If we let $G = C \times R$ where C is the complex line (considered as a complex manifold) and R is a Lie group with trivial f-structure and $D = \{(n+in, n) | \; n$ is an integer$\}$ then G/D is an f-Lie group which is not the product of a complex Lie group and an f-Lie group with trivial f-structure. (G/D is of course diffeomorphic to $C \times S^1$ but the f-structure on G/D is not the product f-structure of $C \times S^1$). This is the example mentioned in the introduction.

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SOUTHERN ILLINOIS UNIVERSITY
Carbondale, Illinois

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Reducing hyperarithmetic sequences

by

Hans Georg Carstens (Hannover)

Abstract. Every a'-sequence is isomorphic to an a^*-sequence. This implies: Every a'-theory T with an a-language has an a^*-model. If T has an infinite normal-model then T has an normal a^*-model.

§ 1. Introduction. If you analyse a mathematical construction to evaluate its complexity e.g. in terms of the hyperarithmetic hierarchy, it is not difficult to get a'-bounds ($a \in O$, O Kleene's system of ordinal notations, $a' = 2^a$), for you can employ recursive processes to describe the construction. If you try to get a^*-bounds (a predicate is a^*-bounded if it is a Boolean combination of $\Sigma_1^0(a)$-predicates) you must analyse some tricky constructions often related to wait and see methods.

In this paper we prove a theorem on hyperarithmetic sequences by which in some cases we can avoid this analysis and get an a^*-bound by means of a'-bound. In § 5 examples regarding models and structures will be discussed.

A model is called *normal* if its universe is the set of natural numbers and the first predicate is the identity. In [3] Hensel and Putnam have shown that every axiomatized consistent theory based on a finite number of predicates which has an infinite model with "=" interpreted as identity, has a normal model in $B^*(1)$, i.e. all predicates are 1^*-bounded. Among its consequences the theorem has an analogue to the Hensel-Putnam result for arbitrary hyperarithmetic theories with a recursive language. We can drop the assumption that the theory must be based on a finite number of predicates, and different to Putnam [5] and Hensel-Putnam [3] the result yields a method which solves Mostowski's problem [4, p. 39] simultaneously for theories with and without identity.

§ 2. The hyperarithmetic hierarchy. Let O be Kleene's system of ordinal notations with the ordering $<_0$, $a' = 2^a$ the successor of a in O, A' the recursive jump of A; we write $A \leqslant B$ if A is recursive in B. $H_1 := \emptyset$, $H_{a'} := H'_a$ for a in O, $H_{3\cdot 5^a} := \{\langle x, y\rangle : y <_0 3\cdot 5^a \;\&\; x \in H_y\}$, where $3\cdot 5^a$ is a notation of a limit ordinal.

$B(a) := \{A\colon A \leqslant H_a\}$, $H := \{B(a)\colon a \in O\}$, $\Sigma_1^0(a) := \{P\colon P$ recursively enumerable in $H_a\}$, $\Pi_1^0(a)$ is defined analogously. $B^*(a) := \{A\colon A$ built up from $\Sigma_1^0(a)$-sets by $\cap, \cup, -\}$. It is well known that "$\subseteq$" is an ω_1 (least non-constructive ordinal) well-ordering of H. Obviously $\langle B(a), \emptyset, \omega, \cap, \cup, -\rangle$ and $\langle B^*(a), \emptyset, \omega, \cap, \cup, -\rangle$ are Boolean algebras with $B^*(a) \subsetneq B(a')$ [6, p. 317]. Let $\langle P\rangle$ be the contraction of the predicate P, f a function, $P^f(x_1, \ldots, x_n) :\Leftrightarrow P(f(x_1), \ldots, f(x_n))$. In the following a bijection $f\colon \omega \to \omega$ will be constructed s.t. for some sequences $(P_i)_{i\in\omega}$ of predicates $\langle P_i\rangle \in B(a')$ and $\langle P_i^f\rangle \in B^*(a)$.

§ 3. a–trial and error predicates. A function f is called a-*(partial) recursive* iff f is (partial) recursive in H_a. By the enumeration theorem such a function has an a-index (denoted by $\langle f, a\rangle$). We choose an indexing coding recursively the schemas of definition for a-partial recursive functions. An a-index is called *recursive* iff it is an index of a total function. If $\langle e, a\rangle$ is the a-index of the a-partial recursive function f we write $[e, a] \simeq f$,

$$\lim_{y\to\infty} f(\vec{x}, y) :\simeq f\big(\vec{x}, \mu y\colon (\forall z \geqslant y\colon f(\vec{x}, z) = f(\vec{x}, y))\big).$$

Let $a \in O$, P is called an a-*trial* and error predicate iff there exists an a-recursive function f such that $P(\vec{x}) \Leftrightarrow \lim_{y\to\infty} f(\vec{x}, y) = 1$ and $\overline{P}(\vec{x}) \Leftrightarrow \lim_{y\to\infty} f(\vec{x}, y) = 0$.

LEMMA. (1) $\exists F$ *recursive function*: $\forall a \in O$: $\forall e$ recursive:

$$[e, a'](\vec{x}) = \lim_{y\to\infty} [F(e), a](\vec{x}, y).$$

(2) $\exists G$ *recursive function*: $\forall a \in O$: $\forall e$ *recursive s.t.* $\forall \vec{x}$: $\lim_{y\to\infty} [e, a](\vec{x}, y)$ *exists*:

$$[G(e), a'](\vec{x}) = \lim_{y\to\infty} [e, a](\vec{x}, y).$$

Proof. (1) By induction on the definition of the class of a'-recursive functions. If e is the index of the characteristic function $K_{H_{a'}}$ of $H_{a'}$ let $F(e)$ be a fixed index of f with

$$f(x, y) = \begin{cases} 0 & \text{if } \neg\exists z \leqslant y\colon T^{H_a}(x, x, z),\\ 1 & \text{otherwise}\end{cases}$$

where T^{H_a} is Kleene's T for H_a. It is now trivial to prove the base of the induction. If e is an index of a function defined by substitution, i.e.

$$[e, a'](\vec{x}) = [e_0, a']\big([e_1, a'](\vec{x}), \ldots, [e_n, a'](\vec{x})\big),$$

take $F(e)$ as an index of

$$[F(e), a](\vec{x}, y) = [F(e_0), a]\big([F(e_1), a](\vec{x}, y), \ldots, [F(e_n), a](\vec{x}, y), y\big).$$

If e is an index of a function defined by μ-recursion, i.e.

$$[e, a'](\vec{x}) = \mu z\colon \big([e_0, a'](\vec{x}, z) = 0\big),$$

take $F(e)$ as an index of

$$[F(e), a](\vec{x}, y) = \mu z \leqslant y\colon \big([F(e_0), a](\vec{x}, z, y) = 0\big).$$

(2) We will give an explicite a'-definition of $f(\vec{x}) = \lim_{y\to\infty} [e, a](\vec{x}, y)$:

$$\begin{aligned} f(\vec{x}) = \lim_{y\to\infty} [e, a](\vec{x}, y) &= \big(\mu n\colon \forall y \geqslant (n)_1\colon [e, a](\vec{x}, y) = (n)_0\big)_0\\ &= \big(\mu n\colon \neg\exists y > (n)_1\colon [e, a](\vec{x}, y) \neq (n)_0\big)_0\\ &= \big(\mu n\colon \neg\exists z\colon T^{H_a}(f_0(e), \langle\vec{x}, n\rangle, z)\big)_0\\ &= \big(\mu n\colon [f_1(f_0(e)), a'](\vec{x}, n) = 0\big)_0\end{aligned}$$

where f_0, f_1 are recursive functions s.t. $[f_0(e), a](\vec{x}, n) \simeq \mu y\colon \big(y \geqslant (n)_1\ \&\ \&\ [e, a](\vec{x}, y) \neq (n)_0\big)$, and $[f_1(e), a']$ is the characteristic function of $\exists z\colon T^{H_a}(e, \langle\ \ \rangle, z)$. Let $G(e)$ be the index of $\big(\mu n\colon [f_1(f_0(e)), a'](\vec{x}, n) = 0\big)_0$. G is recursive and $[G(e), a'](\vec{x}) = \lim_{y\to\infty} [e, a](\vec{x}, y)$.

COROLLARY. *P a-trial and error predicate iff $\langle P\rangle \in B(a')$.*

Proof. "$\Rightarrow$" By the definition of a-trial and error predicate and the theorem of Post [7, p. 167]. "$\Leftarrow$" By (1) of the lemma.

§ 4. The main theorem. A sequence f is called an $a(a')$-*sequence* iff f is $a(a')$-recursive and for all n $f(n)$ is a recursive $a(a')$-index. $P_e(x) :\Leftrightarrow [e, a](\vec{x}) = 0$. Clearly $\langle P_e\rangle \in B(a)$. It is easy to show that all predicates P with $\langle P\rangle \in B(a)$ are equal to a P_e with a suitable recursive a-index e. For any a'-sequence f we will construct a bijection $g\colon \omega \to \omega$ s.t. for all n $\langle P^g_{f(n)}\rangle \in B^*(a)$. That is: if you can get a sequence of hyperarithmetic predicates in $B(a')$ from an a'-sequence you also can get a sequence of predicates in $B^*(a)$ which is "isomorphic" to the original one. Call an a'-sequence f an a^*-sequence iff for all n $\langle P_{f(n)}\rangle \in B^*(a)$. Two a-sequences f and g are isomorphic iff there is a bijective function h s.t. for all n $P_{g(n)} = P^h_{f(n)}$.

THEOREM. *For all $a \in O$ and all a'-sequences f there is a recursive function g s.t. $g \circ f$ is an a^*-sequence isomorphic to f.*

Proof. Let f be an a'-sequence. By the lemma there is a recursive function F s.t. the following function is an a'-sequence of a-indices

$$h_0(n) = \langle F\big((f(n))_0\big), a\rangle.$$

Let p be a recursive function s.t. $[(e)_0, (e)_1]$ is a $p(e)$-place function. Now define

$$h(n, x, y) := \begin{cases} 1 & \text{if} \quad y = 0 \quad \text{or} \quad y = 2, \\ 0 & \text{if} \quad y = 1, \\ [(h_0(n))_0, a]((x)_0, \ldots, (x)_{p(f(n))}, y) & \text{if } y \geqslant 3, \end{cases}$$

$$l(x) := \mu y\colon lh(y) = y \;\&\; \forall n, 0 \leqslant n \leqslant x \dot{-} 1\colon \forall z, z \geqslant (y)_n\colon \forall x_1, \ldots x_{p(f(n))},$$
$$x_1, \ldots, x_{p(f(n))} \leqslant x\colon \big(h(n, \langle x_1, \ldots, x_{p(f(n))}\rangle, z) = h(n, \langle x_1, \ldots, x_{p(f(n))}\rangle, (y)_n)\big),$$

$$x \preccurlyeq y :\Leftrightarrow \big(lh(x) = lh(y) \;\&\; \forall j, 0 \leqslant j \leqslant lh(x) \dot{-} 1\colon (x)_j \leqslant (y)_j\big),$$

$$r(x, y) := \begin{cases} 0 & \text{if} \quad x \not\preccurlyeq y, \\ \mu z\colon \big(\exists x_0, \ldots, x_z\colon \big(x_0 < x_1 < \ldots < x_z \;\&\; \forall i, 0 \leqslant i \leqslant z\colon & \\ \big(x \preccurlyeq x_i \preccurlyeq y \;\&\; \neg \exists u\colon (x \preccurlyeq u \preccurlyeq y \;\&\; x_i < u < x_{i+1})\big)\big)\big) & \text{otherwise}. \end{cases}$$

We can easily verify that the set of numbers x such that $l(lh(x)) \preccurlyeq x$ is not true, is recursively enumerable in H_a. Let s be an a-recursive function which enumerates this set without repetitions. Now define

$$g^*(x) := \begin{cases} (2^{lh(x)} - 1) + 2^{lh(x)+1} r\big(l(lh(x)), x\big) & \text{iff } x \succeq l(lh(x)), \\ 2(\mu z\colon s(z) = x) & \text{otherwise}. \end{cases}$$

By definitions of l, $\preccurlyeq$, r, s, and the properties of F g^* is a bijective a'-recursive function. Thus there is an a'-index $\langle e, a'\rangle$ for g^*. Let g be a recursive function that computes the a'-index of the function defined by

$$[c, a']([e, a'](x_1), \ldots, [e, a'](x_{p(\langle c, a'\rangle)}))$$

from any a'-index $\langle c, a'\rangle$. Clearly $g \circ f$ is an a'-sequence isomorphic to f.

Finally let us show that $g \circ f$ is an a^*-sequence. It suffices to prove that for all n there is an a-recursive function g_n s.t. for all x

$$P_{g \circ f(n)}((x)_0, \ldots, (x)_{p(f(n)) \dot{-} 1}) \Leftrightarrow \lim_{y \to \infty} g_n((x)_0, \ldots, (x)_{p(f(n)) \dot{-} 1}, y) = 0$$

and g_n changes its value at most $p(f(n))$-times. Then $P_{g \circ f(n)}$ with parameters $x_1, \ldots, x_{p(f(n))}$ is true if and only if there is an i, $0 \leqslant i \leqslant p(f(n))$ s.t. g_n changes its value exactly i times. This shows that $\langle P_{g \circ f(n)}\rangle \in B^*(a)$. We will now describe an algorithm which defines the g_n's. First we define

$$l'(x) := \mu y\colon \big(lh(x) = lh(y) \;\&\; \forall n, 0 \leqslant n \leqslant lh(y) \dot{-} 1\colon \forall z, (y)_n \leqslant z$$
$$\leqslant (x)_n\colon \forall u_1, \ldots, u_{p(f(n))}, u_1, \ldots, u_{p(f(n))}$$
$$\leqslant lh(y)\colon h(n, \langle u_1, \ldots, u_{p(f(n))}\rangle, z) = h(n, \langle u_1, \ldots, u_{p(f(n))}\rangle, (x)_n)\big),$$

$$g'(x) := (2^{lh(x)} - 1) + 2^{lh(x)+1} r\big(l'(x), x\big),$$
$$o(x) :\simeq 2\big(\mu z\colon s(z) = x\big),$$
$$m_n := \mu y\colon \big(\forall z, z \geqslant y\colon \forall u_1, \ldots, u_{p(f(n))} \leqslant n\colon h(n, \langle u_1, \ldots, u_{p(f(n))}\rangle, z)$$
$$= h(n, \langle u_1, \ldots, u_{p(f(n))}\rangle, y)\big).$$

Now we will describe the algorithm.

$$g_n(x_1, \ldots, x_{p(f(n))}, 0) := h\big(n, \langle g'(x_1), \ldots, g'(x_{p(f(n))})\rangle, y_0\big),$$

$$y_0 := \begin{cases} \max\limits_{1 \leqslant j \leqslant p(f(n))} \{(x_j)_n\} & \text{iff } \exists j, 1 \leqslant j \leqslant p(f(n))\colon (x_j)_n \neq 0, \\ m_n & \text{otherwise}. \end{cases}$$

Now test in a fixed recursive manner if

$$\exists k, 0 \leqslant k \leqslant lh(x_j) \dot{-} 1\colon \exists u_1, \ldots, u_{p(f(k))}$$
$$\leqslant lh(x_j)\colon h\big(k, \langle u_1, \ldots, u_{p(f(k))}\rangle, (x_j)_k\big) \neq h\big(k, \langle u_1, \ldots, u_{p(f(k))}\rangle, (x_j)_k + y\big);$$
$$y = 1, 2, 3, \ldots$$

The tests are numbered in this way. If the answer is negative for all tests with numbers $\leqslant z$ define

$$g_n(x_1, \ldots, x_{p(f(n))}, z) := g_n(x_1, \ldots, x_{p(f(n))}, 0).$$

If the answer is positive for the first time for x_{j_0} in the test with number z_0 define

$$g_n(x_1, \ldots, x_{p(f(n))}, z_0) :=$$
$$h\big(n, \langle g'(x_1), \ldots, g'(x_{j_0 \dot{-} 1}), o(x_{j_0}), g'(x_{j_0+1}), \ldots, g'(x_{p(f(n))})\rangle, y_1\big),$$

$$y_1 := \begin{cases} \max\limits_{\substack{1 \leqslant j \leqslant p(f(n)) \\ j \neq j_0}} \{(x_j)_n\} & \text{iff } \exists j, 1 \leqslant \underset{j \neq j_0}{j} \leqslant p(f(n))\colon (x_j)_n \neq 0, \\ m_n & \text{otherwise}. \end{cases}$$

Now test again as above. Clearly g_n is a-recursive and changes its value at most $p(f(n))$ times. Now the following holds

$$P_{g \circ f(n)}(x_1, \ldots, x_{p(f(n))})$$
$$\Leftrightarrow [(g \circ f(n))_0, a'](x_1, \ldots, x_{p(f(n))}) = 0$$
$$\Leftrightarrow [(f(n))_0, a'](g^*(x_1), \ldots, g^*(x_{p(f(n))})) = 0$$
$$\Leftrightarrow \lim_{y \to \infty} h(n, \langle g^*(x_1), \ldots, g^*(x_{p(f(n))})\rangle, y) = 0$$
$$\Leftrightarrow \lim_{y \to \infty} g_n(x_1, \ldots, x_{p(f(n))}, y) = 0.$$

The last equivalence holds by the properties of g^* and the definition of the g_n's. This completes the proof.

Remark. By the proof of the theorem for every a'-sequence there is a g^* with the properties mentioned there. Obviously no a'-recursive g^* fulfills these properties for all a'-sequences: Let A be a $B(a')$-$B^*(a)$ set. The assumption $g^* a'$-recursive implies the a'-recursiveness of g^{*-1} and $A^{g^{*-1}}$. But $A^{g \circ g^{*-1}}$ is not $B^*(a)$.

§ 5. Applications of the theorem. The first application will be an extension of the Hensel-Putnam result in [3]. Call a theory T an *a-theory* iff $\mathrm{Thm}_T \in B(a)$, a language L an *a-language* iff its arithmetization is in $B(a)$, and a structure an *$a(a^*)$-structure* iff its universe is ω and all of its predicates and graphs of functions are in $B(a)$ $(B^*(a))$.

THEOREM. *Every a'-theory with an a-language has an a^*-model.*

Proof. The Henkin-Hasenjaeger construction gives an a'-model determined by an a'-sequence.

THEOREM. *Every a'-theory with an a-language and an infinite model for which "=" is interpreted as identity has an a^*-normal-model.*

Proof. Extend the language of the original theory s.t. there is a formula which has only infinite models and is relatively consistent to the theory. Take this formula as a new axiom. Again the Henkin-Hasenjaeger construction gives an a'-normal-model determined by an a'-sequence.

The second application is almost trivial. Call a structure finite iff it has only a finite number of predicates and functions. Clearly the following holds: Every finite a'-structure is isomorphic to an a^*-structure. This is true especially for all algebraic structures i.e. finite structures with functions only. If such a structure has an infinite universe and is $\Sigma_1^0(a)$ then it is obviously an a-structure.

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INSTITUT FÜR MATHEMATIK
Technische Universität

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