A characterization of MAR and MANR-spaces by extendability of mutations

by

Stanisław Godlewski (Warszawa)

Abstract. In this paper two theorems characterizing MAR and MANR-spaces [5] are proved. These theorems are analogous to Borsuk's theorems characterizing $AR(\mathfrak{M})$ and $ANR(\mathfrak{M})$-spaces [1] and FAR and FANR-spaces [2].

The aim of this paper is to prove the following two theorems characterizing MAR and MANR-spaces:

THEOREM 1. *A metrizable space Y is a* MAR*-space if and only if for every quadruple X, X', P, Q such that $P, Q \in ANR(\mathfrak{M})$, Q contains Y as a closed subset, X' is a closed subset of P, and X is a closed subset of X', every mutation $f\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $f'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$.*

THEOREM 2. *A metrizable space Y is a* MANR*-space if and only if for every quadruple X, X', P, Q such that $P, Q \in ANR(\mathfrak{M})$, Q contains Y as a closed subset, X' is a closed subset of P, and X is a closed subset of X', every mutation $f\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $f'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)$, where W is a closed neighborhood of X in X'.*

These theorems are analogous to theorems characterizing $AR(\mathfrak{M})$ and $ANR(\mathfrak{M})$-spaces ([1], p. 87) and to theorems characterizing FAR and FANR-spaces ([2], p. 69).

The definitions of the notions appearing in this paper may be found in [5] and [6] and they will not be recalled here. The notations are the same as in [6]. Let us recall only that $\operatorname{Sh} X$ denotes the shape of X in the sense of Fox ([3]).

We assume that the reader is familiar with papers [5] and [6].

§ 1. Proof of Theorem 1. Suppose $Y \in$ MAR. Take an arbitrary quadruple X, X', P, Q satisfying the conditions formulated in the Theorem. Consider an arbitrary mutation

$$f\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)\,.$$

Since $Y \in \mathrm{MAR}$, then by Theorem (3.5) of [6] we have $\mathrm{Sh}\, Y = \mathrm{Sh}(a)$, where (a) is a space consisting of only one point a. Therefore, there exist mutations

$$\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{W}\big((a), (a)\big) \quad \text{and} \quad \boldsymbol{h}\colon \boldsymbol{W}\big((a), (a)\big) \to \boldsymbol{V}(Y, Q)$$

such that

$$\boldsymbol{gh} \simeq \boldsymbol{w} = \mathrm{Mor}\, \boldsymbol{W}\big((a), (a)\big) \quad \text{and} \quad \boldsymbol{hg} \simeq \boldsymbol{v} = \mathrm{Mor}\, \boldsymbol{V}(Y, Q)\,.$$

Since $\boldsymbol{W}\big((a), (a)\big)$ is a rudimentary system, the mutation $\boldsymbol{gf}\colon \boldsymbol{U}(X, P) \to \boldsymbol{W}\big((a), (a)\big)$ has an extension $(\boldsymbol{gf})'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{W}\big((a), (a)\big)$. Hence, the mutation $\boldsymbol{h}(\boldsymbol{gf})'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ is an extension of the mutation $\boldsymbol{hgf}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$. Since $\boldsymbol{hg} \simeq \boldsymbol{v}$, we have $\boldsymbol{hgf} \simeq \boldsymbol{f}$. Hence, by the homotopy extension theorem for mutations ([6], Theorem (2.3)), the mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$.

Now, suppose that, for every quadruple X, X', P, Q, satisfying the conditions formulated in the Theorem, every mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$.

By the Kuratowski-Wojdysławski theorem ([1]), p. 78) there exists an $\mathrm{AR}(\mathfrak{M})$-space Q containing Y as a closed subset. Let us put $X = Y$ and $X' = P = Q$. Obviously, the quadruple X, X', P, Q satisfies the required conditions. Consider the mutation $\boldsymbol{v} = \mathrm{Mor}\, \boldsymbol{V}(Y, Q)\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q)$. By hypothesis there exists an extension $\boldsymbol{v}'\colon \boldsymbol{V}(Q, Q) \to \boldsymbol{V}(Y, Q)$ of $\boldsymbol{v}$. Let $\boldsymbol{r}$ be a family of all maps $r \in \boldsymbol{v}'$ such that $r(y) = y$ for every $y \in Y$. It is easy to see that $\boldsymbol{r}\colon \boldsymbol{V}(Q, Q) \to \boldsymbol{V}(Y, Q)$ is a mutational retraction. Therefore the space Y is a mutational retract of the space $Q \in \mathrm{AR}(\mathfrak{M})$. Hence, by Theorem (4.9) of [5] we obtain $Y \in \mathrm{MAR}$. Thus, the proof is concluded.

§ 2. Proof of Theorem 2. Suppose $Y \in \mathrm{MANR}$. Take an arbitrary quadruple X, X', P, Q satisfying the conditions formulated in the Theorem. Consider an arbitrary mutation

$$\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)\,.$$

Since $Y \in \mathrm{MANR}$, there exists a closed neighborhood W_0 of Y in Q such that the set Y is a mutational retract of W_0. Let V_0 be the interior of W_0 in Q. It is easy to verify (see the proof of (4.11) in [5]) that there exists a mutational retraction

$$\boldsymbol{r}\colon \boldsymbol{V}(V_0, V_0) \to \boldsymbol{V}(Y, V_0)\,.$$

Let $i\colon Y \to Y$ be the identity on Y and let $j\colon Y \to V_0$ be the inclusion. These maps have extensions

$$\boldsymbol{i}\colon \boldsymbol{V}(Y, V_0) \to \boldsymbol{V}(Y, Q) \quad \text{and} \quad \boldsymbol{j}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(V_0, V_0)\,.$$

It is easy to see (compare the proof of (3.12) in [5]) that

$$\boldsymbol{irj} \simeq \boldsymbol{v} = \mathrm{Mor}\, \boldsymbol{V}(Y, Q)\,.$$

By (2.1) of [4] the mutation $\boldsymbol{jf}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(V_0, V_0)$ is homotopic to a mutation $\boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(V_0, V_0)$, which is an extension of a map $g\colon X \to V_0$. By the first theorem of Hanner ([1], p. 96) we have $V_0 \in \mathrm{ANR}(\mathfrak{M})$, and hence by Theorem (4.2) of [1], (p. 87) the map $g\colon X \to V_0$ has an extension $g'\colon W \to V_0$, where W is a closed neighborhood of X in X'. Consider a mutation $\boldsymbol{g}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(V_0, V_0)$, which is an extension of the map g'. By (2.1) of [5] the mutation $\boldsymbol{g}'$ is an extension of the mutation $\boldsymbol{g}$. Consider the mutations

$$\boldsymbol{irjf}, \boldsymbol{irg}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q) \quad \text{and} \quad \boldsymbol{irg}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)\,.$$

Since $\boldsymbol{g}'$ is an extension of $\boldsymbol{g}$, $\boldsymbol{irg}'$ is an extension of $\boldsymbol{irg}$. Since $\boldsymbol{g} \simeq \boldsymbol{jf}$ and $\boldsymbol{irj} \simeq \boldsymbol{v}$, we have $\boldsymbol{irg} \simeq \boldsymbol{irjf} \simeq \boldsymbol{f}$. Hence, by the homotopy extension theorem for mutations ([6], Theorem (2.3)), the mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $\boldsymbol{f}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)$.

Now, suppose that, for every quadruple X, X', P, Q, satisfying the conditions formulated in the Theorem, every mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ has an extension $\boldsymbol{f}'\colon \boldsymbol{U}'(W, P) \to \boldsymbol{V}(Y, Q)$, where W is a closed neighborhood of X in X'.

By the Kuratowski-Wojdysławski theorem ([1], p. 78) there exists an $\mathrm{ANR}(\mathfrak{M})$-space Q containing the space Y as a closed subset. Let us put $X = Y$ and $X' = P = Q$. Obviously, the quadruple X, X', P, Q satisfies the required conditions. Consider the mutation $\boldsymbol{v} = \mathrm{Mor}\, \boldsymbol{V}(Y, Q)\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q)$. By hypothesis the mutation $\boldsymbol{v}$ has an extension $\boldsymbol{v}'\colon \boldsymbol{V}'(W, Q) \to \boldsymbol{V}(Y, Q)$, where W is a closed neighborhood of Y in Q. Let V_0 be the interior of W in Q. By the first theorem of Hanner, we have $V_0 \in \mathrm{ANR}(\mathfrak{M})$. Consider the systems $\boldsymbol{V}(V_0, V_0)$ and $\boldsymbol{V}(Y, V_0)$. Denote by $\boldsymbol{r}$ the family of all maps $r\colon V_0 \to V$, where $V \in \mathrm{Ob}\, \boldsymbol{V}(Y, V_0)$, such that r is a restriction of map $v' \in \boldsymbol{v}'$ and $r(y) = y$ for every $y \in Y$. It is easy to verify (compare the proof of Theorem (4.14) in [5]) that $\boldsymbol{r}\colon \boldsymbol{V}(V_0, V_0) \to \boldsymbol{V}(Y, V_0)$ is a mutational retraction. Since $V_0 \in \mathrm{ANR}(\mathfrak{M})$, by Theorem (4.11) of [5] we obtain $Y \in \mathrm{MANR}$. Thus the proof is completed.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY PAN

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Curves which are continuous images of tree-like continua are movable

by

J. Krasinkiewicz (Warszawa)

Abstract. This paper contains several results about continuous images of continua which are contractible with respect to graphs. The main result shows that 1-dimensional continuous images of tree-like continua are movable (in the sense of Borsuk's shape theory). We present certain characterizations of continua with trivial shape. These results extend some facts concerned confluent images of continua that was recently obtained.

1. Introduction. Using the notion of movability belonging to shape theory we obtain in this paper some new results concerning curves. The main result of this paper is stated in the title.

In 1968 K. Borsuk [5] began the development of a new theory which compare compacta, i.e., compact metric spaces, from the point of view of their global topological properties. This theory has come to be known as shape theory. Let us recall some basic notions of this theory. Let X and Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q\to Q$ is said to be a *fundamental sequence* from X to Y (in symbols $\underline{f}=\{f_k, X, Y\}$) if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U$ is homotopic to $f_{k+1}|U$, $f_k|U\simeq f_{k+1}|U$ in V for almost all k. If $X=Y$ and f_k is the identity map $1_Q\colon Q\to Q$ for every positive integer k, then the fundamental sequence $\underline{f}$ is said to be the *fundamental identity sequence* for X, and is denoted by $\underline{1}_X$. The composition $\underline{g}\underline{f}$ of fundamental sequences $\underline{f}$ and $\underline{g}=\{g_k, Y, Z\}$ is the fundamental sequence $\underline{g}\underline{f}=\{g_kf_k, X, Z\}$. Two fundamental sequences $\underline{f}$ and $\underline{g}$ from X to Y are said to be *homotopic*, $\underline{f}\simeq\underline{g}$, if for every neighborhood V of Y there exists a neighborhood U of X such that $f_k|U\simeq g_k|U$ in V for almost all k. If there exist two fundamental sequences $\underline{f}$ from X to Y and $\underline{g}$ from Y to X such that $\underline{g}\underline{f}\simeq\underline{1}_X$, then we say that Y *fundamentally dominates* X, $X\underset{F}{\leqslant}Y$. If, in addition, we have $\underline{f}\underline{g}\simeq\underline{1}_Y$ then X and Y are said to be *fundamentally equivalent* — notation: $X\underset{F}{\simeq}Y$. It is known that the relation $\underset{F}{\simeq}$ is a true equivalence relation, and the set of all compacta lying ni Q is therefore partitioned into equivalence classes. The equivalence class containing a compactum X is called the *shape* of X and is denoted