Connectivity points and Darboux points
of real functions

by

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Abstract. For a bounded real-valued function $f$ with domain an open interval, it is shown that the set of points at which $f$ is connected and the set of points at which $f$ is Darboux are $G_{\delta}$-sets.

1. Introduction. In [1], Bruckner and Ceder describe what it means for a real function to be Darboux at a point, and later in [2], Garrett, Nebus, and Kellum introduce the idea of a function connected at a point. It is known that the set of points of continuity for a real-valued function with domain an open interval is a $G_{\delta}$-set. This paper gives a partial answer to a conjecture of Hugh Miller that the set of points at which such a function is connected is also a $G_{\delta}$-set. A similar result is obtained for the set of points at which a function is Darboux.

2. Preliminaries. For any subset $M$ of the plane $\mathbb{R} \times \mathbb{R}$, $(M)_x$ denotes the $X$-projection of $M$ and $(M)_y$ denotes the $Y$-projection. For any subset $K$ of the $X$-axis, $M_K$ denotes the set of points of $M$ which have $X$-projection in $K$. The vertical line through a point $(x, 0)$ is denoted by $l(x)$. All functions in this paper are real-valued with domain an open interval. No distinction is made between a function and its graph.

A function $f$ is said to be connected from the left (right) at a point $z$ of its domain if whenever $(z, a)$ and $(z, b)$ are two limit points of $f$ from the left (right), then the continuum $M$ contains a point of $f$ whenever $(M)_x$ is a non-degenerate set with right (left) end point $z$ and $M_x$ is a subset of the vertical open interval with end points $(z, a)$ and $(z, b)$. The function $f$ is connected at a point $z$ if $(z, f(z))$ is a limit point of $f$ from the left and right and $f$ is connected from both the left and the right at $z$. If each such $M$ is a horizontal interval instead, then one obtains the definitions of Darboux from the left (right) at a point and Darboux at a point.

We first need a result which we apply later to vertical closed intervals which meet the closure, $f$, of a function $f$. These vertical intervals may be bounded or unbounded subsets of the plane.
LEMMA 1. Let \( A \) be an uncountable subset of real numbers, and let \( C = \{ L(a) : a \in A \} \) be a collection of homeomorphic vertical closed intervals such that each \( L(a) \cap x = x \). Then there is a member \( L(a_0) \) of \( C \) that is the limit from one side of a sequence of members of \( C - \{ L(a_0) \} \) and that is contained in the limit from the other side of a sequence of members of \( C - \{ L(a_0) \} \).

Proof. If each member of \( C \) is a vertical line, the result immediately follows from the fact that there is a point \( a_0 \) of \( A \) that is a limit point of \( A \) from both the left and the right. We give the proof for the case when each member of \( C \) is a closed and bounded interval. If each member of \( C \) were a closed ray, the proof would be similar.

It is known that there is an uncountable subcollection \( C' \) of \( C \) with the property that each member \( L(a) \) of \( C' \) is the limit of a sequence of members of \( C - \{ L(a) \} \) from one side, say from the right. This follows from the fact that the plane is separable. For each positive integer \( n \) and for each \( L(a) \) in \( C \), let \( R(a, n) \) denote the rectangle \( [a - 1/n, a] \times [L(a)] \). For each \( n \), define \( C_n \) to be the collection of those members \( L(a) \) of \( C' \) with the property that if \( L(a') \) is in \( C - \{ L(a) \} \) and \( L(a') \) meets \( R(a, n) \), then diameter \( L(a') - \text{diameter} \ L(a) < 1/n \).

Case 1. \( \bigcup_{n=1}^{\infty} C_n \) is countable.

The collection \( B \) of those \( L(a) \) which fail to be in \( \bigcup_{n=1}^{\infty} C_n \) for the reason that no \( L(a') \) in \( C - \{ L(a) \} \) meets some \( R(a, m) \) is countable. Then there is a member \( L(a_0) \) of \( C - \{ \bigcup_{n=1}^{\infty} C_n \} - B \). Therefore for each \( n \), there is some \( L(a_n) \) in \( C - \{ L(a_0) \} \) such that \( L(a_n) \) meets \( R(a_n, n) \) but diameter \( L(a_n) - \text{diameter} \ L(a_n) \cap R(a_n, n) < 1/n \). It now follows that \( L(a_n) \) is contained in the limit from the left of some subsequence of the sequence \( \{ L(a_n) \} \).

Case 2. \( \bigcup_{n=1}^{\infty} C_n \) is uncountable.

Then there is some positive integer \( m \), \( C_m \) is uncountable. Therefore some member \( L(a_m) \) of \( C_m \) is the limit of a sequence \( \{ L(a_{n_k}) \} \) of members of \( C_m - \{ L(a_0) \} \) from either the left or the right. If convergence is from the right, then we can choose an integer \( k \) so large that \( a_k - a_{n_k} < 1/m \), \( L(a_k) \) meets \( R(a_k, m) \), and diameter \( L(a_k) - \text{diameter} \ L(a_k) \cap R(a_k, m) < 1/m \). But this says that \( L(a_k) \) is not in \( C_m \), a contradiction. Therefore convergence must be from the left after all.

But then we can choose an integer \( k \) so large that \( a_k - a_{n_k} < 1/m \), \( L(a_k) \) meets \( R(a_k, m) \), and diameter \( L(a_k) - \text{diameter} \ L(a_k) \cap R(a_k, m) < 1/m \). This says that \( L(a_k) \) is not in \( C_m \), a contradiction. Therefore case 3 cannot occur. This finishes the proof of the lemma.

3. The main results.

THEOREM 1. If \( f \) is a bounded real-valued function with domain an open interval \( (u, v) \), then the set of points at which \( f \) is connected is a \( G_2 \)-set.

Proof. Let \( O_{LR} \) denote the set of points at which \( f \) is connected, \( O_L \) the set of points at which \( f \) is connected just from the left, and \( O_R \) the set of points at which \( f \) is connected just from the right. Let \( x \) be a point in \( O_{LR} \). Then \( f \cap I(x) \) is connected because \( (x, f(x)) \) is a limit point of \( f \) from both the left and the right. For each positive integer \( n \), there is an open interval \( O_n \), \( x \in O_n \) containing \( x \) and having diameter less than \( 1/n \) such that for each \( x \in O_n \), \( f \cap I(x) \) is a subset of the \( 1/n \)-neighborhood of \( f \cap I(x) \). Define \( O_n = \bigcup \{ O(x, u) : x \in O_{LR} \} \). Clearly \( O_{LR} = \bigcap_{n=1}^{\infty} O_n \).

To prove the theorem we need only show \( \bigcap_{n=1}^{\infty} O_n \subset O_{LR} \subset O_L \cup O_R \) and \( O_L \) and \( O_R \) are each countable. For, then it would follow that \( O_{LR} \) is a \( G_2 \)-set because \( O_{LR} = \bigcap_{n=1}^{\infty} O_n \subset O_L \cup O_R \) where \( O_L \subset O_{LR} \) and \( O_R \subset O_{LR} \).

Proof of (1). Let \( x \) be a point in \( \bigcap_{n=1}^{\infty} O_n \), and we may as well suppose \( x \) is not in \( O_{LR} \). Therefore \( f \cap I(x) \) is non-degenerate. For each \( n \), there is an \( x_n \) in \( O_{LR} \) such that \( x \in O(x_n, n) \). Since the diameter of \( O(x_n, n) \) is less than \( 1/n \), the sequence \( (x_n) \) converges to \( x \). We may assume without loss of generality that \( (x_n) \) converges to \( x \) from the left. Since \( x \) is in \( O(x_n, n) \), \( f \cap I(x_n) \) is a subset of the \( 1/n \)-neighborhood of \( f \cap I(x_n) \) for each \( n \). All but finitely many sets \( f \cap I(x_n) \) are non-degenerate. Otherwise, if infinitely many were degenerate, then there would be an integer \( m \) such that \( f \cap I(x_m) \) is degenerate and the diameter of \( f \cap I(x_m) \) is greater than the diameter, \( 2/m \), of the \( 1/m \)-neighborhood of \( f \cap I(x_m) \). This would imply \( x \) is not in \( O(x_m, m) \), a contradiction. We may as well suppose each \( f \cap I(x_n) \) is non-degenerate.

We show now that the sequence \( (f \cap I(x_n)) \) of intervals converges to \( f \cap I(x) \). Let \( P \) and \( Q \) be two points in \( f \cap I(x) \), let \( (x, y) \) be a point between \( P \) and \( Q \), and let \( C_1 \), \( C_2 \), and \( C_3 \) be disjoint open spheres centered at \( P \), \( Q \), and \( (x, y) \), respectively. \( C_1 \) and \( C_2 \) must eventually meet each \( f \cap I(x_n) \); otherwise, an argument similar to the one in the preceding paragraph would result in a similar contradiction. Therefore \( C_3 \) eventually meets each \( f \cap I(x_n) \). Consequently, \( f \cap I(x_n) \) converges to \( f \cap I(x) \), and \( f \cap I(x) \) is connected.

We now show that \( x \) is in \( O_L \). Let \( x \) and \( y \) be two limit points of \( f \) from the left, and let \( M \) be a continuum such that \( (M) \) is a non-
degenerate set with right end point $z$ and $M$ is a subset of the vertical open interval with end points $(z, a)$ and $(z, b)$. Assume $f$ misses $M$. Let $C_1$ and $C_2$ be disjoint open spheres missing $M$ and centered at $(z, b)$ and $(z, b)$ respectively. There is an integer $m$ such that $l(z_m)$ separates two points of $M$ and such that $f \cap l(z_m)$ meets both $C_1$ and $C_2$. Since $M$ separates a point $(z_m, e)$ of $f \cap l(z_m)$ from a point $(z_m, f)$ of $f \cap l(z_m)$ in $(M)_x \times R$, then $M$ separates $(z_m, e)$ from either $(z_m, e)$ or $(z_m, f)$ in $(M)_x \times R$. We may assume that $M$ separates $(z_m, f)$ from $(z_m, e)$ in $(M)_x \times R$ and that $(z_m, e)$ is a limit point of $f$ from the right. Let $C_2$ and $C_3$ be disjoint open spheres in $(M)_x \times R$ with radius $r$, centered at $(z_m, f(z_m))$ and $(z_m, e)$ respectively, and missing $M$. Denote by $S$ the subset of the plane such that $(x, y)$ is in $S$ if and only if $z_m < x < z_m + r$ and $(x, y)$ lies between two points $(z, r_1)$ and $(z, r_2)$ belonging to $C_1$ and $C_2$ respectively. Since $M \cap S$ separates $(z_m, f(z_m))$ from $(z_m, e)$ in $S$, it follows from a lemma of Roberts [3], p. 176, that there is a subcontinuum $N$ of $M$ in $S$ such that $N$ separates $(z_m, f(z_m))$ from $(z_m, e)$ in $S$. $(N)_x \times R$ is a non-degenerate set with left end point $z_m$ and $N$ is a subset of the vertical open interval with end points $(z_m, f(z_m))$ and $(z_m, e)$. Since $z_m$ is in $C_1 \cap C_3$, $N$ meets $f$, a contradiction. Therefore $M$ must miss $f$, and so $z$ is in $C_1$.

Proof of (2). Assume, on the contrary, that $C_2$ is uncountable. First we show that the set $A$ of those points $a$ in $C_2$ for which $f \cap l(a)$ is disconnected is countable. Assume $A$ is uncountable. For each $a$ in $A$, let $L(a)$ be a vertical closed interval with end points $P(a)$ and $Q(a)$ belonging to different components of $f \cap l(a)$ with $P(a)$ lying above $Q(a)$ (written $P(a) > Q(a)$). The collection $C$ of all these $L(a)$ is uncountable. By Lemma 1, there is an $a_0$ in $A$ such that $L(a_0)$ is contained in the limit from left of a sequence $(L(a_n))$ of members of $C \cap -L(a_0)$. There are subsequences $(P(a_n))$ and $(Q(a_n))$ of the sequences $(P(a_n))$ and $(Q(a_n))$ such that $(P(a_n))$ converges to a point $P > P(a_0)$ and $(Q(a_n))$ converges to a point $Q < Q(a_0)$. $P$ and $Q$ are limit points of $f$ from the left and therefore have to lie in the same connected subset $l(a_0) \cap l(a_0) \cap l(a_0)$ of $f \cap l(a_0)$. But since $P > P(a_0) > Q(a_0)$, $P(a_0)$ and $Q(a_0)$ lie in the same component of $f \cap l(a_0)$, a contradiction. Therefore $A$ is countable.

$C_2 \setminus A$ is then uncountable. For each $a$ in $C_2 \setminus A$, let $L(a) = f \cap l(a)$. Let $L$ be an uncountable collection of these $L(a)$ such that each two members of $L$ are homeomorphic. By Lemma 1, there is a member $L(a_0)$ of $C$ that is the limit from one side of a sequence of members of $C \cap \{L(a_n)\}$ and that is contained in the limit from the other side of a sequence of members of $C \setminus \{L(a_n)\}$. In fact, this latter one-sided limit actually equals $L(a_0)$ because $L(a_0) = f \cap l(a_0)$. This shows $l(a_0) \cap l(a_0) = l(a_0) \cap l(a_0) = f \cap l(a_0)$. The set $B$ of all such $a_0$ is uncountable. Therefore there is an $a'$ in $B$ such that some sequence $(L(a_n))$ converges to $(L(a'))$ from the right, where each $a_n$ is in $B \setminus \{a\}$. It follows from the proof of (1) that

$a'$ is in $C_2 \setminus A$, a contradiction to $a'$ belonging to $C_2$. Therefore $C_2$ must be countable. Similarly, $C_3$ is countable.

The proof of the following theorem is similar to the proof of Theorem 1 and is therefore omitted.

**Theorem 2.** If $f$ is a bounded real-valued function with domain an open interval, then the set of points at which $f$ is Darboux is a $G_3$-set.

Since the set of rational numbers is not a $G_3$-set, we obtain the following result.

**Corollary 1.** There is no bounded function $f: \mathbb{R} \to \mathbb{R}$ that is connected at just the rationals, and there is no bounded function $g: \mathbb{R} \to \mathbb{R}$ that is Darboux at just the rationals.

**References**


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