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Curves which are continuous images of tree-like continua are movable

by

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Abstract. This paper contains several results about continuous images of continua which are contractible with respect to graphs. The main result shows that 1-dimensional continuous images of tree-like continua are movable (in the sense of Borsuk's shape theory). We present certain characterizations of continua with trivial shape. These results extend some facts concerned confluent images of continua that was recently obtained.

1. Introduction. Using the notion of movability belonging to shape theory we obtain in this paper some new results concerning curves. The main result of this paper is stated in the title.

In 1968 K. Borsuk [5] began the development of a new theory which compare compacta, i.e., compact metric spaces, from the point of view of their global topological properties. This theory has come to be known as shape theory. Let us recall some basic notions of this theory. Let X and Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q\to Q$ is said to be a *fundamental sequence* from X to Y (in symbols $\underline{f}=\{f_k, X, Y\}$) if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U$ is homotopic to $f_{k+1}|U$, $f_k|U\simeq f_{k+1}|U$ in V for almost all k. If $X=Y$ and f_k is the identity map $1_Q\colon Q\to Q$ for every positive integer k, then the fundamental sequence $\underline{f}$ is said to be the *fundamental identity sequence* for X, and is denoted by $\underline{1}_X$. The composition $\underline{g}\underline{f}$ of fundamental sequences $\underline{f}$ and $\underline{g}=\{g_k, Y, Z\}$ is the fundamental sequence $\underline{g}\underline{f}=\{g_kf_k, X, Z\}$. Two fundamental sequences $\underline{f}$ and $\underline{g}$ from X to Y are said to be *homotopic*, $\underline{f}\simeq\underline{g}$, if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U\simeq g_k|U$ in V for almost all k. If there exist two fundamental sequences $\underline{f}$ from X to Y and $\underline{g}$ from Y to X such that $\underline{g}\underline{f}\simeq\underline{1}_X$, then we say that Y *fundamentally dominates* X, $X\underset{F}{\leqslant}Y$. If, in addition, we have $\underline{f}\underline{g}\simeq\underline{1}_Y$ then X and Y are said to be *fundamentally equivalent* — notation: $X\underset{F}{\simeq}Y$. It is known that the relation $\underset{F}{\simeq}$ is a true equivalence relation, and the set of all compacta lying ni Q is therefore partitioned into equivalence classes. The equivalence class containing a compactum X is called the *shape* of X and is denoted by ShX. We say that X is of *trivial shape*, briefly Sh$X = 1$, if the class ShX contains a one-point space. If X is an arbitrary compactum (not necessarily lying in Q) then all homeomorphic copies of X in Q lie in the same class with respect to $\underset{F}{\simeq}$, and therefore the shape is well defined for every compactum.

If $\underline{X} = \{X_n, p_n\}$ is an inverse sequence of compacta, then as usual we assume that: p_n is a mapping, i.e., a continuous function, from X_{n+1} into X_n which is called a *bonding map*, p_{nn} denotes the identity map of X_n onto itself, and for $n < m$ the map p_{nm} is the composition $p_{nm} = p_n \circ p_{n+1} \circ \ldots \circ p_{m-1}$, where $n = 1, 2, \ldots$ By the inverse limit of this sequence, denoted by $X = \operatorname{invlim} \underline{X}$, we understand the subset of the cartesian product $\prod X_n$ consisting of points $x = \{x_n\}$ such that for every positive integer n we have $p_n(x_{n+1}) = x_n$. The projection $\pi_n\colon X \to X_n$ is defined by $\pi_n(x) = x_n$. The reader is referred to [15], Chap. VIII, for the properties of inverse limits.

S. Mardešić and J. Segal presented in [27] an alternative approach, based on ANR-systems, to the theory of shapes of compacta. We now recall the basic notions of their theory, which will be used in this paper. For our purposes it suffices to consider only ANR-sequences. By an ANR-sequence is meant an inverse sequence $\{X_n, p_n\}$, where each X_n is a (compact) ANR-set. An ANR-sequence $\underline{X}$ is said to be associated with a compactum X if X is homeomorphic to invlim $\underline{X}$. In such a case we sometimes identify X with invlim $\underline{X}$. A map of ANR-sequences $\underline{f}\colon \underline{X} \to \underline{Y}$, where $\underline{X} = \{X_n, p_n\}$ and $\underline{Y} = \{Y_n, q_n\}$, consists of an increasing function $f\colon N \to N$, N being the set of natural numbers, and a collection of mappings $f_n\colon X_{f(n)} \to Y_n$ such that for $n \leqslant m$ we have

$$f_n p_{f(n)f(m)} \simeq q_{nm} f_m .$$

Such a map we denote by $\underline{f} = (f, f_n)$. Two maps $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$, where $\underline{g} = (g, g_n)$, are homotopic, in symbols $\underline{f} \simeq \underline{g}$, if for every $n \in N$ there is an $m \in N$, $m \geqslant f(n), g(n)$, such that

$$f_n p_{f(n)m} \simeq g_n p_{g(n)m} .$$

If $\underline{f} = (f, f_n)\colon \underline{X} \to \underline{Y}$ and $\underline{g} = (g, g_n)\colon \underline{Y} \to \underline{Z}$, then the composition $\underline{g}\underline{f}\colon \underline{X} \to \underline{Z}$ is defined by $\underline{g}\underline{f} = (fg, g_n f_{g(n)})$. The identity map of a sequence $\underline{X}$ is the map $1_{\underline{X}} = (1_N, 1_{X_n})$. Two compacta X and Y are said to be of the *same shape*, briefly $[X] = [Y]$ (in the sense of ANR-systems), if there exist two ANR-sequences $\underline{X}$ and $\underline{Y}$, associated with X and Y, respectively, and two maps $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq 1_{\underline{X}}$ and $\underline{f}\underline{g} \simeq 1_{\underline{Y}}$. The existence of such maps is independent of the choice of ANR-sequences associated with X and Y (see [27], Corollary 1). The following theorem, which was proved in [28], shows that the two approaches to the theory of shapes of compacta are equivalent. Namely,

THEOREM 1. ShX = Sh$Y \Leftrightarrow [X] = [Y]$.

For the Čech (or Vietoris) homology and for the Čech cohomology groups we have the following results.

THEOREM 2. *If* ShX = ShY, *then* $H_n(X; G) \approx H_n(Y; G)$ (see [5], 11.6), *and* $H^n(X; G) \approx H^n(Y; G)$ (see [27], Theorem 16), *for an arbitrary Abelian group* G.

THEOREM 3. *If Y is a retract of X, then the homology and cohomology groups of Y are isomorphic to certain direct divisors of the corresponding groups of the compactum X* (see [4], p. 42).

For plane continua, i.e., connected compacta, K. Borsuk proved the following theorem ([5], 9.1).

THEOREM 4. *Two continua $X, Y \subset E^2$ decomposing the plane into the same number of components are of the same shape.*

In [8] K. Borsuk observed that some continua lying in Q have very singular neighborhoods in Q. In order to distinguish compacta without this singularity he introduced the notion of movable compacta. A compactum $X \subset Q$ is said to be *movable* if for every neighborhood U of X in Q there exists a neighborhood $U_0 \subset U$ of X which is deformable inside U into any neighborhood of X. It is known that this notion is a shape invariant and that all ANR-sets and all plane compacta are movable [8]. In the quoted paper it is also proved that no solenoid of Van Dantzig [13] is movable. Replacing in the definition of movability the compactum X by the pointed compactum (X, x) and also the neighborhoods U, U_0 by the pointed neighborhoods (U, x) and (U_0, x), we obtain the notion of a pointed movable compactum. In [34] A. Trybulec proved a theorem which says that every movable curve has a plane shape. It seems to me that the proof contains a gap. However, it is valid for the following theorem (compare our proof in section 5).

THEOREM 5. *If X is a pointed movable curve, then there is a plane continuum Y such that* ShX = ShY (¹).

We now recall an alternative description of movability given by S. Mardešić and J. Segal [26]. This definition of movability is equivalent to that of Borsuk and will be used in this paper.

An inverse sequence $\{X_n, p_n\}$ is said to be *movable* provided for each integer $n \geqslant 1$ there exists an integer $n_0 \geqslant n$ such that for every $m \geqslant n$ there exists a mapping $g\colon X_{n_0} \to X_m$ such that $p_{nm} g \simeq p_{nn_0}$. If we replace the compacta by pointed compacta and we assume that the maps and homotopies preserve the given points, then we obtain the notion of a pointed movable sequence.

Following Mardešić and Segal, we say that a compactum X is *pointed*

(¹) Added in proof. A. Trybulec has recently proved that this theorem is true for movable curves (not published yet).

movable if there exists a pointed movable ANR-sequence associated with X. It is known that the (pointed) movability of X is independent of the choice of an ANR-sequence associated with X [26]. Since with every compactum we can associate an ANR-sequence (see [18], p. 183), this definition of movability makes sense for any compactum. In the same paper the authors also proved that the curve which has been constructed by Case and Chamberlin in [9] provides an example of an acyclic nonmovable curve. The Case-Chamberlin curve C possesses many other interesting properties. For instance, being acyclic, each map from C into the unit circle S is nullhomotopic, i.e., homotopic to a constant map, however, there is an essential map of C onto the one-point union of two copies of S. In this paper we shall show that C can not be continuously mapped onto any solenoid.

By a graph we understand a curve which is homeomorphic to a polyhedron. The family of all graphs is denoted by G. An acyclic graph is called a *tree*. We say that a space X is *contractible with respect to a space* Y, briefly: X is cr Y (comp. [22], p. 370), if each mapping f from X into Y is nullhomotopic, notation $f \simeq 0$. If F is a collection of spaces, then we say that X is *contractible with respect to* F if X is contractible with respect to each $Y \in F$. In such a case we write: X is cr$F$. For example, "X is crANR" means that X is contractible with respect to each ANR-set.

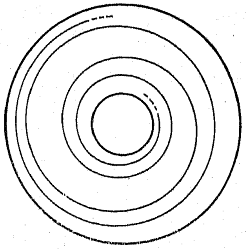
We are now ready to state the main result which will be proved in this paper (for the proof see section 7).

MAIN THEOREM. *If a curve X can be represented as a continuous image of a continuum which is* cr$G$, *then X is pointed movable* (²).

Observe, however, that not all movable curves can be represented as continuous images of continua cr$G$.

EXAMPLE 1. There is a pointed movable curve X such that no continuum cr$S$ can be continuously mapped onto X.

Let X be the curve K described by Fort in [17], p. 542. That is: X is the set of points of the plane having polar coordinates (r, θ), where $r = 1$, $r = 2$ or $r = (2+e^{\theta})/(1+e^{\theta})$.



(²) Added in proof. The collection G can be replaced by the "figure eight".

Since X is a plane continuum, X is pointed movable by a result of Borsuk [8]. Fort has proved that no plane continuum Y which does not separate the plane can be continuously mapped onto X. However the only property of Y which is used in the Fort proof is that Y is cr$S$, which proves our assertion.

The next example shows that the Main Theorem need not be true if we omit the assumption that X is a curve.

EXAMPLE 2. There is a 2-dimensional continuum which is not movable but can be obtained as a continuous image of the unit interval $I = [0, 1]$. Actually, the locally connected continuum constructed by K. Borsuk [7] is such an example, because, by the classical result of Hahn–Mazurkiewicz, every locally connected continuum is a continuous image of I.

2. Certain characterizations of compacta of trivial shape. Given a positive real number $\varepsilon > 0$, a compactum X and a mapping $f\colon X \to Y$ onto Y, f is said to be an ε-mapping provided, for each $y \in Y$, the diameter $\operatorname{diam} f^{-1}(y) < \varepsilon$. A curve X is said to be *tree-like* (*snake-like*, *circle-like*) provided for each $\varepsilon > 0$ there exists an ε-mapping of X onto a tree (onto I, onto S, respectively), see [2].

The following theorem connects some ideas of shape theory with some classical notions of topology.

2.1. (A) *If X is a compactum, then all the conditions* (1)-(6) *are equivalent.*

(1) $\operatorname{Sh} X = 1$.

(2) *If* $X = \operatorname{invlim}\{X_n, p_n\}$, $X_n \in$ ANR, *then for every positive integer n there exists an $m \geqslant n$ such that* $p_{nm} \simeq 0$.

(3) *X is homeomorphic to the limit of an inverse sequence of* AR*-sets.*

(4) *X is the intersection of a decreasing sequence of* AR*-sets.*

(5) *X is* crANR.

(6) *If* $X = \operatorname{invlim}\{X_n, p_n\}$, $X_n \in$ ANR, *then each projection* $\pi_n\colon X \to X_n$ *is nullhomotopic.*

(7) *X is an* FAR*-set* [6].

(B) *If X is a curve, then all the conditions* (1)-(10) *are equivalent.*

(8) *X is* cr$G$.

(9) *X is tree-like.*

(10) *X can be represented as the limit of an inverse sequence of trees with bonding mappings onto.*

(C) *If X is a pointed movable curve, then all the conditions* (1)-(13) *are equivalent.*

(11) *X is* cr$S$.

(12) $H_1(X; G) = 0$ *for some nontrivial Abelian group* G.

(13) $H^1(X; G) = 0$ *for some nontrivial Abelian group* G.

Proof. (1) ⇒ (2). Let P be a one-point space and let $P_n = P$, for every positive integer n. Then $P = \operatorname{invlim}\{P_n, q_n\}$. Let $\underline{X} = \{X_n, p_n\}$ and let $\underline{P} = \{P_n, q_n\}$. By (1) and Theorem 1 there exist two maps $\underline{f} = (f, f_n)$: $\underline{X} \to \underline{P}$ and $\underline{g} = (g, g_n)$: $\underline{P} \to \underline{X}$ such that $\underline{gf} \simeq 1_{\underline{X}}$. Let n be a given positive integer. Since $\underline{gf} = (fg, g_n f_{g(n)})$, it follows that there exists an integer $m \geqslant f \circ g(n)$, n such that

$$g_n \circ f_{g(n)} \circ p_{f\circ g(n)m} \simeq 1_{X_n} \circ p_{nm} = p_{nm}\,.$$

But $f_{g(n)}$: $X_{f\circ g(n)} \to P_{g(n)} = P$ and therefore we see that $p_{nm} \simeq 0$. This finishes the proof.

(2) ⇒ (3). By the quoted result of Freudenthal [18] X can be represented as the limit $X = \operatorname{invlim}\{X_n, p_n\}$, where each X_n is an ANR-set. Using (2) we can construct an increasing sequence of positive integers $n_1 < n_2 < \ldots$ such that $q_k = p_{n_k n_{k+1}}$: $X_{n_{k+1}} \to X_{n_k}$ is nullhomotopic, for $k = 1, 2, \ldots$ It is known that X is homeomorphic to the limit $\operatorname{invlim}\{X_{n_k}, q_k\}$. So, without loss of generality, we can assume that for each n we have $p_n \simeq 0$. Let $\hat{X}_n$ be the cone over X_n, where we regard X_n as the base of this cone. Since $p_n \simeq 0$ we can extend p_n to a mapping $\hat{p}_n$: $\hat{X}_{n+1} \to \hat{X}_n$, where $\hat{p}_n(\hat{X}_{n+1}) \subset X_n$. It is easy to check that X is homeomorphic to $\operatorname{invlim}\{\hat{X}_n, \hat{p}_n\}$. Since $X_n \in$ ANR, it follows that $\hat{X}_n$ is a contractible ANR-set, and therefore an AR-set (see [4], p. 96). The implication follows.

(3) ⇒ (1). This implication can be obtained by combining Theorem 6 of [27] with Theorem 1, because an AR-set is of the same homotopy type as a one-point space.

(3) ⇒ (4). See for instance [20], Theorem 1.4.

(4) ⇒ (5). This follows from [22], p. 373, Theorem 9.

(5) ⇒ (6). The proof is trivial.

In order to show that (6) implies (2) we need the following

LEMMA. *If $\{X_n, p_n\}$ is an inverse sequence of compacta and π_n: $X = \operatorname{invlim}\{X_n, p_n\} \to X_n$ denotes the projection, then there exists a compactum M that is the union of copies of the sets X_n and X (which will be denoted by the same symbols) and the following conditions are fulfilled:*

(i) *for each $n \geqslant 1$ the set $M_n = X \cup \bigcup_{m\geqslant n} X_m$ is a neighborhood of X in M,*

(ii) *if U is a neighborhood of X in M, then U contains X_n for almost all n,*

(iii) *for each positive integer n the map f_n: $M_n \to X_n$ defined by*

$$f_n(x) = \begin{cases} p_{nm}(x) & \text{for} \quad x \in X_m,\ m \geqslant n\,, \\ \pi_n(x) & \text{for} \quad x \in X \end{cases}$$

is continuous.

Proof of Lemma. First of all we can regard $X, X_1, X_2, \ldots$ as disjoint sets (otherwise take the disjoint union of these sets). Let $M' = X \cup \bigcup X_n$ and let B_n be a countable base for open sets of X_n, for $n = 1, 2, \ldots$ The collection of subsets of M' given by

$$B = \bigcup_n \big(B_n \cup \{\pi_n^{-1}(V) \cup \bigcup_{j\geqslant n} p_{nj}^{-1}(V)\colon\ V \in B_n\}\big)$$

constitute a countable base for a topology τ on M'. Now put $M = (M', \tau)$. By a standard calculation we can verify that M is a compactum satisfying all conditions (i)-(iii) (see [18], pp. 153–156).

Remark. A compactum topologically equivalent to M can also be constructed by using the star construction explicitly defined by Overton and Segal in [31], Definition 2, and first mentioned by H. Freudenthal [18], p. 153.

(6) ⇒ (2). Let X be the limit of an inverse ANR-sequence $\{X_n, p_n\}$ and let n be a positive integer. By our hypothesis there is a homotopy F: $X \times I \to X_n$ joining the projection π_n: $X \to X_n$ with a constant map; thus for $x \in X$ we have

$$F(x, 0) = \pi_n(x) \quad \text{and} \quad F(x, 1) = x_0\,.$$

Keeping the notation of the Lemma, let us regard $X \times I$ as a subset of $M_n \times I$. Let G be an extension of F to the set $A = M_n \times \{0\} \cup X \times I \cup M_n \times \{1\}$ defined by the formula

$$G(x, 0) = f_n(x) \quad \text{and} \quad G(x, 1) = x_0 \quad \text{for every } x \in M_n\,.$$

Such an extension exists by (iii). Since $X_n \in$ ANR and A is a closed subset of $M_n \times I$, the map G can be extended to H: $V \to X_n$, where V is a neighborhood of A in $M_n \times I$. By (i), the set V is a neighbourhood of $X \times I$ in $M \times I$. It follows that there exists a neighborhood U of X in M such that $U \times I \subset V$. By (ii) we infer that there is an $m \geqslant n$ such that $X_m \subset U$. Hence $H|X_m \times I$ is a homotopy joining p_{nm} with a constant map, i.e., $p_{nm} \simeq 0$. Thus for each integer $n \geqslant 1$ there exists an $m \geqslant n$ such that $p_{nm} \simeq 0$, which proves the implication.

(4) ⇒ (7). This follows from [6], 9.4, because every AR-set is an FAR-set.

(7) ⇒ (6). Consider X as a subset of the Hilbert space H. Since $X_n \in$ ANR, there is an open neighborhood U of X in H and an extension π_n': $U \to X_n$ of π_n. According to [6], 9.5, X is contractible in U. Let F: $X \times I \to U$ be a contraction of X to a point. Then G: $X \times I \to X_n$, where $G = \pi_n' \cdot F$, defines a homotopy joining π_n with a constant map.

K. Borsuk pointed out to the author that D. M. Hyman [19] has proved some of the above results and other equivalences.

We now pass to part (B). The equivalence (8) $\Leftrightarrow$ (9) has been proved by Case and Chamberlin in [9]. The equivalence (9) $\Leftrightarrow$ (10) follows from [25], p. 147, Lemma 1 and p. 148, Theorem 1. Obviously (10) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (8), which proves part (B).

It remains to prove part (C). Let X be a pointed movable curve. For $n = 1, 2, \ldots$ let $B_n \subset E^2$ be the n-bouquet of circles and let B_0 be a single point. We assume that $B_n \subset B_{n+1} \subset B_\infty$, where B_∞ is a plane continuum consisting of an infinite number of circles, having exactly one point in common, and whose diameters converge to zero. From Theorems 4 and 5 we infer $\operatorname{Sh} X = \operatorname{Sh} B_n$ for some $n = \infty, 0, 1, \ldots$ It follows from a result of Borsuk [3] that condition (11) is equivalent to $H_1(X; R) = 0$, R being the group of rationals. Condition (11) is also equivalent to $H^1(X; Z) = 0$, Z being the group of integers (see [14], p. 226). For each Abelian group G we have $H_1(B_1; G) \approx G \approx H^1(B_1, G)$. Since for $n \geqslant 1$ the set B_1 is a retract of B_n, by combining these results with Theorems 2 and 3 we see that each of the conditions (11), (12) and (13) implies that $n = 0$. This means that $\operatorname{Sh} X = 1$, which is exactly (1). Since the converse implications are obvious, the proof of 2.1 is finished.

Let us note that by the quoted results of Borsuk and Dowker and by Theorems 2 and 3 we obtain the following corollary:

2.2. *Contractibility with respect to S is a shape invariant. Moreover, if Y fundamentally dominates X and Y is* $\operatorname{cr} S$, *then so is X.*

3. Continuous images of tree-like curves. Combining the Main Theorem with 2.1 we obtain the following result:

3.1. *If X is a curve which can be represented as a continuous image of a tree-like curve, then each of the following conditions implies that X is tree-like:*

(i) *X is* $\operatorname{cr} S$,

(ii) $H_1(X, G) = 0$ *for some nontrivial Abelian group G,*

(iii) $H^1(X, G) = 0$ *for some nontrivial Abelian group G.*

In [10] J. J. Charatonik introduced and studied the notion of confluent mappings. Let X and Y be continua. A map $f\colon X \to Y$ onto Y is called confluent if each component of $f^{-1}(C)$ is mapped by f onto C, for every continuum $C \subset Y$. Hence this class contains the class of monotone mappings. Moreover, by a result of G. T. Whyburn [35], p. 148, it follows that each open mapping onto is confluent. In this way the class of confluent mappings comprises all monotone and open mappings onto, and simple examples show that there exist confluent mappings which are neither monotone nor open. It has been proved by A. Lelek [24] that the property $\operatorname{cr} S$ is an invariant of confluent mappings, i.e., if X is $\operatorname{cr} S$ and f is confluent, then $f(X)$ is $\operatorname{cr} S$. Observe that a confluent image of a curve need not be a curve. Actually, R. D. Anderson constructed a monotone and open mapping, hence a fortiori a confluent mapping, of the universal Menger curve onto the Hilbert cube [1] (comp. also [36]). Nevertheless, we find that a confluent image of a $\operatorname{cr} S$ curve is a curve (or a single point) (see [29], p. 472). Combining these results with 3.1 and 2.1, we obtain an alternative proof of the following theorem, which was recently proved by McLean [29].

3.2. *A confluent image of a tree-like curve is a tree-like curve (or a single point).*

The McLean theorem generalizes a result of Rosen [32], who proved that monotone images of tree-like continua are tree-like (comp. also [21]). In [23] A. Lelek defined a class of continua which he calls weakly chainable continua. According to the main result of [23], X is weakly chainable iff the pseudoarc can be continuously mapped onto X (see also [16]). In particular, each locally connected continuum is weakly chainable. Hence the Main Theorem and 3.1 imply:

3.3. *Each weakly chainable curve is movable; if, in addition, it is* $\operatorname{cr} S$, *then it is tree-like.*

3.4. *Each locally connected curve is movable.*

The last result was first proved by S. Nowak. A plane continuum which does not separate the plane has trivial shape (Th. 4). Hence by 2.1 and the Main Theorem we have.

3.5. *A curve which can be represented as a continuous image of a plane continuum which does not separate the plane is pointed movable.*

Since no solenoid of Van Dantzig (see [13] for definition) is movable, the latter result implies one of the principal results of [17]: no plane continuum which does not separate the plane can be continuously mapped onto a solenoid.

4. Inverse limits and covering spaces. For the definitions of the undefined terms used in this section the reader is referred to [33].

4.1. *Let X be a continuum* $\operatorname{cr}\{Y_1, Y_2\}$, *where Y_1 and Y_2 are graphs, and let $p_i\colon \tilde{Y}_i \to Y_i$, $i = 1, 2$, be the universal covering projection. Suppose that commutativity holds in the following diagram, where all maps are continuous.*

$$\begin{tikzcd}
\tilde{Y}_1 \arrow[dd, "p_1"'] & & \tilde{Y}_2 \arrow[dd, "p_2"] \\
& X \arrow[ul, "\tilde{\varphi}_1"'] \arrow[dl, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
Y_1 & & Y_2 \arrow[ll, "g_1"]
\end{tikzcd}$$

Then there exist continuous mappings $\tilde{\varphi}_2\colon X \to \tilde{Y}_2$ *and* $\tilde{g}_1\colon \tilde{Y}_2 \to \tilde{Y}_1$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\tilde{Y}_1 \arrow[dd, "p_1"'] & & \tilde{Y}_2 \arrow[ll, "\tilde{g}_1"'] \arrow[dd, "p_2"] \\
& X \arrow[ul, "\tilde{\varphi}_1"] \arrow[ur, "\tilde{\varphi}_2"'] \arrow[dl, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
Y_1 & & Y_2 \arrow[ll, "g_1"']
\end{tikzcd}$$

Proof. By our assumption we have $\varphi_2 \simeq 0$. Since each constant map can be lifted to $\tilde{Y}_2$ and p_2 is a covering projection, and therefore a fibration ([33], p. 67), φ_2 can be lifted to $\tilde{Y}_2$. Let $\tilde{\varphi}_2$ be a lifting of φ_2. Hence

(1) $$\varphi_2 = p_2 \circ \tilde{\varphi}_2 .$$

Let x_0 be a point of X. Remark that by (1) and by the hypothesis

$$g_1 \circ p_2\big(\tilde{\varphi}_2(x_0)\big) = \varphi_1(x_0) .$$

Consider the following diagram:

$$\begin{tikzcd}
& (\tilde{Y}_1, \tilde{\varphi}_1(x_0)) \arrow[d, "p_1"] \\
(\tilde{Y}_2, \tilde{\varphi}_2(x_0)) \arrow[ur, dashed] \arrow[r, "g_1 \circ p_2"] & (Y_1, \varphi_1(x_0))
\end{tikzcd}$$

We shall show that the dotted arrow in this diagram corresponds to a continuous mapping, which will be denoted by $\tilde{g}_1$, making this diagram commutative. The existence of $\tilde{g}_1$, however, follows from the lifting theorem ([33], p. 76), because $\tilde{Y}_2$ is a connected locally path-connected and simply connected space ([33], p. 83), and p_1 is a fibration with unique path lifting ([33], p. 68, 67). Hence we have a mapping $\tilde{g}_1\colon \tilde{Y}_2 \to \tilde{Y}_1$ and the following conditions are satisfied:

(2) $$g_1 \circ p_2 = p_1 \circ \tilde{g}_1$$

and

(3) $$\tilde{\varphi}_1(x_0) = \tilde{g}_1 \circ \tilde{\varphi}_2(x_0) .$$

By (1) and (2) it remains to show that

(4) $$\tilde{\varphi}_1 = \tilde{g}_1 \circ \tilde{\varphi}_2 .$$

It follows from our assumptions and from (1) and (2) that

$$p_1 \circ \tilde{\varphi}_1 = \varphi_1 = g_1 \circ \varphi_2 = g_1 \circ p_2 \circ \tilde{\varphi}_2 = p_1 \circ \tilde{g}_1 \circ \tilde{\varphi}_2 .$$

Hence $\tilde{\varphi}_1$ and $\tilde{g}_1 \circ \tilde{\varphi}_2$ are two liftings of the same map and by (3) they agree on $x_0 \in X$. Finally, since X is connected, it follows from [33], p. 67, Theorem 2, that the equality (4) is satisfied, which completes the proof.

4.2. *Let Y be a curve represented as the limit of an inverse sequence of graphs $\underline{Y} = \{Y_n, g_n\}$ with bonding mappings onto, and let $\pi_n\colon Y \to \tilde{Y}_n$ be the projection. For each positive integer n let $p_n\colon \tilde{Y}_n \to Y_n$ be the universal projection. Let $f\colon X \to Y$ be a mapping onto, where X is a continuum* cr$\{Y_n\}_{n\geq 1}$. *Then for each integer $n \geq 1$ there exist continuous mappings $\tilde{g}_n\colon \tilde{Y}_{n+1} \to \tilde{Y}_n$, and $\tilde{\varphi}_n\colon X \to \tilde{Y}_n$ such that for $\varphi_n = \pi_n \circ f$ commutativity holds in the following diagram:*

$$\begin{tikzcd}
\tilde{Y}_n \arrow[dd, "p_n"'] & & \tilde{Y}_{n+1} \arrow[ll, "\tilde{g}_n"'] \arrow[dd, "p_{n+1}"] \\
& X \arrow[ul, "\tilde{\varphi}_n"] \arrow[ur, "\tilde{\varphi}_{n+1}"'] \arrow[dl, "\varphi_n"] \arrow[dr, "\varphi_{n+1}"'] & \\
Y_n & & Y_{n+1} \arrow[ll, "g_n"']
\end{tikzcd}$$

Proof. It follows from our assumption that π_n is onto. Since

$$\varphi_n = \pi_n \circ f\colon X \to Y_n ,$$

each φ_n is a mapping onto. By the definition of π_n we have

(1) $$\varphi_n = g_n \circ \varphi_{n+1}.$$

Since $\varphi_1 \simeq 0$, by the same argument as in the proof of 4.1 there exists a continuous mapping $\tilde{\varphi}_1\colon X \to \tilde{Y}_1$ such that

(2) $$\varphi_1 = p_1 \circ \tilde{\varphi}_1 .$$

Now, using (1), (2) and 4.1, we can construct step by step all the mappings with the required properties. This finishes the proof.

4.3. Corollary. *Let Y be a curve represented as the limit of an inverse sequence of graphs $\underline{Y} = \{Y_n, g_n\}$ with bonding maps onto. Let $p_n\colon \tilde{Y}_n \to Y_n$ be a universal covering projection, i.e., p_n is a covering projection and $\tilde{Y}_n$ is simply connected. Then there exist maps $\tilde{g}_n\colon \tilde{Y}_{n+1} \to \tilde{Y}_n$ such that $p_n \circ \tilde{g}_n = g_n \circ p_{n+1}$ for every $n \geq 1$. Suppose that $f\colon X \to Y$ is a continuous map onto Y, where X is a* cr$\{Y_n\}$ *continuum. Then for every $n \geq 1$ there exists a continuum $X_n \subset \tilde{Y}_n$ such that $\tilde{g}_n(X_{n+1}) = X_n$. Hence if the map $k_n\colon X_{n+1} \to X_n$ is defined by the formula $k_n(x) = \tilde{g}_n(x)$, then $\underline{Z} = \{X_n, k_n\}$ is a well-defined inverse sequence. Moreover, if $Z = \operatorname{invlim} \underline{Z}$, then there exist continuous maps $g\colon X \to Z$ onto Z, and $h\colon Z \to Y$ onto Y such that $f = h \circ g$. The map h is induced by the restrictions $h_n = p_n | X_n\colon X_n \to Y_n$, and h_n is onto Y_n.*

Proof. Keeping the notation of 4.2, let us set $X_n = \widetilde{\varphi}_n(X)$. Then the spaces X_n are continua because X is a continuum. The maps $\widetilde{\varphi}_n$ induce a map $g\colon X\to Z$ onto Z. Let $h\colon Z\to Y$ be the map induced by the restrictions $p_n|X_n$. Since the restrictions are onto, h is onto. Since $\varphi_n = \pi_n\circ f$, it is evident that the maps φ_n induce the map f. The equality $f = h\circ g$ easily follows from the constructions of these maps and from 4.2 (for the details see [15], Chap. VIII). This completes the proof.

Remark. If $p\colon \widetilde{Y}\to Y$ is a covering projection and Y is a polyhedron, then $\widetilde{Y}$ is a polyhedron (infinite) of the same dimension as Y, and the projection can be regarded as a simplicial map (see [33], p. 144, Theorem 3). Suppose now that X is a continuum contained in $\widetilde{Y}$ and $\dim\widetilde{Y} = 1$. Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be such triangulations of Y and $\widetilde{Y}$ that the projection is simplicial with respect to $\widetilde{\mathcal{T}}$ and $\mathcal{T}$. It is easy to see that there exist subdivisions $\mathcal{T}'$ of $\mathcal{T}$ and $\widetilde{\mathcal{T}}'$ of $\widetilde{\mathcal{T}}$ such that p is simplicial with respect to $\mathcal{T}'$ and $\widetilde{\mathcal{T}}'$, and X is the space of a finite subcomplex K of $\widetilde{\mathcal{T}}'$, i.e., $X = |K|$. Then $p|X\colon X\to Y$ is a simplicial map with respect to $|K|$ and $\mathcal{T}'$.

Since every curve Y can be represented as the limit of an inverse sequence of one-dimensional connected polyhedra $\underline{Y} = \{Y_n, g_n\}$ with bonding maps onto [25], by combining 4.3 with the above Remark we obtain the following corollaries:

4.4. Corollary. *If a curve Y can be obtained as a continuous image of a* cr$G$ *continuum, then there exist two inverse sequences of pointed polyhedra, $\underline{X} = \{(X_n, x_n), k_n\}$ and $\underline{Y} = \{(Y_n, y_n), g_n\}$, with bonding maps onto such that*:

(1) *X_n is a tree and there exists a finite complex K_n such that $X_n = |K_n|$ and x_n is a vertex of K_n,*

(2) *there exists a finite complex L_n such that $Y_n = |L_n|$ and y_n is a vertex of L_n,*

(3) $Y = \operatorname{invlim}\underline{Y}$,

(4) *for every $n\geqslant 1$ there exists a simplicial map*

$$h_n\colon (X_n, x_n)\to(Y_n, y_n)$$

onto Y_n such that $h_n\circ k_n = g_n h_{n+1}$.

Proof. Let X_n be the spaces considered in 4.3. To prove the above result we need only to note that X_n is a tree. Since the spaces Y_n considered in 4.3 are simply connected one-dimensional polyhedra and X_n is a subpolyhedron of Y_n, X_n is a simply connected compact connected polyhedron, and therefore a tree.

Since the spaces X_n are trees, combining 2.1 with 4.3 we have:

4.5. Corollary. *If f is a map from a* cr$G$ *continuum X onto a curve Y, then there exist a tree-like curve Z and two maps $g\colon X\to Z$ onto Z and $h\colon Z\to Y$ onto Y such that $f = h\circ g$.*

If Y is the unit circle S, then the universal covering space of Y is the real line. Hence the sets X_n in 4.3 are closed intervals (or single points). If Y is a circle-like continuum, then it is the limit of an inverse sequence of circles. Hence by 4.3 (and [23]) we obtain:

4.6. Corollary. *If a circle-like continuum is a continuous image of a* cr$S$ *continuum, then it is a continuous image of a snake-like continuum (the pseudoarc). In particular, it is movable.*

By 4.6 we see that no solenoid is a continuous image of a cr$S$ continuum. The Case-Chamberlin curve C is cr$S$ [9]. Therefore: no solenoid is a continuous image of C.

5. Movable sequences of groups and graphs. If $\underline{A} = \{A_n, p_n\}$ and $\underline{B} = \{B_n, q_n\}$ are inverse sequences of groups and homomorphisms, then by a map $\underline{f} = (f, f_n)\colon \underline{A}\to\underline{B}$ we mean an increasing function $f\colon N\to N$ and a collection of homomorphisms $f_n\colon A_{f(n)}\to B_n$ such that for $n\leqslant m$ we have $f_n\circ p_{f(n)f(m)} = q_{nm}\circ f_m$. The identity map of $\underline{A}$ is the map $\underline{1}_A = (1_N, 1_{A_n})$. Two maps $\underline{f}, \underline{g}\colon \underline{A}\to\underline{B}$, where $\underline{g} = (g, g_n)$, are homotopic, notation: $\underline{f}\approx\underline{g}$, if for every $n\in N$ there exists an $m\in N$ such that $m\geqslant f(n), g(n)$ and $f_n\circ p_{f(n)m} = g_n\circ p_{g(n)m}$. The composition of these maps we define in the same manner as in the case of ANR-sequences. We say that two inverse sequences of group $\underline{A}$ and $\underline{B}$ are equivalent if there exist two maps $\underline{f}\colon \underline{A}\to\underline{B}$ and $\underline{g}\colon \underline{B}\to\underline{A}$ such that $\underline{g}\circ\underline{f}\approx\underline{1}_A$ and $\underline{f}\circ\underline{g}\approx\underline{1}_B$.

One easily shows that

5.1. *The equivalence of sequences is a true equivalence relation.*

If $n_1 < n_2 < \ldots$ is an increasing sequence of positive integers, then $\{A_{n_k}, p_{n_k n_{k+1}}\}$ is called a *subsequence of the sequence* $\underline{A}$. The proof of the following lemma is straightforward.

5.2. *A subsequence of a given sequence $\underline{A}$ is equivalent to $\underline{A}$.*

A sequence $\{A_n, p_n\}$ is said to be *movable* if for every positive integer n there exists an $n_0\geqslant n$ such that for every $m\geqslant n$ there exists a homomorphism $p\colon A_{n_0}\to A_m$ such that $p_{nm}\circ p = p_{nn_0}$.

5.3. *Equivalent sequences are either both movable or both non-movable.*

We say that a sequence $\{A_n, p_n\}$ is *simply movable* if for each integer $n\geqslant 1$ there exists a homomorphism $f_n\colon A_{n+1}\to A_{n+2}$ such that $p_n = p_{nn+2}\circ f_n$. It is an easy exercise to prove that

5.4. *Each movable sequence of groups contains a simply movable subsequence.*

In the subsequent theorems we study sequences of free non-Abelian groups. Some of them are generalizations of the results contained in [34].

For the definition and discussion of free groups the reader is referred to Crowell and Fox [12]. Now we recall the notion of the free product.

Let A and B be two subgroups of a given group G. We say that G is the *free product* of A and B, notation: $G = A * B$, if every element of G, different from the identity element 1, is uniquely expressible in the form $c_1 \cdot c_2 \dots \cdot c_n$, where $c_i \neq 1$, $c_i \in A \cup B$, and no two consecutive elements c_i, c_{i+1} belong to A or B. The subgroups are called free factors of G. It follows from the definition that the assignment $b \to 1$, for $b \in B$, and $a \to a$, for $a \in A$, yields an r-homomorphism of G onto A. This homomorphism will be referred to as the retraction of G onto A. Hence each free factor of G is a retract of G. The following theorem, proved by H. Federer and B. Jónsson (Trans. Amer. Math. Soc. 68.(1950), pp. 1–27), is an important fact in the theory of free groups.

5.5. *If h is a homomorphism of a free non-Abelian group G into a free non-Abelian group H, then G can be represented as the free product $G = A * B$ in such a way that B is mapped by h onto the identity element of H and h restricted to A is a monomorphism. In particular, if h is an epimorphism, then the restriction is an isomorphism between A and H.*

An inverse sequence of groups $\underline{A} = \{A_n, p_n\}$ is said to be an *r-sequence* if each bonding homomorphism p_n is an r-homomorphism. The sequence $\underline{A}$ is finitely generated (free) if each group A_n is finitely generated (free, respectively). It is clear that each r-sequence is movable, and the following theorem states, in some sense, the converse result (comp. [34]).

5.6. *Each movable free sequence is equivalent to a free r-sequence.*

Proof. Let $\underline{G} = \{G_n, h_n\}$ be a movable free sequence. By 5.1, 5.2 and 5.4 we can assume that it is simply movable. Hence for each integer $n \geqslant 1$ there exists a homomorphism $f_n\colon G_{n+1} \to G_{n+2}$ such that

(1) $$h_n = h_{nn+2} \circ f_n .$$

According to 5.5 each group G_{n+1}, $n \geqslant 1$, can be represented as the free product $G_{n+1} = G'_{n+1} * G''_{n+1}$ such that

(2) $h_n|G'_{n+1}$ is a monomorphism, and $G''_{n+1} \subset \ker h_n$.

For $n \geqslant 2$ let $r_n\colon G_n \to G'_n$ be the retraction and for $n = 1$ let $G'_n = G_n$ and let r_n be the identity homomorphism of G_1. Denote by i_n the inclusion map of G'_n into G_n. Then by (2) we obtain

(3) $$h_n \circ i_{n+1} \circ r_{n+1} = h_n , \quad n \geqslant 1 .$$

Observe that by (1) and (3) we have $h_n \circ i_{n+1} \circ r_{n+1} \circ h_{n+1} \circ f_n = h_n \circ h_{n+1} \circ f_n = h_n = h_n \circ i_{n+1} \circ r_{n+1}$. Since $h_n \circ i_{n+1}$ is a monomorphism (see (2)), this implies that

(4) $$r_{n+1} \circ h_{n+1} \circ f_n = r_{n+1} .$$

For each positive integer n we now define two homomorphisms,

$$h'_n\colon G'_{n+1} \to G'_n \quad \text{and} \quad f'_n\colon G'_{n+1} \to G'_{n+2} ,$$

by the following formula:

$$h'_n = r_n \circ h_n \circ i_{n+1} \quad \text{and} \quad f'_n = r_{n+2} \circ f_n \circ i_{n+1} .$$

It follows from (3) that

(5) $$h'_n \circ r_{n+1} = r_n \circ h_n .$$

By using (3) and (4) it is easy to check that

(6) $h'_{n+1} \circ f'_n$ is the identity map of G'_{n+1}.

We now show that the sequence $\underline{\hat{G}} = \{G'_n, h'_n\}$ is equivalent to $\underline{G}$.

It follows from (5) that $\underline{p} = (1_N, r_n)$ is a map of $\underline{G}$ into $\underline{\hat{G}}$. For each $n \in N$ let $q(n) = n+1$ and let $q_n\colon G'_{n+1} \to G_n$ be defined by $q_n = h_n \circ i_{n+1}$. Since by (3) we have $h_n \circ q_{n+1} = h_n \circ h_{n+1} \circ i_{n+2} = h_n \circ i_{n+1} \circ r_{n+1} \circ h_{n+1} \circ i_{n+2} = q_n \circ h'_{n+1}$, we infer that $\underline{q} = (q, q_n)$ constitutes a map from $\underline{\hat{G}}$ into $\underline{G}$. Since $\underline{q} \circ \underline{p} = (q, h_n) \approx \underline{1_G}$ and $\underline{p} \circ \underline{q} = (q, h'_n) \approx \underline{1_{\hat{G}}}$, $\underline{\hat{G}}$ is equivalent to $\underline{G}$. Setting $\underline{G'} = \{G'_{n+1}, h'_{n+1}\}$, $n \geqslant 1$, we conclude by 5.2, 5.1, (6) and by the preceding remark that $\underline{G'}$ is the required r-sequence, because, by the Nielsen–Schreier theorem, a subgroup of a free group is free (see [12], p. 36).

5.7. *If $\{G_n, h_n\}$ is a finitely generated free r-sequence, then for each positive integer n there exists a finite subset $G^*_n = \{g_{n1}, \dots, g_{nr_n}, \dots, g_{ns_n}\}$ of G_n (where $s_1 = r_1$) freely generating G_n, and the following conditions are satisfied:*

$(1)_n$ $$r_{n+1} = s_n ,$$

$(2)_n$ $$h_n(g_{n+1,j}) = g_{nj} \quad \textit{for} \quad j = 1, 2, \dots, r_{n+1} ,$$

$(3)_n$ $$h_n(g_{n+1,j}) = 1 \quad \textit{for} \quad j > r_{n+1} .$$

Proof. According to 5.5 each group G_{n+1} can be represented as a free product $G_{n+1} = A_{n+1} * B_{n+1}$ such that $B_{n+1} \subset \ker h_n$ and $h_n|A_{n+1}$ is an isomorphism between A_{n+1} and G_n. The basis will be constructed successively. We define $G^*_1 = \{g_{11}, \dots, g_{1s_1}\}$ to be an arbitrary free basis of G_1. Now we define G_2. Put $r_2 = s_1$. The elements g_{2j} for $j \leqslant r_2$ are defined by

$$g_{2j} = (h_1|A_2)^{-1}(g_{1j}) .$$

Since B_2 is a retract of G_2 and G_2 is finitely generated, B_2 is a finitely generated free group. Let the elements $g_{2r_2+1}, \dots, g_{2s_2}$ form a free basis of B_2 and put: $G^*_2 = \{g_{21}, \dots, g_{2r_2}, g_{2r_2+1}, \dots, g_{2s_2}\}$. It is easy to see that this set constitutes a free basis of G_2 and satisfies conditions $(1)_1$-$(3)_1$.

In the same way we define all the other sets G_n^* satisfying the required conditions.

The next notion will play an essential role in the proof of the Main Theorem. We say that a sequence of groups $\underline{G} = \{G_n, h_n\}$ has a *regular system of generators* if for each integer $n \geqslant 1$ there exists a finite set $G_n^* \subset G_n$ generating G_n and satisfying the following condition:

(i) $h_n(G_{n+1}^*) \subset G_n^*$.

5.8. *Each free sequence having a regular system of generators is movable.*

To prove 5.8 we need the following simple lemma.

LEMMA. *Let A be a free group, let A^* be a set generating A and let B and C be arbitrary groups. If $f\colon A \to B$ and $h\colon C \to B$ are homomorphisms such that*

(1) $$f(A^*) \subset h(C),$$

then there is a homomorphism $f'\colon A \to C$ such that

(2) $$h \circ f' = f.$$

Proof. Since $h(C)$ is a group and A^* generates A, it follows from (1) that

(3) $$f(A) \subset h(C).$$

Let $F = \{g\}$ be a free basis of A. By (3) there is an element $c_g \in C$ such that $h(c_g) = f(g)$. Let $p\colon F \to C$ be defined by $p(g) = c_g$. Then for $g \in F$ we have $h \circ p(g) = f(g)$. Since F is a free basis, we can extend p to a homomorphism $f'\colon A \to C$. Hence f and $h \circ f'$ are two homomorphisms from A into B which agree on F. Hence these homomorphisms are equal, because F generates A. This proves (2).

Proof of 5.8. Let $\underline{G} = \{G_n, h_n\}$ be a free sequence and let $\{G_n^*\}$ be a regular system of generators for $\underline{G}$.

Since G_n^* is finite, it follows from (i) that there is an integer $n_0 \geqslant n$ such that

(1) $$h_{nn_0}(G_{n_0}^*) = h_{nm}(G_m^*) \quad \text{for each } m \geqslant n_0.$$

Now we show that for a given integer $m \geqslant n$ there is a homomorphism h from G_{n_0} into G_m such that

(2) $$h_{nm} \circ h = h_{nn_0},$$

and this will finish the proof.

We can assume that $m \geqslant n_0$ (otherwise we put $h = h_{mn_0}$). Hence by (1) we have $h_{nn_0}(G_{n_0}^*) \subset h_{nm}(G_m)$. Applying the lemma, we obtain a homomorphism $h\colon G_{n_0} \to G_m$ satisfying (2).

The above results imply following theorem.

5.9. *A finitely generated free sequence is movable if and only if it is equivalent to a finitely generated free sequence having a regular system of generators.*

Now we introduce the concept of the realization of a sequence of groups. Let $\underline{G} = \{G_n, h_n\}$ be a finitely generated free sequence. We say that a pointed inverse sequence of graphs $(\underline{X}, x) = \{(X_n, x_n), f_n\}$ is a realization of $\underline{G}$ if for each positive integer n we have $G_n = \pi(X_n, x_n)$, the fundamental group, and $h_n = (f_n)_{\#}$, the induced homomorphism.

5.10. *A finitely generated free sequence is movable iff any its realization is pointed movable.*

Proof. Let $\underline{G}$, $\underline{X}$ and $x = (x_1, x_2, \ldots)$ be as in the definition. Suppose first that $\underline{G}$ is movable and let n be a given positive integer. To prove that $(\underline{X}, x)$ is movable it suffices to find an integer $n_0 \geqslant n$ such that for $m \geqslant n$ there is a mapping $f\colon (X_{n_0}, x_{n_0}) \to (X_m, x_m)$ such that

(1) $$f_{nm} \circ f \simeq f_{nn_0} \operatorname{rel} x_{n_0}.$$

But by the movability of $\underline{G}$ there is an integer $n_0 \geqslant n$ such that for $m \geqslant n$ there is a homomorphism $h\colon G_{n_0} \to G_m$ such that $h_{nm} \circ h = h_{nn_0}$. By [33], p. 141, Theorem 8, there is a mapping $f\colon (X_{n_0}, x_{n_0}) \to (X_m, x_m)$ such that $f_{\#} = h$. Again applying the quoted theorem, we see that the map f satisfies (1), which proves the movability of $(\underline{X}, x)$.

Suppose next that $(\underline{X}, x)$ is movable and let n be a positive integer. It follows that there is an integer $n_0 \geqslant n$ such that for each $m \geqslant n$ there is a mapping f from (X_{n_0}, x_{n_0}) into (X_m, x_m) such that $f_{nm} \circ f \simeq f_{nn_0} \operatorname{rel} x_{n_0}$. Setting $h = f_{\#}$, we conclude that $h_{nm} \circ h = h_{nn_0}$ (see [33], p. 141, Theorem 8). This proves the movability of $\underline{G}$.

In a similar manner one shows that

5.11. *The realizations of equivalent sequences are equivalent.*

Using the above results, we may simply prove the Trybulec theorem (Theorem 5 from the Introduction). This proof is similar to that of Trybulec.

Proof of Theorem 5. Let $B_0 = \{x\} \subset B_1 \subset \ldots \subset B_\infty$ be the bouquets considered in § 2. Let p_n be a retraction of B_{n+1} onto B_n, $n \geqslant 1$, under which the set $B_{n+1} \setminus B_n$ is mapped onto B_0, and let p_{nm} have the same meaning as in the case of ANR-sequences.

Let (X', x') be a movable curve. Represent it as the limit of an inverse sequence of graphs, $X' = \operatorname{invlim}\{X'_n, f'_n\}$, and let $x' = (x'_1, x'_2, \ldots)$. Now put $(\underline{X}', x') = \{(X'_n, x'_n), f'_n\}$, $G'_n = \pi(X'_n, x'_n)$ and $h'_n = (f'_n)_{\#}$, for $n = 1, 2, \ldots$ By a result of Mardešić and Segal [28] the ANR-sequence $(\underline{X}', x')$ is movable. Each G'_n is a finitely generated free group (see [33], p. 141, Corollary 5). Since $(\underline{X}', x')$ is a realization of $\underline{G}' = \{G'_n, h'_n\}$, it follows from 5.10 that $\underline{G}'$ is movable. By 5.6 there is a finitely generated free r-sequence $\underline{G} = \{G_n, h_n\}$ equivalent to $\underline{G}'$. Applying 5.7, we obtain the free basis $G_n^* = \{g_{n1}, g_{n2}, \ldots, g_{nr_n}, \ldots, g_{ns_n}\}$ satisfying conditions $(1)_n$-$(3)_n$ for $n = 1, 2, \ldots$ From the definition of p_n it follows that the ANR-sequence $(\underline{X}, x) = (B_{s_1}, x) \xleftarrow{p_{s_1s_2}} (B_{s_2}, x) \xleftarrow{p_{s_2s_3}} \ldots$ is a realization of $\underline{G}$. Hence, by 5.11, $(\underline{X}, x)$ is equivalent to $(\underline{X}, x)$. Since $\operatorname{invlim} \underline{X}$ is a continuum equivalent to B_n or B_∞, according as $\lim s_k = n$ or $\lim s_k = \infty$, the shape of X' is planar by Theorem 1.

6. Torn loops. Let X be a topological space and let x_0 be a point of X. If ω^0 and ω^1 are mappings from the unit interval I into X such that

$$\omega^0(0) = x_0 = \omega^1(1),$$

then we say that the pair (ω^0, ω^1) form a *torn loop* in X based at x_0, briefly: *t-loop*. In such a case we write $(\omega^0, \omega^1)\colon I \to (X, x_0)$.

Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map of pointed spaces and let ω be a loop in Y (in the usual sense) based at y_0. We say that a t-loop $(\omega^0, \omega^1)\colon I \to (X, x_0)$ is a t_f-lifting of ω (and ω is generated by (ω^0, ω^1) by means of f) iff the following conditions are fulfilled:

(i) $$f(\omega^0(1)) = f(\omega^1(0)),$$

(ii) $$\omega(t) = \begin{cases} f(\omega^0(2t)) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f(\omega^1(2t-1)) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Throughout this section by a polyhedron we mean the space $|K|$ of a finite (simplicial) complex K. If x_0 is a vertex of K, then $(|K|, x_0)$ is called a *pointed polyhedron*. A *subpolyhedron* of $X = |K|$ is the space of a subcomplex of K. Let $\omega\colon I \to X$ be a path. The path ω is said to be a *simplicial path*, briefly: a *s-path*, if there is a partition $t_0 = 0 < t_1 < \ldots < t_n = 1$ of I such that each interval $[t_j, t_{j+1}]$ is linearly mapped by ω onto a simplex of X. If, in addition, ω is a homeomorphism or $\omega(I)$ is a one-point set, then we say that ω is a *sh-path*. In these cases $\omega(0)$ and $\omega(1)$ are vertices of K. In a similar way we define a *simplicial loop*, briefly: a *s-loop*, and a *simplicial torn loop*, briefly: a *st-loop*. By a k-simplex we mean a k-dimensional one.

The following proposition is obvious.

6.1. *Let X and Y be 1-dimensional polyhedra and f a simplicial map from X into Y. If D is a subpolyhedron of Y containing all vertices of Y, then $X \setminus f^{-1}(D)$ decomposes into the union of open 1-simplexes each of which goes through f onto an open 1-simplex in Y.*

6.2. *Let (X, x_0) and (Y, y_0) be 1-dimensional pointed polyhedra and suppose that X is a tree. Let $f\colon (X, x_0) \to (Y, y_0)$ be a simplicial map and let (ω^0, ω^1) be a t-loop in (X, x_0) generating a loop ω in (Y, y_0). Then there exist a s-loop $\hat\omega$ in (Y, y_0) equivalent to ω and a t-loop $(\hat\omega^0, \hat\omega^1)$ in (X, x_0) which generates $\hat\omega$ and is such that for $i = 0, 1$, the map $\hat\omega_i$ is a sh-path in X (such a t-loop $(\hat\omega^0, \hat\omega^1)$ we denote by sht-loop).*

Proof. By the assumptions we have

(1) $$\omega^0(0) = x_0 = \omega^1(1), \quad f(\omega^0(1)) = \omega(\tfrac{1}{2}) = f(\omega^1(0)),$$

(2) $$\omega(t) = \begin{cases} f(\omega^0(2t)) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f(\omega^1(2t-1)) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

First we shall prove the lemma in the case where

(I) *$\omega^0(1)$, $\omega^1(0)$ are vertices of X.* Since X is connected, there exists a sht-loop $(\hat\omega^0, \hat\omega^1)$ in (X, x_0) such that

(3) $$\hat\omega^0(1) = \omega^0(1) \quad\text{and}\quad \hat\omega^1(0) = \omega^1(0).$$

Let us note that

(4) $$\hat\omega^0(0) = x_0 = \omega^0(0) \quad\text{and}\quad \hat\omega^1(1) = x_0 = \omega^1(1).$$

Since X is simply connected, by (3) and (4) there exist homotopies $F_i\colon I \times I \to X$, $i = 0, 1$, satisfying the conditions

(5) $F_0(t, 0) = \omega^0(t)$, $F_0(t, 1) = \hat\omega^0(t)$, $F_0(0, s) = x_0$ and $F_0(1, s) = \omega^0(1)$ for every $0 \leqslant t, s \leqslant 1$;

(6) $F_1(t, 0) = \omega^1(t)$, $F_1(t, 1) = \hat\omega^1(t)$, $F_1(0, s) = \omega^1(0)$ and $F_1(1, s) = x_0$ for every $0 \leqslant t, s \leqslant 1$.

Setting

$$F(t, s) = \begin{cases} f(F_0(2t, s)) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f(F_1(2t-1, s)) & \text{for } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

we obtain a homotopy joining ω with a s-loop. Indeed, this formula defines a continuous map, because for $t = \frac{1}{2}$ we have by (5), (1) and (6):

$$f(F_0(1, s)) = f(\omega^0(1)) = f(\omega^1(0)) = f(F_1(0, s)).$$

Moreover,

$$F(0, s) = f(F_0(0, s)) = y_0 = f(F_1(1, s)) = F(1, s).$$

Again applying (5), (6) and (2), we obtain

$$F(t, 0) = \omega(t)$$

and

$$F(t, 1) = \begin{cases} f \circ \hat\omega^0(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f \circ \hat\omega^1(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Hence if $\hat\omega$ denotes the loop generated by $(\hat\omega^0, \hat\omega^1)$, then F establishes an equivalence between ω and $\hat\omega$ in (Y, y_0). Since f is simplicial, it follows from (3) and (4) that $\hat{\omega}$ is a s-loop in (Y, y_0) and $(\hat{\omega}^0, \hat{\omega}^1)$ is a sht-loop in (X, x_0) which is a t_f-lifting of $\hat{\omega}$. This completes the proof in the case (I).

Now suppose that

(II) *$\omega(\frac{1}{2})$ lies in an open 1-simplex (ab)*. Then by (1) and 6.1 there exist two open 1-simplexes (pp'), (qq') in X containing $\omega^0(1)$ and $\omega^1(0)$, respectively. Let t_0 be the first point of I that is not in $(\omega^0)^{-1}(pp')$ when going from 1 to 0. Likewise, let t_1 be the first point of I that is not in $(\omega^1)^{-1}(qq')$ when going from 0 to 1. Since (pp') is an open subset of X we infer that either $\omega^0(t_0) = p$ or $\omega^0(t_0) = p'$. Likewise, either $\omega^1(t_1) = q$ or $\omega^1(t_1) = q'$. Changing the notations if necessary we can assume that

(7) $$\omega^0(t_0) = p \quad \text{and} \quad \omega^1(t_1) = q .$$

Now we shall reduce this case to the case (I) by constructing a loop ω_1 equivalent to ω and moreover having a t_f-lifting (ω_1^0, ω_1^1) such that

(8) $\omega_1^0(1)$ and $\omega_1^1(0)$ are vertices of X.

Consider two subcases:

$(\mathrm{II})_1$ $f(p) = f(q)$. For $(t, s) \in I \times I$ put

$$F_0(t, s) = \begin{cases} \omega^0(t) & \text{for} \quad 0 \leqslant t \leqslant t_0 , \\ (1-s)\,\omega^0(t) + sp & \text{for} \quad t_0 \leqslant t \leqslant 1 , \end{cases}$$

and

$$F_1(t, s) = \begin{cases} (1-s)\,\omega^1(t) + sq & \text{for} \quad 0 \leqslant t \leqslant t_1 , \\ \omega^1(t) & \text{for} \quad t_1 \leqslant t \leqslant 1 . \end{cases}$$

By the construction of t_0 and t_1 and by (7) it is easy to check that these formulas define continuous maps from $I \times I$ into X. Setting $\omega_1^0(t) = F_0(t, 1)$ and $\omega_1^1(t) = F_1(t, 1)$, for $t \in I$, we obtain two paths in X satisfying condition (8). Obviously, we have even more:

(9) $$\omega_1^0(1) = p \quad \text{and} \quad \omega_1^1(0) = q .$$

Note that $\omega_1^0(0) = x_0$ and $\omega_1^1(1) = x_0$; hence this together with $(\mathrm{II})_1$ implies

(10) (ω_1^0, ω_1^1) is a t-loop in (X, x_0) generating a loop ω_1 in (Y, y_0).

Setting

$$F(t, s) = \begin{cases} f(F_0(2t, s)) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2} , \\ f(F_1(2t-1, s)) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1 , \end{cases}$$

for $(t, s) \in I \times I$, we obtain a homotopy establishing an equivalence between ω and ω_1. In fact, since f is a linear map of the closed simplex $\langle pp' \rangle$ (or $\langle qq' \rangle$) onto $\langle ab \rangle$, for $t = \frac{1}{2}$ we have

$$f(F_0(1, s)) = f((1-s)\,\omega^0(1) + sp) = (1-s) f \circ \omega^0(1) + sf(p)$$
$$= (1-s) f \circ \omega^1(0) + sf(q) = f((1-s)\,\omega^1(0) + sq) = f(F_1(0, s))$$

(see (1), (II) and $(\mathrm{II})_1$). This shows that F is well defined and continuous. Using (1) and (2), one easily shows that $F(0, s) = F(1, s) = y_0$ and $F(t, 0) = \omega(t)$. Finally, by the definition of ω_1^i we have

$$F(t, 1) = \begin{cases} f \circ \omega_1^0(2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2} , \\ f \circ \omega_1^1(2t-1) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1 ; \end{cases}$$

hence $\omega_1(t) = F(t, 1)$. Thus $\omega \sim \omega_1$, and therefore (10) proves the lemma in the case $(\mathrm{II})_1$.

$(\mathrm{II})_2$ $f(p) \neq f(q)$. Since $\langle pp' \rangle$ is simply connected and by (1) and (II) we have $t_0 \neq 1$, there exists a mapping

$$G\colon [t_0, 1] \times I \to \langle pp' \rangle$$

with the following properties (see (7)):

(11) $$G(t, 0) = \omega^0(t) , \quad G(t, 1) = \frac{1-t}{1-t_0} p + \frac{t-t_0}{1-t_0} p' ,$$
$$G(t_0, s) = p \quad \text{and} \quad G(1, s) = (1-s)\,\omega^0(1) + sp' .$$

Now we define two maps from $I \times I$ into X given by

$$F_0(t, s) = \begin{cases} \omega^0(t) & \text{for} \quad 0 \leqslant t \leqslant t_0 , \\ G(t, s) & \text{for} \quad t_0 \leqslant t \leqslant 1 , \end{cases}$$

and

$$F_1(t, s) = \begin{cases} (1-s)\,\omega^1(t) + sq & \text{for} \quad 0 \leqslant t \leqslant t_1 , \\ \omega^1(t) & \text{for} \quad t_1 \leqslant t \leqslant 1 . \end{cases}$$

By (11) and (7) these maps are well defined and continuous. Put

$$F(t, s) = \begin{cases} f(F_0(2t, s)) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2} , \\ f(F_1(2t-1, s)) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1 . \end{cases}$$

This is also a well-defined and continuous map from $I \times I$ into X. Indeed, for $t = \frac{1}{2}$ we have

$$f(F_0(1, s)) = f((1-s)\,\omega^0(1) + sp') = (1-s) f(\omega^0(1)) + sf(p')$$
$$= (1-s) f(\omega^1(0)) + sf(q) = f((1-s)\,\omega^1(0) + sq) = f(F_1(0, s)) ,$$

which is a consequence of (11), (1) and the fact, following from $(\mathrm{II})_2$, that $f(p') = f(q)$. Again by (1) we have $F(0, s) = x_0 = F(1, s)$, and by (11) and (2), $F(t, 0) = \omega(t)$. It is also true that $F_0(0, 1) = x_0 = F_1(1, 1)$, $F_0(1, 1) = G(1, 1) = p'$ and $F_1(0, 1) = q$. Since p' and q are vertices of X and $f(p') = f(q)$, then setting

$$\omega_1^0(t) = F_0(t, 1), \quad \omega_1^1(t) = F_1(t, 1) \quad \text{and} \quad \omega_1(t) = F(t, 1),$$

we see that ω_1 is equivalent to ω and (ω_1^0, ω_1^1) is a t-loop in (X, x_0) generating ω_1, and condition (8) is fulfilled. This proves the lemma in the case $(\mathrm{II})_2$.

It remains to consider the case where $\omega(\frac{1}{2})$ is a vertex of Y. However, the proof in this case can easily be obtained by a slight modification of the argument used in the case (II), and therefore we omit it. This completes the proof of 6.2.

6.3. *Let (X, x_0) and (Y, y_0) be 1-dimensional pointed polyhedra and suppose that X is a tree. Let $f: (X, x_0) \to (Y, y_0)$ be a simplicial map. Then there exist an integer N and a finite set $F = \{g_1, g_2, \ldots, g_n\}$ freely generating the fundamental group $\pi(Y, y_0)$ and satisfying the following condition:*

If $g \in \pi(Y, y_0)$ and g has some representative loop ω, i.e., $g = [\omega]$, such that there exists a t_f-lifting (ω^0, ω^1) of ω in (X, x_0), then g can be written as a word w in the alphabet F with length $l(w) < N$ [3].

In particular,

$$G^* = \{g \in \pi(Y, y_0)\colon \text{there exists a t-loop in } (X, x_0) \text{ generating such a loop } \omega \text{ that } g = [\omega]\}$$

is a finite set.

Proof. First we shall construct the set $F \subset \pi(Y, y_0)$. Let D be such a subpolyhedron of Y which is a maximal tree in Y. It follows that each vertex of Y belongs to D, in particular

$$y_0 \in D. \tag{1}$$

Since the lemma trivially holds if $D = Y$, we may assume, and we do, that $D \neq Y$. Let $L_i = (a_i b_i)$, $i = 1, 2, \ldots, n$, be all distinct open 1-simplexes in $Y \setminus D$. Hence we have

$$Y \setminus D = \bigcup_{i=1}^{n} L_i. \tag{2}$$

Consider a fixed integer i. Since $a_i, b_i \in D$ and D is a connected polyhedron, by (1) there exists a s-loop ω_i in (Y, y_0) such that

$$\omega_i([0, \tfrac{1}{3}]) \subset D \supset \omega_i([\tfrac{2}{3}, 1]) \tag{3}$$

and ω_i maps the interval $[\frac{1}{3}, \frac{2}{3}]$ linearly onto the simplex $\langle a_i b_i \rangle$.

---

[3] Let w be a word in the alphabet F and let $w' = g_{i_1}^{\alpha_1} \cdot \ldots \cdot g_{i_k}^{\alpha_k}$ be the reduced word equivalent to w (see [12], Chap. III, 3). Then we define $l(w) = l(w') = |\alpha_1| + \ldots + |\alpha_k|$.

The elements $g_i \in F$, $i = 1, 2, \ldots, n$, are now defined as follows: $g_i = [\omega_i]$. It is known that the set F is a system of free generators of $\pi(Y, y_0)$ (see [33], p. 141). It follows from 6.1 that

$$X \setminus f^{-1}(D) = \bigcup_{j=1}^{m} A_j, \tag{4}$$

where A_j is an open 1-simplex in X. Let N be an integer such that

$$N > 2m. \tag{5}$$

Now we shall show that F and N defined above satisfy the conclusion of the lemma.

Let $g \in G^*$. According to 6.2 we can assume that (ω^0, ω^1) is a sht-loop in (X, x_0), ω is a s-loop in (Y, y_0) generated by (ω^0, ω^1), and $g = [\omega]$. Hence there exist two sequences,

$$u_0' = 0 < u_1' < \ldots < u_k' = 1 \quad \text{and} \quad v_k' = 0 < v_{k+1}' < \ldots < v_l' = 1,$$

constituting two triangulations of I under which ω^0 and ω^1, respectively, are sh-paths in X. Let us adopt the following notations:

$$\begin{aligned} u_j &= \omega^0(u_j') \quad \text{and} \quad p_j = f(u_j) \quad \text{for} \quad 0 \leqslant j \leqslant k, \\ v_j &= \omega^1(v_j') \quad \text{and} \quad p_j = f(v_j) \quad \text{for} \quad k \leqslant j \leqslant l. \end{aligned} \tag{6}$$

Let us note that

$$f(u_k) = f(\omega^0(u_k')) = f \circ \omega^0(1) = f \circ \omega^1(0) = f(\omega^1(v_k')) = f(v_k). \tag{7}$$

This justifies our notations for $j = k$. It follows that u_j and v_j are vertices of X and therefore for each $j = 0, 1, \ldots, l$ the point p_j is a vertex of Y (f is simplicial). Hence $p_j \in D$. Let $\hat{\omega}_j$ denote a sh-path in D from y_0 to p_j. Furthermore, let $\overline{\omega}_j$ be a linear map of I onto the simplex $\langle p_j p_{j+1} \rangle$ (perhaps degenerate however by (6) and (7) the points p_j and p_{j+1} are indeed vertices of a simplex in Y), such that

$$\overline{\omega}_j(0) = p_j \quad \text{and} \quad \overline{\omega}_j(1) = p_{j+1}. \tag{8}$$

Now put

$$\omega^* = (\hat{\omega}_0 * \overline{\omega}_0 * \hat{\omega}_1^{-1}) * (\hat{\omega}_1 * \overline{\omega}_1 * \hat{\omega}_2^{-1}) * \ldots * (\hat{\omega}_{l-1} * \overline{\omega}_{l-1} * \hat{\omega}_l^{-1}).$$

Since $\hat{\omega}_0(I) = y_0 = \hat{\omega}_l(I)$ and since (ω^0, ω^1) is a t_f-lifting of ω, by (6), (7) and (8) and by the construction of $\hat{\omega}_i$ it is easy to verify that

$$\omega^* \sim \omega, \tag{9}$$

i.e., ω^* is equivalent to ω in (Y, y_0).

For $j = 0, 1, \ldots, l$ let $\tau_j = \hat{\omega}_j * \overline{\omega}_j * \hat{\omega}_{j+1}^{-1}$. Then each τ_j is a loop in (Y, y_0). Suppose that

$$\tau_j \nsim 1, \tag{10}$$

i.e., τ_j is not equivalent to the trivial loop. Then $\langle p_j p_{j+1}\rangle$ must be a simplex of the form $\langle a_i b_i\rangle$ ($\not\subset D$), for otherwise we would have $\tau_j(I) \subset D$ and therefore $\tau_j \sim 1$ because D is simply connected. In such a case condition (6) implies

$$\langle p_j p_{j+1}\rangle = \begin{cases} f(\langle u_j u_{j+1}\rangle) & \text{for} \quad j < k, \\ f(\langle v_j v_{j+1}\rangle) & \text{for} \quad j \geqslant k. \end{cases}$$

So we have the following implication:

$$\tau_j \nsim 1 \Rightarrow \begin{cases} \langle u_j u_{j+1}\rangle = \bar{A}_{r_j} & \text{for} \quad j < k, \\ \langle v_j v_{j+1}\rangle = \bar{A}_{s_j} & \text{for} \quad j \geqslant k. \end{cases}$$

Since (ω^0, ω^1) is a sht-path in (X, x_0), we obtain by the above property the following implications:

$$j \neq j' \wedge j, j' < k \Rightarrow r_j \neq r_{j'},$$
$$j \neq j' \wedge j, j' \geqslant k \Rightarrow s_j \neq s_{j'}.$$

Consequently, condition (4) implies

$$\begin{aligned} &\mathrm{Card}\{j\colon j < k \wedge \tau_j \nsim 1\} \leqslant m, \\ &\mathrm{Card}\{j\colon j \geqslant k \wedge \tau_j \nsim 1\} \leqslant m. \end{aligned} \tag{11}$$

It follows from (3), (8), (10) and from the above considerations that

$$\tau_j \sim \omega_i \quad \text{or} \quad \tau_j \sim \omega_i^{-1}, \tag{12}$$

for some $i = 1, 2, \ldots, n$. Since $g = [\omega] = [\omega^*] = [\tau_0] \cdot \ldots \cdot [\tau_{l-1}]$ (see (9)), from (5), (11) and (12) it follows that g can be written as a word in F with length less then N. To complete the proof we need only to note that from a finite set we can form only a finite number of words with length less or equal to a fixed positive integer.

6.4. *Let (X, x_0) and (Y, y_0) be connected pointed polyhedra and let $f\colon (X, x_0) \to (Y, y_0)$ be a simplicial map onto. Let*

$G^* = \{g \in \pi(Y, y_0)$: there exists a t-loop in (X, x_0) which generates a loop ω in (Y, y_0) being a representative of $g\}$.

Then G^* is a set of generators of $\pi(Y, y_0)$.

Proof [4]. Let g be an element of $\pi(Y, y_0)$. We have to show that g can be written as a word in the alphabet G^*. Let ω be a loop in (Y, y_0) which is a representative of g. We may assume that ω is a s-loop. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be such points on I that ω maps linearly each interval $[t_i, t_{i+1}]$ onto a simplex in Y. Put

$$p_i = \omega(t_i), \quad i = 0, 1, \ldots, n.$$

Then $\langle p_i p_{i+1}\rangle$ is a simplex in Y (perhaps degenerate). By our assumptions there is a simplex $\langle q_i \bar{q}_i\rangle$ in Y such that $f(q_i) = p_i$ and $f(\bar{q}_i) = p_{i+1}$, for $i = 0, 1, \ldots, n-1$ (if $p_i = p_{i+1}$ we choose $\bar{q}_i = q_i$). Let τ_j be a path in X from x_0 to q_j, and let σ_j be a linear map of I onto $\langle q_j \bar{q}_j\rangle$ such that $\sigma_j(0) = q_j$ and $\sigma_j(1) = \bar{q}_j$, for $0 \leqslant j < n$. Let τ_{-1} and τ_n be two paths in X such that $\tau_{-1}(I) = x_0 = \tau_n(I)$. Then (τ_{-1}, τ_0^{-1}) and $(\tau_j * \sigma_j, \tau_{j+1}^{-1})$, $0 \leqslant j < n$, are t-loops in (X, x_0). Let us note that $f(\tau_{-1}(1)) = y_0 = p_0 = f(q_0) = f(\tau_0(1)) = f(\tau_0^{-1}(0)), f(\tau_j * \sigma_j(1)) = f(\bar{q}_j) = p_{j+1} = f(q_{j+1}) = f(\tau_{j+1}(1)) = f(\tau_{j+1}^{-1}(0))$, $0 \leqslant j < n-1$, and $f(\tau_n^{-1}(0)) = f(x_0) = y_0 = p_n$. It follows that the t-loops generate the following loops in (Y, y_0):

$$\omega_0(t) = \begin{cases} f \circ \tau_{-1}(2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ f \circ \tau_0^{-1}(2t-1) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

and

$$\omega_{j+1}(t) = \begin{cases} f \circ \tau_j * \sigma_j(2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ f \circ \tau_{j+1}^{-1} & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

for $j = 0, 1, \ldots, n-1$. By the definitions it follows that $[\omega_j] \in G^*$, for $j = 0, 1, \ldots, n$. By standard calculations we can show that

$$\omega \sim \omega_0 * \omega_1 * \ldots * \omega_n.$$

Therefore $g = [\omega] = [\omega_0] \cdot [\omega_1] \cdot \ldots \cdot [\omega_n]$ is a word in G^*, which proves the lemma.

[4] The idea of the proof is due to A. Trybulec.

7. Proof of the Main Theorem. Let Y be a curve which can be represented as a continuous image of a continuum cr G. We adopt the notations of 4.4. For each positive integer n define G_n to be the fundamental group of (Y_n, y_n), and $\varphi_n = (g_n)_\#$ the induced homomorphism. Each G_n is a free group, and hence $\{G_n, \varphi_n\}$ constitute a free sequence. Let G_n^* be the set of $g \in G_n$ for which there exists a t-loop in (X_n, x_n) generating (by means of h_n) a loop ω in (Y_n, y_n) which is a representative of g. It follows from 4.4 and 6.3 that G_n^* is a finite set. Moreover, by 4.4 and 6.4, the set G_n^* is a set of generators for G_n. By 4.4 and by the way φ_n is defined we see that for each positive integer n

$$\varphi_n(G_{n+1}^*) \subset G_n^*.$$

In fact, if $(\omega_{n+1}^0, \omega_{n+1}^1)$ is a t-loop in (X_{n+1}, x_{n+1}) which generates (by means of h_{n+1}) a loop ω_{n+1} in (Y_{n+1}, y_{n+1}), then we see that

$(k_n \circ \omega_{n+1}^0, k_n \circ \omega_{n+1}^1)$ is a t-loop in (X_n, x_n) generating (by means of h_n) the loop $g_n \circ \omega_{n+1}$ in (Y_n, y_n). Hence we have $\varphi_n([\omega_{n+1}]) = [g_n \circ \omega_{n+1}] \in G_n$.

Combining these results with 5.8, we infer that the sequence $\{G_n, \varphi_n\}$ is movable. Hence, using 5.10, we see that the inverse ANR-sequence $\{Y_n, g_n\}$ is pointed movable. Finally, by 4.4(3) this sequence is associated with Y, and hence Y is pointed movable. This completes the proof of the Main Theorem.

8. Problems. As far as the author knows, the following problems concerning movability and continuous images of tree-like curves are open.

PROBLEM 1. Is it true that a curve which is a continuous image of a movable continuum is movable (5)?

PROBLEM 2. Is it true that a curve which is a continuous image of a plane continuum is movable (5)?

PROBLEM 3. Is it true that hereditarily decomposable curves are movable (5)?

PROBLEM 4. Is it true that arcwise connected curves are movable?

PROBLEM 5. How can we characterize movable curves which cannot be obtained as continuous images of tree-like curves?

PROBLEM 6. Is it true that movable continua are pointed movable?

There is a partial solution of Problem 3. Actually, H. Cook [11] proved that every hereditarily decomposable and hereditarily unicoherent curve is tree-like, and hence such a curve is movable.

It is known that there exists a snake-like curve which can be continuously mapped onto any snake-like curve (see [16], [23] and [30]). In the next question we ask about an analogue in the class of tree-like curves. Namely,

PROBLEM 7. Is there a tree-like curve which can be continuously mapped onto any tree-like curve?

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A topological collapse number for all spaces

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Abstract. We here assign to each topological X a symbol $C_T(X)$, the collapse number of X, that may be a non-negative integer or ∞. In calculating $C_T(X)$ we permit in a space Y with at least two points two possible operations at each stage. These are a pseudo-isotopy of Y to a compact set K, say, such that K is pointwise fixed throughout the deformation or one may remove a point from Y prior to performing the pseudo-isotopy. If X is an arbitrary space such that a finite number of such operations terminates in a point X, then X has a finite collapse number, $C_T(X)$, and this number is the least number of nonidentity pseudo-isotopies required to "shrink" X to a point. If X has but one point $C_T(X) = 0$, while all other spaces without a finite $C_T(X)$ are given the collapse number ∞.

It is pointed out here that $C_T(X)$ has been defined in a purely topological manner and that restriction of the above deformations could be made to the PL category $C_{\mathrm{PL}}(X)$ or the various differential ones. Further an analogous notion could be defined using homotopies with weaker restrictions.

THEOREM 1. *$C_T(X)$ is a topological invariant, but $C_T(X)$ is not an invariant of simple homotopy type or homotopy type.*

Proof. That $C_T(X)$ is a topological invariant follows from its definition. The rest comes from the following observation. Both the closed 3-cell I^3 and the dunce cap D [2] are of the same simple homotopy. One has $C_T(I^3) = 1$. Since there is no pseudo-isotopy shrinking D to a point, one must first remove a point of D to begin the collapsing. Regardless of the point p removed, $D-p$ is no longer homologically trivial. Thus $C_T(D) > 1$.

EXAMPLE 1. Let X be the 1-point union of r circles ($r \geqslant 2$) and let Y be the space got by attaching to a closed 2-cell r arcs in such a way that each arc meets the 2-cell at its endpoints in two fixed points of the boundary of the 2-cell, but are otherwise disjoint in pairs. Then X and Y are the same simple homotopy type. Upon removing from Y one of the two boundary fixed points, the resulting space shrinks to a point by pseudo-isotopy and $C_T(Y) = 1$. Now $C_T(X) = r$.

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* Reference [2] could as well be K. Borsuk, *Über das Phänomen der Unzerlegbarkeit in der Polyeden Topologie*, Comm. Math. Helv. 8 (1935), pp. 142—148.