak{V}_{k+1}$ is a refinement of $\mathfrak{V}_k$.
(iv) $\operatorname{mesh}\mathfrak{V}_k<\frac{1}{2}d(X,Q-W_k)$.
(v) The order of $\mathfrak{V}_k\cap\hat{A}\leqslant n+2$.

Let $\{U_k\}$ be a sequence of open neighborhoods of $\hat{X}$ in Q such that $\overline{U}_k\subset\bigcup\{V\colon V\in\mathfrak{V}_k\}$ and $\overline{U}_{k+1}\subset U_k$ for $k=1,2,\ldots$ Denote by M_k the nerve of $\mathfrak{V}_k\cap\overline{U}_k$. Note that M_k is also the nerve of $\mathfrak{V}_k$ by (3.16) (ii). We define a map $\psi_k\colon M_k\to W_{k-1}$ as follows. Let N_k be the subcomplex of M_k spanned by vertices which correspond to elements $V\in\mathfrak{V}_k$ such that $V\cap X\neq\emptyset$. For each vertex v of N_k, choose a point x_v of $V\cap X$, where V is the element of $\mathfrak{V}_k$ corresponding to v. Define $f\colon N_k^0\to X$ $(\subset W_k)$ by $f(v)=x_v$, where N_k^0 is the 0-skeleton of N_k. Extend f linearly to a map $g\colon N_k\to Q$. Then by (3.16) (iv) we know that $g(N_k)\subset W_k$. For each vertex v of M_k-N_k, choose an arbitrary point x_v of X and define $h\colon N_k\cup M_k^0\to W_k$ by $h|N_k=g$ and $h(v)=x_v$ for a vertex v of M_k-N_k, where M_k^0 is the 0-skeleton of M_k. By (3.14) extend h to a map $\psi_k\colon M_k\to W_{k-1}$. (We see by (3.16) (v) $\dim(M_k-N_k)\leqslant n+1$.) This completes the construction of ψ_k, $k=2,3,\ldots$

Let $\varphi_k\colon\overline{U}_k\to M_k$ be a canonical map and define $h_k\colon\overline{U}_k\to W_{k-1}$ by $h_k=\psi_k\varphi_k$, $k\geqslant 2$. From the construction of ψ_k, it is easy to know the following.

(3.17) If B_k is a neighborhood of X in Q such that $B_k\subset\overline{U}_k\cap(\bigcup\{V\colon V\in\mathfrak{V}_k,\ V\cap X\neq\emptyset\})$, then $d\big(h_k(x),x\big)<d(X,Q-W_k)$ for each $x\in B_k$.

Moreover the following assertion holds.

(3.18) For each $k\geqslant 3$, $h_{k-1}|\overline{U}_k\simeq h_k$ in W_{k-2}, that is, there is a homotopy $H_k\colon\overline{U}_k\times I\to W_{k-2}$ connecting $h_{k-1}|\overline{U}_k$ and h_k.

To prove (3.18), let $\pi\colon M_k\to M_{k-1}$ be a projection (cf. (3.16) (iii)). Since for each $x\in\overline{U}_k$ both the points $\varphi_{k-1}(x)$ and $\pi\varphi_k(x)$ belong to some closed simplex of M_{k-1}, we know that $h_{k-1}|\overline{U}_k\simeq\psi_{k-1}\pi\varphi_k$ in W_{k-1}. Consider the maps $\psi_{k-1}\pi$ and ψ_k of M_k to W_{k-1}. For each point $x\in N_k$, we know by (3.16) (iv) that $d\big(\psi_{k-1}\pi(x),\psi_k(x)\big)<d(X,Q-W_{k-1})$. Hence there is a homotopy $H\colon N_k\times I\to W_{k-1}$ connecting $\psi_{k-1}\pi|N_k$ and $\psi_k|N_k$. Moreover, note that for each closed simplex σ of M_k the set $\psi_{k-1}\pi(\sigma)\cup\psi_k(\sigma)$ is contained in W_{k-1}. Hence, by (3.15), there is an extension $H'\colon M_k\times I\to W_{k-2}$ of H connecting $\psi_{k-1}\pi$ and ψ_k. Therefore we know that there is a homotopy H_k satisfying (3.18). Finally, for each $k=3,4,\ldots$, consider the maps $h_{k-1}\colon\overline{U}_{k-1}\to W_{k-2}$, $h_k\colon\overline{U}_k\to W_{k-2}$ and the homotopy $H_k\colon\overline{U}_k\times I\to W_{k-2}$ in (3.18). Since W_{k-2} is an ANR, there is a homotopy $\tilde{H}_k\colon\overline{U}_{k-1}\times\times I\to W_{k-2}$ such that $\tilde{H}_k(x,0)=h_{k-1}(x)$ for $x\in\overline{U}_{k-1}$, $\tilde{H}_k|\overline{U}_k\times I=H_k$ and $\tilde{H}_k(x,t)=h_{k-1}(x)$ for $(x,t)\in(\overline{U}_{k-1}-U_{k-1})\times I$. Define $h_k^{k-1}\colon\overline{U}_{k-1}\to W_{k-2}$ by $h_k^{k-1}(x)=\tilde{H}_k(x,1)$ for $x\in\overline{U}_{k-1}$. Then h_k^{k-1} is an extension of h_k and $h_k^{k-1}|\overline{U}_{k-1}-U_{k-1}=h_{k-1}|\overline{U}_{k-1}-U_{k-1}$. Let us define $\bar{h}_k\colon\overline{U}_2\to W_1$, $k=3,4,\ldots$, by $\bar{h}_k(x)=h_{i+1}^i(x)$ for $x\in\overline{U}_i-U_{i+1}$ and $i=2,3,\ldots,k-1$. For each $k\geqslant 3$, extend $\bar{h}_k$ to a map of Q into Q and denote it by $\bar{h}_k$ again. From the construction of h_k, we know that for each i, $3\leqslant i\leqslant k$,

$$\bar{h}_i|\overline{U}_k\simeq\bar{h}_k|\overline{U}_k\quad\text{in } W_{i-1}. \tag{3.19}$$

Consider a sequence $\underline{h}=\{\bar{h}_k\colon k=3,4,\ldots\}$ of maps of Q into Q. From (3.19) it follows that $\underline{h}$ is a fundamental sequence of X into X. Let us prove that $\underline{h}\underline{i}_X\simeq\underline{1}_X$. Let G be a neighborhood of X in Q. Since $\{W_k\}$ is a complete system of neighborhoods of X in Q, there is a $k\geqslant 3$ such that $W_k\subset G$. Let B_k be a neighborhood of X contained in $\overline{U}_k\cap\cap(\bigcup\{V\colon V\in V_k,\ V\cap X\neq\emptyset\})$. From (3.17) it follows that $h_k|B_k\simeq 1_Q|B_k$ in W_k. This shows that $\underline{h}\underline{i}_X\simeq\underline{1}_X$. Thus we proved that X is a fundamental retract of $\hat{X}$ and hence $\operatorname{Sh}(\hat{X})\geqslant\operatorname{Sh}(X)$ if the condition (1) or (2) in the theorem holds.

To complete the proof, consider the quotient spaces X/A and $\hat{X}/\hat{A}$. Since $\hat{A}$ is contractible, $\hat{A}$ is approximatively k-connected for every $k=0,1,\ldots$, by [4, (2.7)]. Since $\hat{X}$ is finite dimensional, (3.12) implies $\operatorname{Sh}(\hat{X})=\operatorname{Sh}(\hat{X}/\hat{A})$. Since $\hat{X}/\hat{A}$ and X/A are homeomorphic, it holds that $\operatorname{Sh}(X/A)=\operatorname{Sh}(\hat{X}/\hat{A})=\operatorname{Sh}(\hat{X})\geqslant\operatorname{Sh}(X)$. Hence $\operatorname{Fd}(X/A)\geqslant\operatorname{Fd}(X)$. This completes the proof.

In Theorem 5 the equality does not hold generally. If X is a finite dimensional and contractible compactum and A is a closed subset of X such that $\operatorname{Fd}(A)=n$ and $\check{H}^n(A\colon G)\neq 0$ for some abelian group G, then $\operatorname{Fd}(X/A)>\operatorname{Fd}(X)+n=n$, because $\check{H}^{n+1}(\hat{X}\colon G)\neq 0$ and hence $\operatorname{Fd}(X/A)=\operatorname{Fd}(\hat{X})>n$, where $\hat{X}$ is the compactum obtained from X and a cone over A in the proof of Theorem 5.

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Kurepa's hypothesis and the continuum

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Abstract. Silver [5] proved that Con(ZFC + "there is an inaccessible cardinal") implies Con(ZFC+ CH+ "there are no Kurepa trees"). In order to obtain this result, he generically collapses an inaccessible cardinal to ω_2. Hence CH necessarily holds in his final model. In this paper we sketch Silver's proof, and then show how it can be modified to obtain a model in which there are no Kurepa trees and the continuum is anything we wish.

Introduction. We work in ZFC and use the usual notation and conventions. For details concerning the forcing theory we require, see Jech [3] or Shoenfield [4]. A *tree* is a poset $\underset{\sim}{T} = \langle T, \leqslant_T \rangle$ such that $\hat{x} = \{y \in T|\ y <_T x\}$ is well-ordered by $<_T$ for any $x \in T$. The order-type of $\hat{x}$ is the *height* of x in $\underset{\sim}{T}$, $ht(x)$. The αth *level* of $\underset{\sim}{T}$ is the set $T_\alpha = \{x \in T|\ ht(x) = \alpha\}$. $\underset{\sim}{T}$ is an ω_1*-tree* iff:

(i) $(\forall \alpha < \omega_1)(T_\alpha \neq \emptyset)\ \&\ (T_{\omega_1} = \emptyset)$;
(ii) $(\forall \alpha < \beta < \omega_1)(\forall x \in T_\alpha)(\exists y_1, y_2 \in T_\beta)(x <_T y_1, y_2\ \&\ y_1 \neq y_2)$;
(iii) $(\forall \alpha < \omega_1)(\forall x, y \in T_\alpha)(\lim(\alpha) \to [x = y \leftrightarrow \hat{x} = \hat{y}])$;
(iv) $(\forall \alpha < \omega_1)(|T_\alpha| \leqslant \omega)\ \&\ |T_0| = 1$.

For further details of ω_1-trees, see Jech [2].

If $\underset{\sim}{T}$ is an ω_1-tree, a *branch* of $\underset{\sim}{T}$ is a maximal totally ordered subset of $\underset{\sim}{T}$. A branch b of $\underset{\sim}{T}$ is *cofinal* if $(\forall \alpha < \omega_1)(T_\alpha \cap b \neq \emptyset)$. $\underset{\sim}{T}$ is *Kurepa* if it has at least ω_2 cofinal branches. If $V = L$, then there is a Kurepa tree. This result is due to Solovay. For a proof, see Devlin [1] or Jech [2]. More generally, if $V = L[A]$, where $A \subseteq \omega_1$, then there is a Kurepa tree, from which it follows that if there are no Kurepa trees, then ω_2 is inaccessible in L. (All of this is still due to Solovay, and is proved in [1] and [2].). Hence, in order to establish Con(ZFC+K), where K denotes the statement "there are no Kurepa trees", one must at least assume Con(ZFC+I), where I denotes the statement "there is an inaccessible cardinal".

Now, if M is any cardinal absolute extension of L, and if $\underset{\sim}{T}$ is a Kurepa tree in L, then $\underset{\sim}{T}$ will clearly be a Kurepa tree in M. Hence, if $\varkappa$ is any cardinal of cofinality greater than ω, we can, by standard arguments, find a generic extension of L, with the same cardinals as L, such that,