If we let $G = C \times R$ where $C$ is the complex line (considered as a complex manifold) and $R$ is a Lie group with trivial $f$-structure and $D = \{(n+in, n) \mid n \text{ is an integer}\}$ then $G/D$ is an $f$-Lie group which is not the product of a complex Lie group and an $f$-Lie group with trivial $f$-structure. ($G/D$ is of course diffeomorphic to $C \times S^1$ but the $f$-structure on $G/D$ is not the product $f$-structure of $C \times S^1$). This is the example mentioned in the introduction.

References


Reducing hyperarithmetical sequences

by

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Abstract. Every $\alpha'$-sequence is isomorphic to an $\alpha^*$-sequence. This implies: Every $\alpha'$-theory $T$ with an $\alpha$-language has an $\alpha^*$-model. If $T$ has an infinite normal-model then $T$ has an normal $\alpha^*$-model.

§ 1. Introduction. If you analyze a mathematical construction to evaluate its complexity e.g. in terms of the hyperarithmetical hierarchy, it is not difficult to get $\alpha'$-bounds ($\alpha \in O$, $O$ Kleene's system of ordinal notations, $\alpha^* = 2^\alpha$), for you can apply recursive processes to describe the construction. If you try to get $\alpha^*$-bounds (a predicate is $\alpha^*$-bounded if it is a Boolean combination of $\Sigma^0_n$-predicates) you must analyze some tricky constructions often related to wait and see methods.

In this paper we prove a theorem on hyperarithmetical sequences by which in some cases we can avoid this analysis and get an $\alpha^*$-bound by means of $\alpha'$-bound. In §5 examples regarding models and structures will be discussed.

A model is called normal if its universe is the set of natural numbers and the first predicate is the identity. In [3] Hensel and Putnam have shown that every axiomatized consistent theory based on a finite number of predicates which has an infinite model with $\alpha^*$ interpreted as identity, has a normal model in $B^*(1)$, i.e. all predicates are $1^*$-bounded. Among its consequences the theorem has an analogue to the Hensel-Putnam result for arbitrary hyperarithmetical theorems with a recursive language. We can drop the assumption that the theory must be based on a finite number of predicates, and different to Putnam [5] and Hensel-Putnam [3] the result yields a method which solves Mostowski's problem [4, p. 39] simultaneously for theories with and without identity.

§ 2. The hyperarithmetical hierarchy. Let $O$ be Kleene's system of ordinal notations with the ordering $<_\alpha$, $\alpha^* = 2^\alpha$ the successor of $\alpha$ in $O$, $\alpha'$ the recursive jump of $\alpha$; we write $\alpha \leq B$ if $\alpha$ is recursive in $B$. $H^*_a := \emptyset$, $H^*_a := H^*_a$ for $a \in O$, $H^*_a := \{(x, y) : y <_\alpha 3 \cdot 5^a \amp \alpha \in H^*_a\}$, where $3 \cdot 5^a$ is a notation of a limit ordinal.
§ 3. a-trial and error predicates. A function \( f \) is called a (partial) recursive function iff \( f \) is partial and recursive in \( H_a \). By the enumeration theorem such a function has an a-index (denoted by \( \langle f, a \rangle \)). We choose an indexing coding recursively the schemas of definition for a partial recursive functions. An a-index is called recursive if it is an index of a total function. If \( \langle f, a \rangle \) is the a-index of the partial recursive function \( f \) we write \( [e, a] \simeq f \),

\[
\lim_{y \to \infty} f(x, y) =: f(x, y) \quad \forall x \geq y : f(x, y) = f(x, y)
\]

Let \( a \in O \), \( P \) is called an a-trial and error predicate iff there exists an a-recursive function \( f \) such that \( P(x) \iff f(x, y) = 1 \) and \( P(x) \iff f(x, y) = 0 \).

**Lemma.** (1) \( \exists f \) recursive function: \( \forall x \in O \): \( f \) recursive:

\[
\langle e, a \rangle(x) = \lim_{y \to \infty} f(x, a)(x, y)
\]

(2) \( \exists G \) recursive function: \( \forall x \in O : \forall y : \exists x : \lim_{y \to \infty} f(x, a)(x, y) \)

**Proof.** (1) By induction on the definition of the class of a'-recursive functions. If \( \langle e, a \rangle \) is the index of the characteristic function \( \mathcal{K}_{a'} \) of \( H_a \) let \( F(e) \) be a fixed index of \( f \) with

\[
f(x, y) = \begin{cases} 0 & \text{if } \forall x \geq y : f(x, y) = f(x, y) \\ 1 & \text{otherwise} \end{cases}
\]

where \( T_{H_a} \) is Kleene's \( T \) for \( H_a \). It is now trivial to prove the base of the induction. If \( e \) is an index of a function defined by substitution, i.e.

\[
[e, a' \rangle(x) = \langle a_0, a' \rangle(e_0, a' \rangle(x), \ldots, a_n, a')(x),
\]

take \( F(e) \) as an index of

\[
[\mathcal{F}(e), a](x, y) = [\mathcal{F}(e), a]([\mathcal{F}(e), a](x, y), \ldots, [\mathcal{F}(e), a](x, y), y).
\]

If \( e \) is an index of a function defined by \( \mu \)-recursion, i.e.

\[
[e, a \rangle(x) = \mu z : [e, a \rangle(x, z) = 0,
\]

take \( F(e) \) as an index of

\[
[\mathcal{F}(e), a](x, y) = [\mathcal{F}(e), a](x, y) = \mu z : [\mathcal{F}(e), a](x, z, y) = 0.
\]

(2) We will give an explicit a'-definition of \( f(x, y) = \lim_{y \to \infty} f(x, a)(x, y) \):

\[
f(x, y) = \lim_{y \to \infty} f(x, a)(x, y) = (\mu y : \forall x \geq y : [e, a](x, y) = (n)_y)
\]

where \( f_0, f_1 \) are recursive functions s.t. \( [f_0(e), a](x, n) = (n)_y \) and \( [f_1(e), a'] \) is the characteristic function of \( y : \exists x : T_{H_a}(f_0(x), x, n, y) = 0 \).

**Corollary.** A a'-trial and error predicate iff \( P \in B(a') \).

**Proof.** By the definition of a-trial and error predicate and the theorem of Post [7, p. 167]. "By (1) of the lemma.

§ 4. The main theorem. A sequence \( f \) is called an a\((a')\)-sequence iff \( f \) is a\((a')\)-recursive and for all \( f(n) \) is a recursive a\((a')\)-index. \( \mathcal{P}(a) \) \( \iff \langle e_0, a \rangle(x, y) = 0 \). Clearly \( \mathcal{P}(a) \in B(a) \). It is easy to show that all predicates \( P \) with \( \mathcal{P}(a) \in B(a) \) are equal to \( P \) with a suitable recursive a\((a')\)-index \( e \). For any a\((a')\)-sequence \( f \) we will construct a bijection \( g \) with s.t. for all \( \mathcal{P}(g) \in B(a') \). That is: If you can get a sequence of hyperarithmetic predicates in B\((a')\) you also can get a sequence of predicates in B\((a')\) which is isomorphic in the original one. Call an a\((a')\)-sequence \( f \) an a\((a')\)-sequence iff for all \( \mathcal{P}(g) \in B(a') \). Two a\((a')\)-sequences \( f \) and \( g \) are isomorphic iff there is a recursive function \( h \) s.t. for all \( \mathcal{P}(h) \in B \).

**Theorem.** For all \( a \in O \) and all a\((a')\)-sequences \( f \) there is a recursive function \( h \) s.t. \( g = f \) is an a\((a')\)-sequence isomorphic to \( f \).

**Proof.** Let \( f \) be an a\((a')\)-sequence. By the lemma there is a recursive function \( F \) s.t. the following function is an a\((a')\)-sequence of a-indices

\[
h_a(n) = (f \langle (n) \rangle, a).
\]
Let $p$ be a recursive function s.t. $[(e_0, (e))]$ is a $p(e)$-place function. Now define

$$
\begin{align*}
&\quad h(n, x, y) := \begin{cases} 1 & \text{if } y = 0 \text{ or } y = 2, \\
0 & \text{if } y = 1, \\
\left[\{h(n), a\}, (x_1, \ldots, x_{p(n)}\}, y\right] & \text{if } y \geq 3,
\end{cases} \\
&\quad l(x) := \forall y : l(n) = y \& \forall n, 0 < n < x - 1: \forall y, 0 < y < l(n) < y.
\end{align*}
$$

Lastly, we can define the sequence recursively. Now define

$$
[\left(\left\{g(n), a\right\}, \left(x_1, \ldots, x_{p(n)}\right), y\right)] := \begin{cases} 1 & \text{if } l(n) < y, \\
0 & \text{if } l(n) = y, \\
\left[\left\{h(n), a\right\}, \left(x_1, \ldots, x_{p(n)}\right), y\right] & \text{if } y < l(n).
\end{cases}
$$

Now we will describe the algorithm.

$$
\begin{align*}
g_0(a_0, \ldots, x_{p(n)}), 0 := & \left[\right] \left[\right] (n), \left[\right] (a_0, \ldots, x_{p(n)}), 0 \right], \\
\max & \left\{\left\{a_0\right\}\right\} \text{ if } \exists y, 1 < j \leq p(f(n)), (a_0) \neq 0, \\
y_0 := & \left[\right] \left[\right] (a_0).
\end{align*}
$$

Now test in a fixed recursive manner if

$$
\exists k, 0 \leq k \leq l(n) - 1 : \exists a_k, \ldots, x_{p(n)} \\
\leq a_k (a_0) : h(n, \langle a_1, \ldots, a_k \rangle, (a_{k+1}) \neq h(n, \langle a_1, \ldots, a_k \rangle, (a_k + y)) \\
y_1 := \left[\right] \left[\right] (a_0).
$$

The tests are numbered in this way. If the answer is negative for all tests with numbers $\leq j$ define

$$
g_n(a_0, \ldots, x_{p(n)}), 0 := g_n(a_1, \ldots, x_{p(n)}), 0.
$$

If the answer is positive for the first time for $a_k$ in the test with number $a_0$ define

$$
g_n(a_1, \ldots, x_{p(n)}), 0 := h(n, \langle a_1, \ldots, a_k \rangle, (a_{k+1}) \right\} (a_{k+1}) \neq 0, \\
y_1 := \left[\right] \left[\right] (a_0).
$$

Now test again as above. Clearly $g_n$ is $n$-recursive and changes its value at most $p(f(n))$ times. Now the following holds

$$
\begin{align*}
P_{p(n)}(a_1, \ldots, x_{p(n)}), a & := \left[\right] \left[\right] (a_1, \ldots, x_{p(n)}), a \right], \\
[p(f(n))] & \left[\right] (a_1, \ldots, x_{p(n)}), a \right] = 0, \\
\lim & \left[\right] \left[\right] (a_1, \ldots, x_{p(n)}), y \right] = 0, \\
\lim & \left[\right] \left[\right] (a_1, \ldots, x_{p(n)}), y \right] = 0.
\end{align*}
$$
The last equivalence holds by the properties of $g^*$ and the definition of the $g_n$'s. This completes the proof.

Remark. By the proof of the theorem for every $a^*$-sequence there is a $g^*$ with the properties mentioned there. Obviously no $a^*$-recursive $g^*$ fulfills these properties for all $a^*$-sequences: Let $A$ be a $B(a)^* - B^*(a)$ set. The assumption $g^*a^*$-recursive implies the $a^*$-recursiveness of $g^*a^*$ and $A^*a^*$. But $A^*a^*$ is not $B^*(a)$.

§ 5. Applications of the theorem. The first application will be an extension of the Hensel-Putnam result in [3]. Call a theory $T$ an $a$-theory iff $Th(T) \in B(a)$, a language $L$ an $a$-language iff its arithmetization is in $B(a)$, and a structure an $a(a^*)$-structure iff its universe is $\omega$ and all of its predicates and graphs of functions are in $B(a)$ ($B^*(a)$).

Theorem. Every $a^*$-theory with an $a$-language has an $a^*$-model.

Proof. The Henkin-Hasenjaeger construction gives an $a^*$-model determined by an $a^*$-sequence.

Theorem. Every $a^*$-theory with an $a$-language and an infinite model for which "=" is interpreted as identity has an $a^*$-normal-model.

Proof. Extend the language of the original theory $T$ s.t. there is a formula which has only infinite models and is relatively consistent to the theory. Take this formula as a new axiom. Again the Henkin-Hasenjaeger construction gives an $a^*$-normal-model determined by an $a^*$-sequence.

The second application is almost trivial. Call a structure finite iff it has only a finite number of predicates and functions. Clearly the following holds: Every finite $a^*$-structure is isomorphic to an $a^*$-structure. This is true especially for all algebraic structures i.e. finite structures with functions only. If such a structure has an infinite universe and is $\Sigma^2(a)$ then it is obviously an $a$-structure.

References