$= f_1(D_1(0) \cup D_1(1))$. But, by applying Lemma 2 repeatedly, we find there also exist nests

$$A_a \supset A_{am_1} \supset A_{am_1m_2} \supset \ldots \supset A_{am_1m_2\ldots m_j}, \qquad D_0 \supset D_0^1(0) \cup D_0^1(1) \supset D_0^2 \supset \ldots \supset D_0^j,$$

and

$$D_1(0) \cup D_1(1) \supset D_1^1 \supset D_1^2(0) \cup D_1^2(1) \supset \ldots \supset D_1^j(0) \cup D_1^j(1)$$

such that $f_0(D_0(k))$, $f_1|D_1^k(0)$, and $f_1|D_1^k(1)$ are regular with respect to $(A^0_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k} \cup D_{am_1^{(1)}\ldots m_k})$, $(A^1_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k})$ and $(A^1_{am_1^{(1)}\ldots m_k}, D_{am_1^{(1)}\ldots m_k})$ respectively, if $k = 2, 4, \ldots, j$, and $f_0|D_0^k(0)$, $f_0|D_0^k(1)$, and $f_1|D_1^k$ are regular with respect to $(A^0_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k})$, $(A^0_{am_1^{(1)}\ldots m_k}, D_{am_1^{(1)}\ldots m_k})$, and $(A^1_{am_1\ldots m_k}, D_{am_1^{(0)}\ldots m_k} \cup D_{am_1^{(1)}\ldots m_k})$, respectively, if $k = 1, 3, \ldots, j-1$. Thus, $|f_i| \cap A_{am_1\ldots m_j} \neq \emptyset$ for $i = 0, 1$ and we have a contradiction to (4).

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Examples of statisch and finite-statisch AC-lattices

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Abstract. The purpose of this paper is to introduce a class of examples of statisch and finite-statisch atomistic lattices having the covering property. It will follow that any weakly modular atomistic lattice with the covering property is statisch, hence M-symmetric.

1. Basic terminology. Though the terminology will essentially follow that of [2], we introduce its more salient features here. An AC-*lattice* is an atomistic lattice with the covering property:

$$p \text{ an atom, } p \nleqslant a \text{ implies } p \vee a \text{ covers } a.$$

An element of a lattice with 0 is called a *finite* element if it is either zero or the join of a finite number of atoms; an *infinite* element is simply an element that is not finite.

For complete atomistic lattices the notion of *statisch* was introduced by Wille in [3] and extended by the author in [1] to the more general concept of a *finite-statisch* lattice. In [2], p. 65, S. Maeda shows how these ideas may be generalized to an arbitrary atomistic lattice, and it is his idea that leads us to adopt the following definition:

DEFINITION 1. Let L be an atomistic lattice. Then L is called *statisch* if p an atom, $p \leqslant a \vee b$ implies the existence of finite elements a_1 and b_1 such that $p \leqslant a_1 \vee b_1$, $a_1 \leqslant a$ and $b_1 \leqslant b$; it is called *finite-statisch* if p, q atoms with $p \leqslant q \vee a$ implies $p \leqslant q \vee a_1$ for some finite element $a_1 \leqslant a$.

It should be noted that any modular atomistic lattice as well as any compactly generated atomistic lattice is statisch, and any finite-modular AC-lattice ([2], Lemma 15.11, p. 65) is finite-statisch.

2. The examples. We now present a pair of theorems that provide a large number of examples of statisch and finite-statisch atomistic lattices. In connection with this we shall write $[a, \rightarrow]$ for an element a of a lattice L to denote $\{x \in L;\ x \geqslant a\}$. Before proceeding we shall need the following preliminary lemma whose proof is omitted since it is essentially a restatement of [2], Lemma 8.18, p. 39.

LEMMA 2. *Let $a < b$ in an AC-lattice L. Then:*

(1) *$[a, \to]$ is an AC-lattice.*

(2) *An element $c \in L$ is an atom of $[a, \to]$ if and only if there exists an atom p of L such that $c = a \vee p$, $p \nleqslant a$.*

(3) *An element $c \in L$ is a finite element of $[a, \to]$ if and only if $c = a \vee d$ for some finite element d of L. In particular, if a is itself finite in L, then $c \in L$ is a finite element of $[a, \to]$ if and only if $c \geqslant a$ and c is finite in L.*

We are now ready to state our theorems.

THEOREM 3. *Let L be an AC-lattice such that every infinite element dominates a finite element a having the property that $[a, \to]$ is finite-statisch. Then L is finite-statisch.*

Proof. Let $p, q, b \in L$ with p, q atoms and $p \leqslant q \vee b$. We must produce a finite element $b_1 \leqslant b$ such that $p \leqslant q \vee b_1$. If b is itself finite we may take $b_1 = b$ and be done, so assume b infinite. By hypothesis there is a finite element $a < b$ such that $[a, \to]$ is finite-statisch. If $p \leqslant a$ we may take $b_1 = a$, and if $q \leqslant a$ then $p \leqslant q \vee b = b$ and we may take $b_1 = p$. Thus we may assume that $p \nleqslant a$ and $q \nleqslant a$. By Lemma 2, $p \vee a$, $q \vee a$ are atoms of $[a, \to]$. Working in $[a, \to]$, $p \vee a \leqslant (q \vee a) \vee b$ so there must exist a finite element $b_1 \leqslant b$ in $[a, \to]$ such that $p \vee a \leqslant (q \vee a) \vee b_1$. By Lemma 2, b_1 is finite in L so $p \leqslant p \vee a \leqslant q \vee a \vee b_1 = q \vee b_1$, thereby completing the proof.

THEOREM 4. *Let L be an AC-lattice such that every infinite element dominates a finite element a having the property that $[a, \to]$ is statisch. Then L is statisch.*

Proof. By Theorem 3, L is finite-statisch. Let $p, a, b \in L$ with p an atom and $p \leqslant a \vee b$. If $p \leqslant a_1 \vee b$ with $a_1 \leqslant a$ and a_1 finite, then by [2], Lemma 15.11, p. 65 there is a finite element $b_1 \leqslant b$ such that $p \leqslant a_1 \vee b_1$, and we are done. Let us assume that $p \nleqslant a_1 \vee b$ for any finite element $a_1 \leqslant a$, and try to arrive at a contradiction. First of all, this forces a to be infinite, so there must exist a finite element $a_1 < a$ having the property that $[a_1, \to]$ is statisch. By assumption, $p \nleqslant a_1 \vee b$, so $p \nleqslant a_1$, and $p \vee a_1$ is an atom of $[a_1, \to]$. Now $p \vee a_1 \leqslant a \vee (a_1 \vee b)$ with $[a_1, \to]$ statisch implies the existence of finite elements a_2, b_2 of $[a_1, \to]$ such that $p \vee a_1 \leqslant a_2 \vee b_2$, $a_2 \leqslant a$ and $b_2 \leqslant a_1 \vee b$. By Lemma 2, a_2 and b_2 are also finite in L, so we now have

$$p \leqslant p \vee a_1 \leqslant a_2 \vee b_2 \leqslant a_2 \vee (a_1 \vee b) = (a_1 \vee a_2) \vee b$$

with $a_1 \vee a_2 \leqslant a$ and finite, a contradiction. We conclude that the theorem must indeed be true.

Observing now that every weakly modular AC-lattice has the property that for every atom p, $[p, \to]$ is modular (hence statisch), we have the next result.

COROLLARY 5. *Every weakly modular AC-lattice is statisch.*

It follows from [2], Corollary 15.13, p. 66 that every weakly modular AC-lattice is in fact M-symmetric.

3. Concluding remarks. We close by mentioning the characterization provided by Wille [4] of incidence geometries of grade n. He shows that a geometry is of this type if and only if its lattice of flats is a matroid lattice having the property that for every element a of height n, $[0, a]$ is distributive and $[a, 1]$ is modular. Because of this, it seems eminently reasonable to study AC-lattices in which $[a, \to]$ is modular for every element a of height n. By Theorem 4, every such lattice is statisch.

The notion of a strongly planar AC-lattice may also be generalized by recalling ([2], Lemma 14.4, p. 59) that an AC-lattice L is strongly planar if and only if for each atom p of L, the lattice $[p, \to]$ is finite-modular. If one examines an AC-lattice in which for every element a of height n, $[a, \to]$ is finite-modular, then by Theorem 3 the resulting lattice is at least finite-statisch.

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