[33] E. H. Spanier, *Algebraic Topology*, New York 1966.
[34] A. Trybulec, *On shapes of movable curves*, Bull. Acad. Polon. Sci. 21 (1973), pp. 727–733.
[35] G. T. Whyburn, *Analytic Topology*, Providence 1963.
[36] D. C. Wilson, *Open mappings of the universal curve onto continuous curves*, Trans. Amer. Math. Soc. 168 (1972), pp. 497–515.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Accepté par la Rédaction le 16. 1. 1974

A topological collapse number for all spaces

by

P. H. Doyle* (East Lansing, Mich.)

Abstract. We here assign to each topological X a symbol $C_T(X)$, the collapse number of X, that may be a non-negative integer or ∞. In calculating $C_T(X)$ we permit in a space Y with at least two points two possible operations at each stage. These are a pseudo-isotopy of Y to a compact set K, say, such that K is pointwise fixed throughout the deformation or one may remove a point from Y prior to performing the pseudo-isotopy. If X is an arbitrary space such that a finite number of such operations terminates in a point X, then X has a finite collapse number, $C_T(X)$, and this number is the least number of nonidentity pseudo-isotopies required to "shrink" X to a point. If X has but one point $C_T(X) = 0$, while all other spaces without a finite $C_T(X)$ are given the collapse number ∞.

It is pointed out here that $C_T(X)$ has been defined in a purely topological manner and that restriction of the above deformations could be made to the PL category $C_{\mathrm{PL}}(X)$ or the various differential ones. Further an analogous notion could be defined using homotopies with weaker restrictions.

THEOREM 1. *$C_T(X)$ is a topological invariant, but $C_T(X)$ is not an invariant of simple homotopy type or homotopy type.*

Proof. That $C_T(X)$ is a topological invariant follows from its definition. The rest comes from the following observation. Both the closed 3-cell I^3 and the dunce cap D [2] are of the same simple homotopy. One has $C_T(I^3) = 1$. Since there is no pseudo-isotopy shrinking D to a point, one must first remove a point of D to begin the collapsing. Regardless of the point p removed, $D-p$ is no longer homologically trivial. Thus $C_T(D) > 1$.

EXAMPLE 1. Let X be the 1-point union of r circles ($r \geqslant 2$) and let Y be the space got by attaching to a closed 2-cell r arcs in such a way that each arc meets the 2-cell at its endpoints in two fixed points of the boundary of the 2-cell, but are otherwise disjoint in pairs. Then X and Y are the same simple homotopy type. Upon removing from Y one of the two boundary fixed points, the resulting space shrinks to a point by pseudo-isotopy and $C_T(Y) = 1$. Now $C_T(X) = r$.

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* Reference [2] could as well be K. Borsuk, *Über das Phänomen der Unzerlegbarkeit in der Polyeden Topologie*, Comm. Math. Helv. 8 (1935), pp. 142—148.

THEOREM 2. *Let M^2 be a closed 2-manifold. Then $C_T(M^2) \leqslant 2$.*

Proof. Upon removing a point from M^2 one can pseudo-isotop the resulting space to a copy of Y in Example 1. Hence $C_T(M^2) \leqslant 2$.

COROLLARY. *The 2-sphere is characterized among closed 2-manifolds by having collapse number* 1.

The corollary to Theorem 2 generalizes in all dimensions.

THEOREM 3. *Let M^n be a closed n-manifold. Then M^n is an n-sphere if and only if $C_T(M^n) = 1$.*

Proof. Certainly $C_T(S^n) = 1$. Hence assume $C_T(M^n) = 1$. This means that $M^n - p$ (for some point p) shrinks by pseudo-isotopy to a point x in $M^n - p$ with x fixed throughout the deformation. The given any compact set C in $M^n - p$ and any $\varepsilon > 0$ there is a stage in the pseudo-isotopy such that C is in the ε-neighborhood of x. If for C one selects the residual set in a standard decomposition it follows that C has a euclidean neighborhood in M^n and M^n is thus a sphere [1].

There is a matter not settled by Theorem 3 and it is the following. Is there an AR in E^3 (topologically embedded) that will serve as a residual set for a counter-example to the Poincaré conjecture. A similar question may be asked in dimension 4.

THEOREM 4. *If K is a finite simplicial complex then if K is connected $0 \leqslant C_T(K) < \infty$.*

Proof. One notes that K can certainly be "shrunk" to a point by a finite number of removals of points from principal simplices of dimension at least two so that eventually a graph or point is obtained. If a graph results that is not a tree, either a point or tree may be got in a finite number of steps. Hence $C_T(K)$ is finite.

THEOREM 5. *If X is a space with a finite number of components, $\{C_i\}$, while at least one component is not a point and has a finite collapse number, while all others have a finite collapse number, then X has a finite collapse number (and conversely).*

Proof. That infinitely many components are impossible is a matter of the definition of $C_T(X)$.

Thus it is sensible to consider only the category of connected spaces with a finite collapse number. There exist continua X with $C_T(X) = 1$ while their euler characteristics are infinite. A simple example is the suspension of the sequence $\{1/n\}$ plus its limit 0. Even for connected spaces path connectedness is not required. An example of this is the graph in E^2 of $f(x) = \sin(1/x)$ $(0 < x < 1)$ plus the point $(0, 0)$. For upon removal of $(0, 0)$ from this space one has an open interval topologically. The original space has two path components. For an arbitrary connected space X with infinitely many path components $C_T(X) = \infty$ for

if $C_T(X)$ were finite our admissible operations would not permit collapsing in a finite number of steps. Thus it appears that the essential questions about spaces X, $C_T(X)$ finite, should be asked within the category of path connected spaces. A special case in this category is the subcategory $\mathcal{S}_1$ of those spaces for which $C_T(X) = 1$.

$\mathcal{S}_1$ splits into two parts, that is, the contractible spaces and those spaces that become contractible upon removal of a single point. Here, of course, the contractibility must be achieved by a pseudo-isotopy. Furthermore, in the general case if $C_T(X) < \infty$, the last step in shrinking takes place in this category.

Thus let $\mathcal{S}_1$ be the category of spaces that contain at least two points and may be shrunk to a point by pseudo-isotopy with or without excision. If X is in $\mathcal{S}_1$ let w be a point not in X and assume Y is some space formed by annexing w to X. Regardless of the neighborhoods of w in Y we note that $Y - w = X$ and $C(Y) < \infty$. There are cases in which the proper annexation of w to X above results in spaces Y that is topologically X. It is well known that this occurs in such a frequently used space as L^2. So it is not difficult in the language of categories to characterize those spaces with $C_T(X) = 1$.

Proceeding inductively, let $\mathcal{S}_2$ be those path connected spaces X for which $C_T(X) = 2$. If X is in $\mathcal{S}_2$, the first pseudo-isotopy yields a space X_1 in $\mathcal{S}_1$ where X_1 must be compact. From the definition of $C_T(X) < \infty$ one notes that the step of going from X to X_1 by allowable moves is typical of the inductive process for $C_T(X) = n \geqslant 2$. If X is in $\mathcal{S}_n$, then one obtains after one pseudo-isotopy an element X_{n-1} in $\mathcal{S}_{n-1}$ that is compact.

References

[1] P. H. Doyle and J. G. Hocking, *A decomposition theorem for n-dimensional manifolds*, Proc. Amer. Math. Soc. 13 (1962), pp. 469–471.

[2] E. C. Zeeman, *On the dunce hat*, Topology 2, 1964, pp. 341–358.

Accepté par la Rédaction le 11. 3. 1974