Connectivity points and Darboux points of real functions

by

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Abstract. For a bounded real-valued function f with domain an open interval, it is shown that the set of points at which f is connected and the set of points at which f is Darboux are G_δ-sets.

1. Introduction. In [1], Bruckner and Ceder describe what it means for a real function to be Darboux at a point, and later in [2], Garrett, Nelms, and Kellum introduce the idea of a function connected at a point. It is known that the set of points of continuity for a real-valued function with domain an open interval is a G_δ-set. This paper gives a partial answer to a conjecture of Hugh Miller that the set of points at which such a function is connected is also a G_δ-set. A similar result is obtained for the set of points at which a function is Darboux.

2. Preliminaries. For any subset M of the plane $R\times R$, $(M)_X$ denotes the X-projection of M and $(M)_Y$ denotes the Y-projection. For any subset K of the X-axis, M_K denotes the set of points of M which have X-projection in K. The vertical line through a point $(z, 0)$ is denoted by $l(z)$. All functions in this paper are real-valued with domain an open interval. No distinction is made between a function and its graph. A function f is said to be *connected from the left* (*right*) *at a point* z of its domain if whenever (z, a) and (z, b) are two limit points of f from the left (right), then the continuum M contains a point of f whenever $(M)_X$ is a non-degenerate set with right (left) end point z and M_z is a subset of the vertical open interval with end points (z, a) and (z, b). The function f is *connected at a point* z if $(z, f(z))$ is a limit point of f from the left and right and f is connected from both the left and the right at z. If each such M is a horizontal interval instead, then one obtains the definitions of *Darboux from the left* (*right*) *at a point* and *Darboux at a point*.

We first need a result which we apply later to vertical closed intervals which meet the closure, $\bar{f}$, of a function f. These vertical intervals may be bounded or unbounded subsets of the plane.

LEMMA 1. *Let A be an uncountable subset of real numbers, and let $C = \{L(a): a \in A\}$ be a collection of homeomorphic vertical closed intervals such that each $(L(a))_X = a$. Then there is a member $L(a_0)$ of C that is the limit from one side of a sequence of members of $C - \{L(a_0)\}$ and that is contained in the limit from the other side of a sequence of members of $C - \{L(a_0)\}$.*

Proof. If each member of C is a vertical line, the result immediately follows from the fact that there is a point a_0 of A that is a limit point of A from both the left and the right. We give the proof for the case when each member of C is a closed and bounded interval. If each member of C were a closed ray, the proof would be similar.

It is known that there is an uncountable subcollection C' of C with the property that each member $L(a)$ of C' is the limit of a sequence of members of $C - \{L(a)\}$ from one side, say from the right. This follows from the fact that the plane is separable. For each positive integer n and for each $L(a)$ in C, let $R(a, n)$ denote the rectangle $[a-1/n, a] \times \times (L(a))_Y$. For each n, define C_n to be the collection of those members $L(a)$ of C' with the property that if $L(a')$ is in $C - \{L(a)\}$ and $L(a')$ meets $R(a, n)$ then diameter $L(a)$ − diameter $L(a') \cap R(a, n) > 1/n$.

Case 1. $\bigcup_{n=1}^{\infty} C_n$ is countable.

The collection B of those $L(a)$ which fail to be in $\bigcup_{n=1}^{\infty} C_n$ for the reason that no $L(a')$ in $C - \{L(a)\}$ meets some $R(a, m)$ is countable. Then there is a member $L(a_0)$ of $(C' - \bigcup_{n=1}^{\infty} C_n) - B$. Therefore for each n, there is some $L(a_n)$ in $C - \{L(a_0)\}$ such that $L(a_n)$ meets $R(a_0, n)$ but diameter $L(a_0)$ − diameter $L(a_n) \cap R(a_0, n) \leqslant 1/n$. It now follows that $L(a_0)$ is contained in the limit from the left of some subsequence of the sequence $\{L(a_n)\}$.

Case 2. $\bigcup_{n=1}^{\infty} C_n$ is uncountable.

Then for some positive integer m, C_m is uncountable. Therefore some member $L(a_0)$ of C_m is the limit of a sequence $\{L(a_n)\}$ of members of $C_m - \{L(a_0)\}$ from either the left or the right. If convergence is from the right, then we can choose an integer k so large that $a_k - a_0 < 1/m$, $L(a_0)$ meets $R(a_k, m)$, and diameter $L(a_k)$ − diameter $L(a_0) \cap R(a_k, m) < 1/m$. But this says that $L(a_k)$ is not in C_m, a contradiction. Therefore convergence must be from the left after all.

But then we can choose an integer k so large that $a_0 - a_k < 1/m$, $L(a_k)$ meets $R(a_0, m)$, and diameter $L(a_0)$ − diameter $L(a_k) \cap R(a_0, m) < 1/m$. This says that $L(a_0)$ is not in C_m, a contradiction. Therefore case 2 cannot occur. This finishes the proof of the lemma.

3. The main results.

THEOREM 1. *If f is a bounded real-valued function with domain an open interval (u, v), then the set of points at which f is connected is a G_δ-set.*

Proof. Let C_{LR} denote the set of points at which f is connected, C_L the set of points at which f is connected just from the left, and C_R the set of points at which f is connected just from the right. Let x be a point in C_{LR}. Then $\bar{f} \cap l(x)$ is connected because $(x, f(x))$ is a limit point of f from both the left and the right. For each positive integer n, there is an open interval $O(x, n)$ containing x and having diameter less than $1/n$ such that for each z in $O(x, n)$, $(\bar{f} \cap l(z))_Y$ is a subset of the $1/n$-neighborhood of $(\bar{f} \cap l(x))_Y$. Define $O_n = \bigcup \{O(x, n): x \in C_{LR}\}$. Clearly $C_{LR} \subset \bigcap_{n=1}^{\infty} O_n$.

To prove the theorem we need only show (1) $\bigcap_{n=1}^{\infty} O_n \subset C_{LR} \cup C_L \cup C_R$ and (2) C_L and C_R are each countable. For, then it would follow that C_{LR} is a G_δ-set because $C_{LR} = \bigcap_{n=1}^{\infty} O_n - (C_L^* \cup C_R^*)$ where $C_L^* \subset C_L$ and $C_R^* \subset C_R$.

Proof of (1). Let z be a point in $\bigcap_{n=1}^{\infty} O_n$, and we may as well suppose z is not in C_{LR}. Therefore $\bar{f} \cap l(z)$ is non-degenerate. For each n, there is an x_n in C_{LR} such that z is in $O(x_n, n)$. Since the diameter of $O(x_n, n)$ is less than $1/n$, the sequence $\{x_n\}$ converges to z. We may assume without loss of generality that $\{x_n\}$ converges to z from the left. Since z is in $O(x_n, n)$, $(\bar{f} \cap l(z))_Y$ is a subset of the $1/n$-neighborhood of $(\bar{f} \cap l(x_n))_Y$ for each n. All but finitely many sets $\bar{f} \cap l(x_n)$ are non-degenerate. Otherwise, if infinitely many were degenerate, then there would be an integer m such that $\bar{f} \cap l(x_m)$ is degenerate and the diameter of $\bar{f} \cap l(z)$ is greater than the diameter, $2/m$, of the $1/m$-neighborhood of $\bar{f} \cap l(x_m)$. This would imply z is not in $O(x_m, m)$, a contradiction. We may as well suppose each $\bar{f} \cap l(x_n)$ is non-degenerate.

We show next that the sequence $\{\bar{f} \cap l(x_n)\}$ of intervals converges to $\bar{f} \cap l(z)$. Let P and Q be two points in $\bar{f} \cap l(z)$, let (z, w) be a point between P and Q, and let C_1, C_2, and C_3 be disjoint open spheres centered at P, Q, and (z, w) respectively. C_1 and C_2 must eventually meet each $\bar{f} \cap l(x_n)$; otherwise, an argument similar to the one in the preceding paragraph would result in a similar contradiction. Therefore C_3 eventually meets each $\bar{f} \cap l(x_n)$. Consequently, $\{\bar{f} \cap l(x_n)\}$ converges to $\bar{f} \cap l(z)$, and $\bar{f} \cap l(z)$ is connected.

We now show that z is in C_L. Let (z, a) and (z, b) be two limit points of f from the left, and let M be a continuum such that $(M)_X$ is a non-degenerate set with right end point z and M_z is a subset of the vertical open interval with end points (z, a) and (z, b). Assume f misses M. Let C_1 and C_2 be disjoint open spheres missing M and centered at (z, a) and (z, b) respectively. There is an integer m such that $l(x_m)$ separates two points of M and such that $\bar{f} \cap l(x_m)$ meets both C_1 and C_2. Since M separates a point (x_m, s) of $\bar{f} \cap C_1$ from a point (x_m, t) of $\bar{f} \cap C_2$ in $(M)_X \times R$, then M separates $(x_m, f(x_m))$ from either (x_m, s) or (x_m, t) in $(M)_X \times R$. We may assume that M separates $(x_m, f(x_m))$ from (x_m, s) in $(M)_X \times R$ and that (x_m, s) is a limit point of f from the right. Let C_3 and C_4 be disjoint open spheres in $(M)_X \times R$ with radius r, centered at $(x_m, f(x_m))$ and (x_m, s) respectively, and missing M. Denote by S the subset of the plane such that (x, y) is in S if and only if $x_m \leqslant x \leqslant x_m + r$ and (x, y) lies between two points (x, r_3) and (x, r_4) belonging to C_3 and C_4 respectively. Since $M \cap S$ separates $(x_m, f(x_m))$ from (x_m, s) in S, it follows from a lemma of Roberts [3], p. 176, that there is a subcontinuum N of M in S such that N separates $(x_m, f(x_m))$ from (x_m, s) in S. $(N)_X$ is a non-degenerate set with left end point x_m, and N_{x_m} is a subset of the vertical open interval with end points $(x_m, f(x_m))$ and (x_m, s). Since x_m is in C_{LR}, N meets f, a contradiction. Therefore M must meet f, and so z is in C_L.

Proof of (2). Assume, on the contrary, that C_L is uncountable. First we show that the set A of those points a in C_L for which $\bar{f} \cap l(a)$ is disconnected is countable. Assume A is uncountable. For each a in A, let $L(a)$ be a vertical closed interval with end points $P(a)$ and $Q(a)$ belonging to different components of $\bar{f} \cap l(a)$ with $P(a)$ lying above $Q(a)$ (written $P(a) > Q(a)$). The collection C of all these $L(a)$ is uncountable. By Lemma 1, there is an a_0 in A such that $L(a_0)$ is contained in the limit from left of a sequence $\{L(a_n)\}$ of members of $C - \{L(a_0)\}$. There are subsequences $\{P(a_{ni})\}$ and $\{Q(a_{ni})\}$ of the sequences $\{P(a_n)\}$ and $\{Q(a_n)\}$ such that $\{P(a_{ni})\}$ converges to a point $P \geqslant P(a_0)$ and $\{Q(a_{ni})\}$ converges to a point $Q \leqslant Q(a_0)$. P and Q are limit points of f from the left and therefore have to lie in the same connected subset $\overline{f_{(u,a_0)}} \cap l(a_0)$ of $\bar{f} \cap l(a_0)$. But since $P \geqslant P(a_0) > Q(a_0) \geqslant Q$, $P(a_0)$ and $Q(a_0)$ lie in the same component of $\bar{f} \cap l(a_0)$, a contradiction. Therefore A is countable.

$C_L - A$ is then uncountable. For each a in $C_L - A$, let $L(a) = \bar{f} \cap l(a)$. Let C be an uncountable collection of these $L(a)$ such that each two members of C are homeomorphic. By Lemma 1, there is a member $L(a_0)$ of C that is the limit from one side of a sequence of members of $C - \{L(a_0)\}$ and that is contained in the limit from the other side of a sequence of members of $C - \{L(a_0)\}$. In fact, this latter one-sided limit actually equals $L(a_0)$ because $L(a_0) = \bar{f} \cap l(a_0)$. This shows $\overline{f_{(u,a_0)}} \cap l(a_0) = \overline{f_{(a_0,v)}} \cap l(a_0) = \bar{f} \cap l(a_0)$. The set B of all such a_0 is uncountable. Therefore there is an a' in B such that some sequence $\{L(a_n)\}$ converges to $L(a')$ from the right, where each a_n is in $B - \{a'\}$. It follows from the proof of (1) that a' is in C_R, a contradiction to a' belonging to C_L. Therefore C_L must be countable. Similarly, C_R is countable.

The proof of the following theorem is similar to the proof of Theorem 1 and is therefore omitted.

THEOREM 2. *If f is a bounded real-valued function with domain an open interval, then the set of points at which f is Darboux is a G_δ-set.*

Since the set of rational numbers is not a G_δ-set, we obtain the following result.

COROLLARY 1. *There is no bounded function $f\colon R \to R$ that is connected at just the rationals, and there is no bounded function $g\colon R \to R$ that is Darboux at just the rationals.*

References

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