Applications of a mismatch theorem to decomposition spaces*

by

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Abstract. We apply the mismatch theorem [12] to several upper semi-continuous decompositions of S^3. In addition to unifying these results we establish a conjecture of R. H. Bing by showing that his straight line interval decomposition [7] is not E^3 — a simpler example of the same phenomenon is also given.

1. Introduction. The purpose of this paper is to apply Theorem 1, stated below and proved in [11] and [12], to certain upper semi-continuous decompositions of E^3 to determine whether or not the associated decomposition space is topologically E^3. The decomposition spaces for which Theorem 1 is applicable are those equivalent to E^3/G where G is a monotone upper semi-continuous decomposition of E^3 situated properly with respect to the x, y — coordinate plane Q. More precisely, if U_0 and U_1 are the components of $E^3 - Q$ then we require that each element of G intersects Q in a connected set so that $G(Q) = \{g \cap Q|\ g \in G\}$, and $G^i = \{g \cap \mathrm{Cl}\, U_i|\ g \in G\} \cup U_{1-i}$ represent monotone upper semi-continuous decompositions of Q and E^3, respectively. We also require that $Q/G(Q) \approx Q$ (by a theorem of R. L. Moore [16], it is sufficient to require that $g \cap Q$ fails to separate Q for all $g \in G$) and $E^3/G^i \approx E^3$ so that $(E^3/G) + \infty$ represents the sum of the two crumpled cubes $(E^3/G^0) - U_1 + \infty$ and $(E^3/G^1) - U_0 + \infty$ sewn together by the identity homeomorphism on their common boundary $\big(Q/G(Q)\big) + \infty$.

THEOREM 1. *If K_0 and K_1 are crumpled cubes and h: $\mathrm{Bd}\, K_0 \to \mathrm{Bd}\, K_1$ is a homeomorphism, then the sum $K_0 \cup_h K_1$ (the disjoint union of K_0 and K_1 with x identified with $h(x)$ for each $x \in \mathrm{Bd}\, K_0$) is topologically S^3 if and only if there exist disjoint 0-dimensional F_σ sets F_0 and F_1 in $\mathrm{Bd}\, K_0$ such that $F_0 \cup \mathrm{Int}\, K_0$ and $h(F_1) \cup \mathrm{Int}\, K_1$ are 1-ULC.*

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The specific decompositions G of E^3 that we include in this paper have many additional properties. In the remainder of this section we discuss some of these properties and introduce notation that will be used in Sections 2 and 3.

We use H to stand for the collection of non-degenerate elements in G, H^* for the union of the elements in H, and P for the natural projection map from E^3 to E^3/G. For our decompositions $P(H^*)$ is always compact 0-dimensional; thus, G is definable by cubes-with-handles [13], [15] in the sense that there exists a sequence $\{M_j\}$ such that $M_{j+1} \subset \operatorname{Int} M_j$, $H^* = \bigcap M_j$, and each component of each M_j is a PL cube-with-handles. We use the following notation for the defining sequence $\{M_j\}$.

The components of M_1 are denoted by $A_1, A_2, \ldots$, the components of M_2 that lie in $\operatorname{Int} A_k$ are denoted by $A_{k1}, A_{k2}, \ldots$, and, inductively, $A_{m_1m_2\ldots m_j}$ denotes a component of M_j that lies in the component $A_{m_1m_2\ldots m_{j-1}}$ of M_{j-1}. Thus, for each non-degenerate element $g \in H \subset G$ there exists a sequence $m_1, m_2, \ldots$ of positive integers such that g is the intersection of the nest $A_{m_1} \supset A_{m_1m_2} \supset \ldots$ To avoid unnecessary subscripts we use Greek letters to denote juxtaposition of subscripts. For example, when α represents $m_1m_2\ldots m_j$ we write A_α to denote $A_{m_1m_2\ldots m_j}$ and $A_{\alpha 1}$ to denote $A_{m_1m_2\ldots m_j1}$.

For the decompositions G of this paper it is easy to establish that $E^3/G^i \approx E^3$ ($i = 0$ or 1). Let $A_\alpha^i = A_\alpha \cap \operatorname{Cl} U_i$. The cubes-with-handles A_α have the property that for $i = 0$ or 1 there exist a finite collection $X_\alpha^i(1)$, $X_\alpha^i(2), \ldots$ of disjoint 3-cells, and a compact 3-manifold with boundary $Y_\alpha^i \subset U_i$ such that

(1) $$A_\alpha^i = Y_\alpha^i \cup X_\alpha^i(1) \cup X_\alpha^i(2) \cup \ldots,$$

(2) $A_\alpha \cap Q$ is the union of a finite collection $\{D_\alpha(j)\}$ of disjoint disks and $X_\alpha^i(j) \cap Q = Q \cap \operatorname{Bd} X_\alpha^i(j) = D_\alpha(j)$, and

(3) $X_\alpha^i(j) \cap Y_\alpha^i = \operatorname{Bd} X_\alpha^i(j) \cap \operatorname{Bd} Y_\alpha^i$ is a disk $Z_\alpha^i(j)$.

Furthermore, if $\varepsilon > 0$ then there exists a positive integer j such that $\operatorname{Diam} Y_{m_1\ldots m_j}^i < \varepsilon$. The 3-manifolds-with-boundary $M_j^i = \bigcup A_{m_1\ldots m_j}^i = M_j \cap \operatorname{Cl} U_i$ form a defining sequence for the non-degenerate elements H^i of the upper semi-continuous decomposition G^i of E^3. If V is an open set in E^3 containing H^{i*} and $\varepsilon > 0$ then there exists an isotopy of E^3 onto E^3 that is the identity on $E^3 - V$ and takes each element of H^i to a set of diameter less than ε. We construct the isotopy by first taking j large enough to insure that each $A_{m_1\ldots m_j}^i$ lies in V and $\operatorname{Diam} Y_{m_1\ldots m_j}^i < \varepsilon$, and then pushing each 3-cell $X_{m_1\ldots m_j}^i$ into a thin shell neighborhood of $Z_{m_1\ldots m_j}^i$. It now follows from the shrinking argument in [4] that $E^3/G^i \approx E^3$. Thus,

$K_i = (E^3/G^i) - U_{1-i} + \infty$ is a crumpled cube and Theorem 1 is applicable to the sum $(E^3/G) + \infty$.

Since A_α is a cube-with-handles and $A_\alpha \cap Q$ is a union of disjoint disk $D_\alpha(1), D_\alpha(2), \ldots$, it follows that $A_\alpha^i = A_\alpha \cap \operatorname{Cl} U_i$ is also a cube-with-handles. For the decompositions of this paper, we also have that $A_{\alpha j}^i$ is embedded in A_α^i inessentially in the sense that each loop in $A_{\alpha j}^i$ is null-homotopic in A_α^i. These facts are used in Sections 2 and 3 to lift loops in E^3/G^i to E^3.

2. Decompositions that yield E^3.

In this section we specialize Theorem 1 to the decomposition spaces of Section 1. We then apply the theorem to several decompositions from the literature to show that the associated decomposition spaces are topologically E^3.

Notation. If D is a disk and J is a simple closed curve in D then $D(J)$ stands for the subdisk of D that J bounds. Also, if $\{J_i\}_{i=1}^n$ is a collection of disjoint simple closed curves in $\operatorname{Int} D(J)$ such that $D(J_i) \cap D(J_j) = \emptyset$ if $i \neq j$, then $D(J; J_1, \ldots, J_n)$ or $D(J; \{J_i\})$ represent the disk-with-holes in D whose boundary in $J \cup J_1 \cup \ldots \cup J_n$. By a loop u in the space X is meant a map $u\colon J \to X$, where J is a simple closed curve. For a map F, by $|F|$ is meant the image of the map F.

Definition. If $\mathfrak{C}$ and $\mathfrak{D}$ are collections of loops in a space M (we always assume that the preimage simple closed curves are disjoint) then $\mathfrak{C}$ *carries* $\mathfrak{D}$ *null-homotopically in* M if for each loop $u \in \mathfrak{D}$ there exists a map F from a disk-with-holes $D(J; J_1, \ldots, J_n)$ into M such that $F|J = u$ and $F|J_i \in \mathfrak{C}$ ($i = 1, \ldots, n$).

The following theorem gives a necessary and sufficient condition for the special decompositions of Section 1 to have a decomposition space equivalent to E^3. We use the notation of Section 1, and for $i = 0$ or 1 and each α, we let $\mathfrak{L}_\alpha^i$ be the collection of loops in $(\operatorname{Bd} A_\alpha^i) - \bigcup_j D_\alpha(j)$ null-homotopic in A_α^i, and let $\mathcal{M}_\alpha^i$ be any subcollection of $\mathfrak{L}_\alpha^i$ which carries $\mathfrak{L}_\alpha^i$ null-homotopically in $(\operatorname{Bd} A_\alpha^i) - \bigcup_j D_\alpha(j)$.

Theorem 2. *For the upper semi-continuous decompositions G of Section 1, $E^3/G \approx E^3$ if and only if for each α there exist an integer j and collections $\mathfrak{L}^i \subset \bigcup_{m_1\ldots m_j} \mathfrak{L}_{\alpha m_1\ldots m_j}^i$ ($i = 0, 1$) such that*

(1) *if $u_0 \in \mathfrak{L}^0$ and $u_1 \in \mathfrak{L}^1$ then $|u_0|$ and $|u_1|$ are not subsets of the same $A_{\alpha m_1\ldots m_j}$, and*

(2) *for $i = 0, 1$, $\mathfrak{L}^i$ carries $\mathcal{M}_\alpha^i$ null-homotopically in $A_\alpha^i - \bigcup_{m_1\ldots m_j} \operatorname{Int} A_{\alpha m_1\ldots m_j}^i$.*

Proof. We first show that the condition is necessary for $E^3/G \approx E^3$. Suppose $E^3/G \approx E^3$ and α is given. For $i = 0, 1$, since $\mathcal{M}_\alpha^i \subset \mathfrak{L}_\alpha^i$, we may assume without loss of generality that F_0^i is a map from the disjoint union $D_1^i \cup D_2^i \cup \ldots$ of disks D_j^i into $A_a^i - \bigcup_i D_a(j)$ such that $\mathcal{M}_a^i = \{F_0^i|D_j^i\}_j$. The images $|PF_0^0|$ and $|PF_0^1|$ may intersect in $P(H^* \cap A_a)$; however, by Theorem 1 there exist disjoint 0-dimensional F_σ sets F_0 and F_1 in $P(Q \cap H^*)$ such that $F_i \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ is 1-ULC $(i = 0, 1)$. Since F_i is 0-dimensional, and $F_i \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ is 1-ULC, there exists a finite collection $\{C_j^i\}$ of disjoint subdisk of $\operatorname{Int}(D_1^i \cup D_2^i \cup \ldots)$ such that $(PF_0^i)^{-1}(P(H^*)) \subset \operatorname{Int} C_j^i$ and $PF_0^i|\operatorname{Bd} C_j^i$ is null-homotopic in $F_i \cup \operatorname{Int} P(A_a^i)$. Thus, we may extend $PF_0^i|((D_1^i \cup D_2^i \cup \ldots) - \bigcup_j \operatorname{Int} C_j^i)$ to $D_1^i \cup D_2^i \cup \ldots$ to obtain a map F_1^i from $D_1^i \cup D_2^i \cup \ldots$ into $F_i \cup (P(A_a^i) \cap \operatorname{Int} P(\operatorname{Cl} U_i))$. Since $F_0 \cap F_1 = \emptyset$, we now have $|F_1^0| \cap |F_1^1| = \emptyset$. Since $|F_1^0| \cap |F_1^1| = \emptyset$, there exists an integer j such that if $|F_1^i| \cap P(A_{am_1\ldots m_j}) \neq \emptyset$ then $|F_1^{1-i}| \cap P(A_{am_1\ldots m_j}) = \emptyset$. We may assume that $|F_1^i|$ and $P(\operatorname{Bd} A_{am_1\ldots m_j m_{j+1}})$ are in general position so that there exists a finite collection $\{E_j^i\}$ of disjoint subdisks of $\operatorname{Int}(D_1^i \cup D_2^i \cup \ldots)$ such that $(F_1^i)^{-1}(P(H^*)) \subset \bigcup_j \operatorname{Int} E_j^i$, $F_1^i|\operatorname{Bd} E_j^i$ is a loop in $\bigcup_{m_1\ldots m_{j+1}} P(\operatorname{Bd} A^i_{am_1\ldots m_j m_{j+1}})$, and

$$F_1^i((D_1^i \cup D_2^i \cup \ldots) - \bigcup_j \operatorname{Int} E_j^i) \subset P(A_a^i - \bigcup_{m_1\ldots m_{j+1}} \operatorname{Int} A^i_{am_1\ldots m_{j+1}}) - P(\bigcup_j D_a^i).$$

Since P is a homeomorphism on $A_a^i - \bigcup_{m_1\ldots m_{j+1}} \operatorname{Int} A^i_{am_1\ldots m_{j+1}}$ and each $A_{am_1\ldots m_{j+1}}$ is inessentially embedded in $A_{am_1\ldots m_j}$, we may extend $P^{-1}F_1^i|((D_1^i \cup D_2^i \cup \ldots) - \bigcup_j \operatorname{Int} E_j^i)$ to $D_1^i \cup D_2^i \ldots$ to obtain a map F_2^i from $D_a^i \cup D_2^i \cup \ldots$ into $A_a^i - \bigcup_j D_a(j)$ such that $F_2^i(E_j^i)$ lies in the same $A^i_{am_1\ldots m_j}$ that contains $F_2^i(\operatorname{Bd} E_j^i)$. Since $A^i_{am_1\ldots m_{j+1}} \subset A^i_{am_1\ldots m_j}$, and j was chosen so that $|F_1^i| \cap P(A_{am_1\ldots m_j}) \neq \emptyset$ implies $|F_1^{1-i}| \cap P(A_{am_1\ldots m_j}) = \emptyset$, it follows that if $|F_2^i| \cap A_{am_1\ldots m_j} \neq \emptyset$ then $|F_2^{1-i}| \cap A_{am_1\ldots m_j} = \emptyset$. Some of the loops

$$F_2^i|J \subset D_1^i \cup D_2^i \cup \ldots: J \to \operatorname{Bd} A^i_{am_1\ldots m_j}$$

may not be null-homotopic in $A^i_{am_1\ldots m_j}$, but since $A^i_{am_1\ldots m_j}$ is a cube-with-handles there exists a 1-dimensional spine of $A^i_{am_1\ldots m_j}$ in general position with respect to $|F_2^i|$. Let h^i be a homeomorphism of E^3 onto E^3 that pushes the boundary of a small regular neighborhood of these 1-spines in the $A^i_{am_1\ldots m_j}$'s radially onto $(\bigcup_{m_1\ldots m_j} \operatorname{Bd} A^i_{am_1\ldots m_j})$. Let $F_3^i = h^i F_2^i$ and let $D_j'^i = D_j^i - (F_3^i)^{-1}(\bigcup_{m_1\ldots m_j} \operatorname{Int} A^i_{am_1\ldots m_j})$. It follows that we may take the collection of loops $\mathfrak{L}^i$ of the conclusion to be $\{(F_3^i|J)|J$ is a component of $\operatorname{Bd}(\bigcup_j D_j'^i) - \operatorname{Bd}(\bigcup_j D_j^i)\}$.

We next show that the condition is sufficient to insure that $E^3/G \approx E^3$. We will find disjoint 0-dimensional F_σ sets F_0 and F_1 that satisfy the hypothesis of Theorem 1 for the crumpled cubes $K_0 = P(\operatorname{Cl} U_0) + \infty$ and $K_1 = P(\operatorname{Cl} U_1) + \infty$. To assist us in the description of these F_σ sets we use our hypothesis to obtain a new defining sequence for the non-degenerate elements $H \subset G$ by deleting some of the A_a's. Let $B_{n_1} = A_{n_1}$, and let $N_1 = M_1 = \bigcup_{n_1} B_{n_1}$. Inductively, assume that B_β has been defined equal to some A_a. By hypothesis there exist an integer j and collection $\mathfrak{L}^0$ and $\mathfrak{L}^1$ satisfying (1) and (2). Let $B_{\beta 1}, B_{\beta 2}, \ldots$ be a relabeling of the elements of the collection $\{A_{am_1\ldots m_j}\}$ and let $J_\beta^i = \{k|\ |u| \subset B_{\beta k}$ for some $u \in \mathfrak{L}^i\}$. Let $N_j = \bigcup_{n_1\ldots n_j} B_{n_1\ldots n_j}$ and if $B_\beta = A_a$ let $\mathfrak{L}_\beta^{i*} = \mathfrak{L}_a^i$, $\mathcal{M}_\beta^{i*} = \mathcal{M}_a^i$, $B_\beta^i = A_a^i$, and $D_\beta^*(j) = D_a(j)$. Note that by (1) $J_\beta^0 \cap J_\beta^1 = \emptyset$ for all β, and in terms of the new defining sequence $\{N_j\}$ for H and the notation just introduced, the condition in Theorem 2 becomes the following.

(3) For each a there exist disjoint collections J_a^0 and J_a^1 of integers such that for $i = 0, 1$ and for each loop $u \in \mathcal{M}_a^{i*}$ there exists a map F from a disk-with-holes $D(J_0; J_1, \ldots, J_n)$ into $B_a^i - \bigcup_j \operatorname{Int} B_{aj}^i$ such that $F|J_0 = u$ and $F|J_j \in \mathfrak{L}^{i*}_{ak_j}$ for some $k_j \in J_a^i$ $(j = 1, \ldots, n)$.

Since for $i = 0, 1$, and for each a $\mathcal{M}_a^{i*}$ carries $\mathfrak{L}_a^{i*}$ null-homotopically in $(\operatorname{Bd} B_a^i) - \bigcup_j D_a^*(j)$, the existence of the map F in the following statement is established by induction using (3). For $i = 0, 1$, for each a, and each loop $u \in \mathfrak{L}_a^{i*}$ there exist (a) a disk D, (b) a collection $\{J_1\} \cup \{J_{1m_1}\} \cup \{J_{1m_1m_2}\} \cup \ldots$ of disjoint simple closed curves in D such that $D(J_1; \{J_{1m_1}\})$, $D(J_{1m_1}, \{J_{1m_1m_2}\}, \ldots$ are disk-with-holes and $D(J_1) \supset D(J_{1m_1}) \supset D(J_{1m_1m_2}) \supset \ldots$ is a null-sequence or a finite sequence, and (c) a map F from $D(J_1; \{J_{1m_1}\}) \cup (\bigcup_{m_1} D(J_{1m_1}; \{J_{1m_1m_2}\})) \cup \ldots$ into $B_a^i - \bigcup_j D_a^*(j)$ such that $F|J_1 = u$, $F|J_{1m_1\ldots m_j} \in \operatorname{Bd} B^i_{an_1\ldots n_j}$ where $n_k \in J^i_{an_1\ldots n_{k-1}}$ $(k = 1, \ldots, j)$, and

$$F(D(J_{1m_1\ldots m_j}; \{J_{1m_1\ldots m_{j+1}}\})) \subset (B^i_{an_1\ldots n_j} - \bigcup_k D^{*(k)}_{an_1\ldots n_j}) - \bigcup_{an_1\ldots n_{j+1}} \operatorname{Int} B_{an_1\ldots n_{j+1}}.$$

It is clear that we may extend the map PF to $D(J_1)$ by defining

$$PF(D(J_1) \cap D(J_{1m_1}) \cap D(J_{1m_1m_2}) \cap \ldots) = P(B_a^i \cap B_{an_1}^i \cap B_{an_1n_2}^i \cap \ldots)$$

for each infinite sequence $m_1, m_2, \ldots$ Thus,

(4) for $i = 0, 1$, each a, and each loop $u \in \mathfrak{L}_a^{i*}$, Pu is null-homotopic in $|Pu| \cup \operatorname{Int} P(B_a^i) \cup C_a^i$ where C_a^i is the compact 0-dimensional set $\{P(B_{an_1} \cap B_{an_1n_2} \cap \ldots)|\ n_j \in J_{an_1\ldots n_{j-1}}, j = 1, 2, \ldots\}$.

We now define F_0 and F_1. For $i = 0, 1$, let $C_k^i = \{P(B_{n_1} \cap B_{n_1n_2} \cap B_{n_1n_2n_3} \cap \ldots)|\ n_j \in J^i_{n_1\ldots n_{j-1}}$ if $j > k\}$. Note that for $i = 0, 1$, and $k = 1, 2, \ldots$, C_k^i is a compact 0-dimensional subset of $P(H^*)$, $C_k^i \subset C_{k+1}^i$, and $C_k^0 \cap C_k^1 = \emptyset$ since $J_\alpha^0 \cap J_\alpha^1 = \emptyset$ for all α. For $i = 0, 1$, let $F_i = \bigcup_{k=1}^{\infty} C_k^i$. It is clear that F_0 and F_1 are disjoint 0-dimensional F_σ sets in $P(H^*) \subset P(Q)$.

We next show that $F_i \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ is 1-ULC ($i = 0, 1$). Let $\varepsilon > 0$ be given. Since $P(\operatorname{Cl} U_i)$ is 1-ULC and $P(Q)$ is locally tame in $P(\operatorname{Cl} U_i)$ modulo $P(H^*)$, there exist a $\delta > 0$ such that δ-loops in $\operatorname{Int} P(\operatorname{Cl} U_i)$ are null-homotopic in an ε-subset of $P(H^*) \cup \operatorname{Int} P(\operatorname{Cl} U_i)$. Let f be a map of a disk D into an ε-subset of $P(H^*) \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ such that $f|\operatorname{Bd} D$ is a δ-loop in $\operatorname{Int} P(\operatorname{Cl} U_i)$. There exists an integer j such that

$$\operatorname{Diam}\big(f(D) \cup \bigcup \{P(B^i_{n_1\ldots n_j})|\ f(D) \cap P(B^i_{n_1\ldots n_j}) \neq \emptyset\}\big) < \varepsilon$$

and

$$f(\operatorname{Bd} D) \cap \big(\bigcup_{n_1\ldots n_j} P(B^i_{n_1\ldots n_j})\big) = \emptyset.$$

Since $P(H^*)$ is compact and 0-dimensional, there exists a finite collection $\{E_j\}$ of disjoint subdisk of $\operatorname{Int} D$ such that $f(D - \bigcup_j \operatorname{Int} E_j) \subset \operatorname{Int} P(\operatorname{Cl} U_i)$ and $f(E_j) \subset \bigcup_{n_1\ldots n_{j+1}} P(B^i_{n_j\ldots n_{j+1}})$. Since each $B^i_{n_1\ldots n_{j+1}}$ is inessentially embedded in $B^i_{n_1\ldots n_j}$ and P^{-1} is defined on $\operatorname{Int} P(\operatorname{Cl} U_i)$, it follows that $P^{-1}f|D - \bigcup_j \operatorname{Int} E_j$ may be extended to a map $f_1\colon D \to U_i$ such that $f_1(E_k) \subset B^i_{n_1\ldots n_j}$ for some $n_1, \ldots, n_j$. We may assume that $|f_1|$ is in general position with the 1-spine of any $B_{n_1\ldots n_j}$ such that $|f_1| \cap B^i_{n_1\ldots n_j} \neq \emptyset$. Let h be a homeomorphism of E^3 onto E^3 that takes the boundary of a small regular neighborhood of these 1-spines onto $\bigcup \operatorname{Bd} B^i_{n_1\ldots n_j}$ and let $f_2 = hf_1$. It follows that there exists a finite collection $\{D_j\}$ of disjoint subdisks of $\operatorname{Int} D$ such that $f_3(D - \bigcup D_j) \subset U_i - \bigcup B^i_{n_1\ldots n_j}$, $f_3|\operatorname{Bd} D = P^{-1}f|\operatorname{Bd} D$, and $f_3|\operatorname{Bd} D_k \in \mathfrak{L}^{i*}_{n_1\ldots n_j}$ for some $n_1, \ldots, n_j$. Since $C^i_{n_1\ldots n_j} \subset F_i$, it follows by (4) that there exists an extension of $Pf_3|D - \bigcup \operatorname{Int} D_j$ to a map $f_4\colon D \to F_i \cup \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ such that $f_4(D_k) \subset P(B^i_{n_1\ldots n_j})$. By our restrictions on $\operatorname{Diam} P(B^i_{n_1\ldots n_j})$, we have that $\operatorname{Diam}|f_4| < \varepsilon$. Thus, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that δ-loops in $\operatorname{Int} P(\operatorname{Cl} U_i)$ are null-homotopic in an ε-subset of $F_i \cup \operatorname{Int} P(\operatorname{Cl} U_i)$. It now follows that $F_i \cup \operatorname{Int} P(\operatorname{Cl} U_i)$ is 1-ULC by the argument in [9; Theorem 4.2].

We now apply Theorem 2 to several decompositions in the literature to show that the associated decomposition spaces are equivalent to E^3. Note that we may take $\mathcal{M}_\alpha^i$ to be representatives of a set of generators for the kernel of the natural homomorphism $j_*\colon \pi_1(\operatorname{Bd} A_\alpha^i - \bigcup_k D_\alpha(k)) \to \pi_1(A_\alpha^i)$ induced by injection $j\colon \operatorname{Bd} A_\alpha^i - \bigcup_k D_\alpha(k) \to A_\alpha^i$.

Figures 1–5 represent the iterative step for the defining sequences of decompositions G of E^3 whose decomposition spaces E^3/G have been established to be E^3 by other methods in [4], [1], [10], [14], and [7], re-

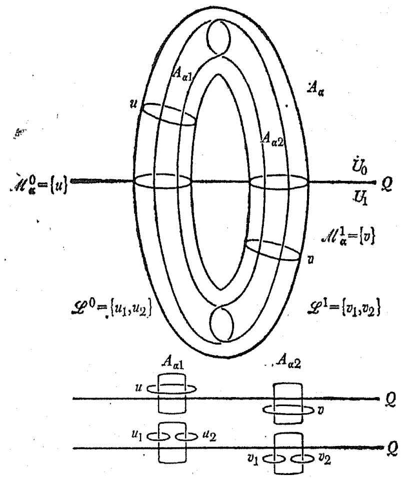


Fig. 1. R. H. Bing's Alexander horned sphere decomposition [4]

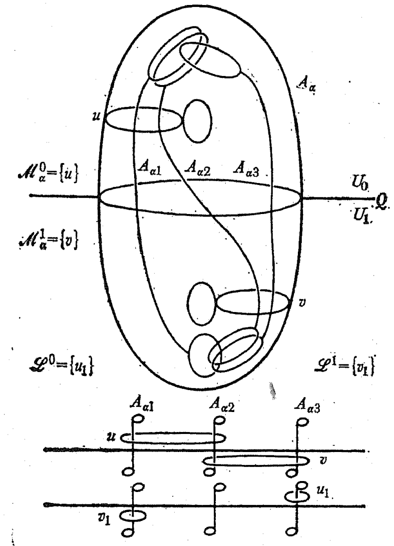


Fig. 2. E. H. Anderson's modification of the dog bone decomposition [1]

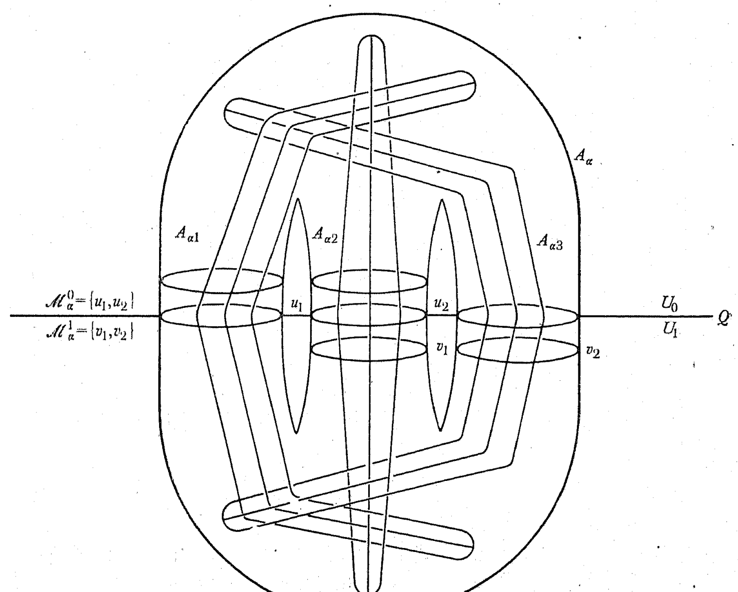


Fig. 3. L. O. Cannon's 3-horned sphere decomposition [10]

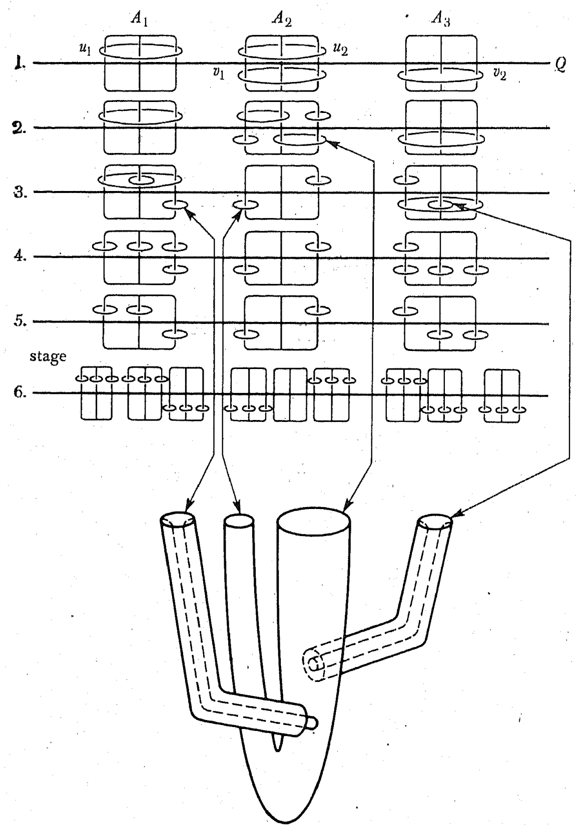


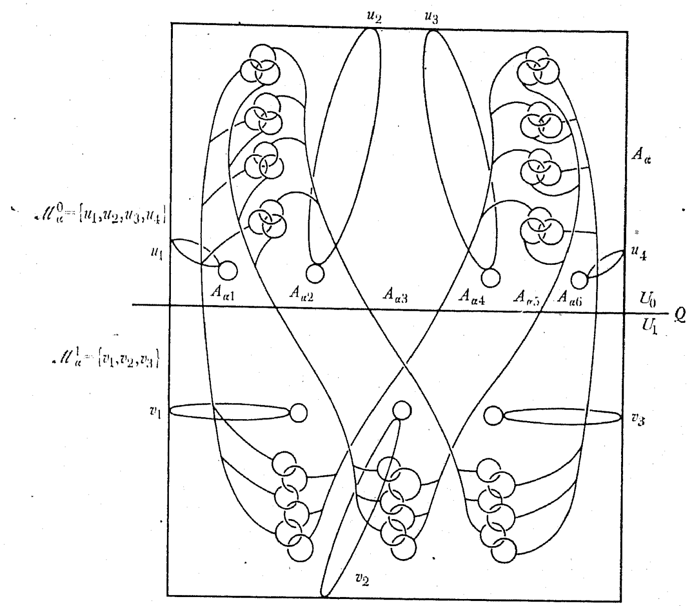
Fig. 3a

spectively. Our proof that $E^3/G \approx E^3$ is indicated by the hieroglyphics accompanying each figure. The loops in $\mathcal{M}_a^i$ are indicated in both the figure and Line 1 of the proof. The collection of loops shown above (below) the plane Q in the $(i+1)$-th line carries the collection of loops shown above (below) Q in the ith line null-homotopically in

$$A_a^0 - \bigcup_j \operatorname{Int} A_{aj}^0 (A_a^1 - \bigcup_j \operatorname{Int} A_{aj}^k) .$$

The last line of the proof shows the loops in $\mathfrak{L}^i$ required to apply Theorem 2.

Each line is intended to be a schematic representation of the A_{aj}'s in A_a with the horizontal line representing the plane Q. No linking of



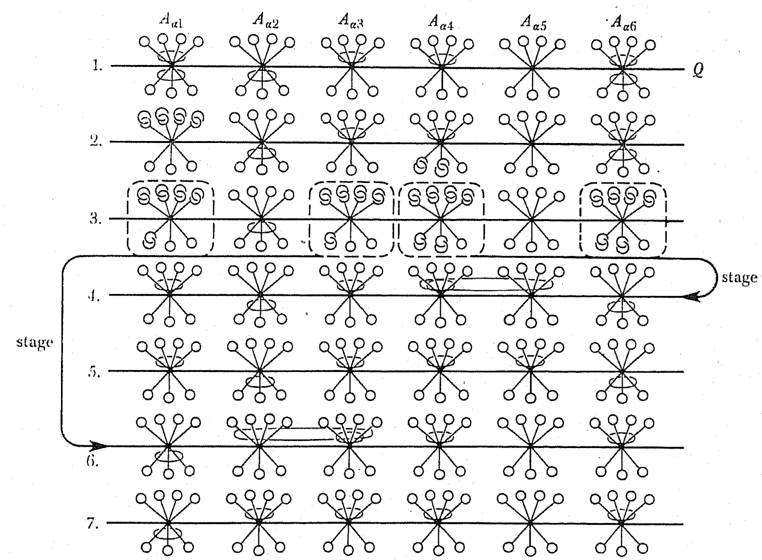


Fig. 4. L. F. McAuley's straight line interval decomposition [14]

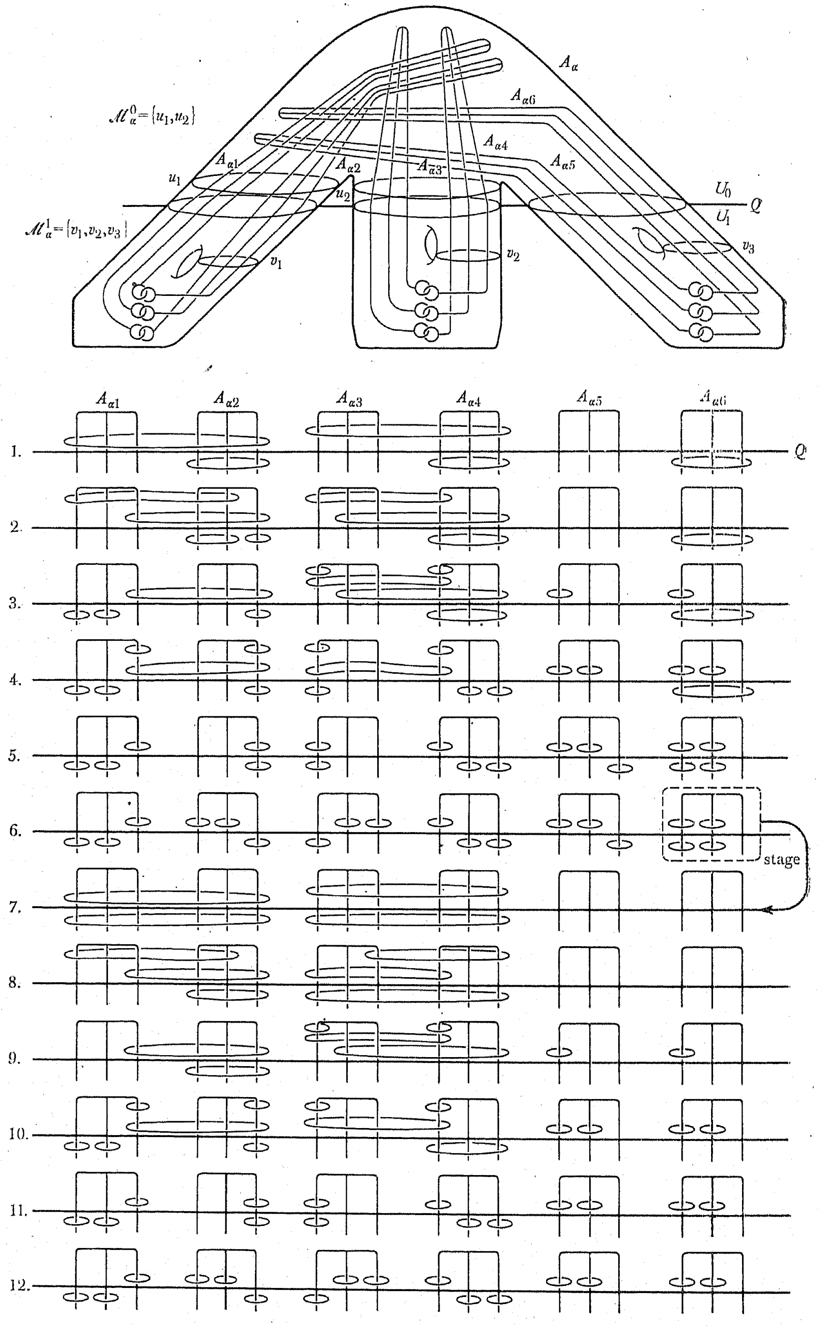


Fig. 5. R. H. Bing's three fingered decomposition [7]

the $A_{\alpha j}$'s is shown, because we only need to indicate the location of relevant loops. The reader should translate the information given in the hieroglyphics into the figure upon reading adjacent lines of the proof. For the reader's convenience, in most cases only one singular disk-with-holes is required to proceed from the top (bottom) of line i to the top (bottom) of line $i+1$. For example, in Figure 3 to proceed from the loops below Q in Line 2 to the loops below Q in Line 3 only the singular disk-with-holes in Figure 3a is required. For the other steps we leave the construction of the appropriate singular disks-with-holes to the reader; however, in all steps this is essentially easy since the location of the boundary components can be determined by the information in the hieroglyphics. The word "stage" is used in the proof to indicate that the loops in the next line are in the $A_{\alpha j}$'s as well as A_α. Since, for the decompositions of this paper, the embedding of the $A_{\alpha j}$'s in A_α is independent of α, the reader may also visualize these loops in the accompanying figure with an appropriate adjustment of notation.

3.0. Decompositions that fail to yield E^3. In this section we apply Theorem 2 to several decompositions in the literature to establish that the associated decomposition spaces are not topologically E^3. For each decomposition we find an α (since the embeddings of the $A_{\alpha j}$'s in A_α are independent of α, any α will suffice for the decompositions in this paper) and loops in $\mathfrak{L}^0_\alpha$ and $\mathfrak{L}^1_\alpha$ for which there exists no j and collections $\mathfrak{L}^0$ and $\mathfrak{L}^1$ satisfying (1) and (2) of Theorem 2. To accomplish this, we find a "pattern" of loops in $\mathfrak{L}^0_\alpha$ and $\mathfrak{L}^1_\alpha$ for which the hieroglyphic methods of Section 2 fail to produce a mismatch. To prove that a mismatch is impossible and we are not just poor players of the loop trading game, we use the concept of a regular map and reduce each decomposition to a combinatorial selection problem by using the following lemma.

Definition. Let A be a compact PL 3-manifold with boundary in E^3 and C be a compact set in $\mathrm{Bd}A$. A PL map f from a disk D into E^3 is *regular with respect to* (A, C) if

1. $f|\mathrm{Bd}D$ is a non-trivial loop in $(\mathrm{Bd}A)-C$,
2. $f(D)$ and $\mathrm{Bd}(A)$ are in general position,
3. there exists a collar B of $\mathrm{Bd}D$ in D such that $f(B) \subset A$, and
4. if K is a component (necessarily a simple closed curve by (2)) of $f^{-1}(f(\mathrm{Int}D) \cap \mathrm{Bd}A)$ then $f|K$ is trivial in $(\mathrm{Bd}A)-C$.

Let A be any one of the four PL 3-manifolds with boundary in Figure 6, let A^* be the indicated sub-manifold of A (A^* may have more than one component), and let C and C^* be the indicated compact subsets of $\mathrm{Bd}A$ and $\mathrm{Bd}A^*$, respectively (C and C^* are disks or unions of disjoint disks).

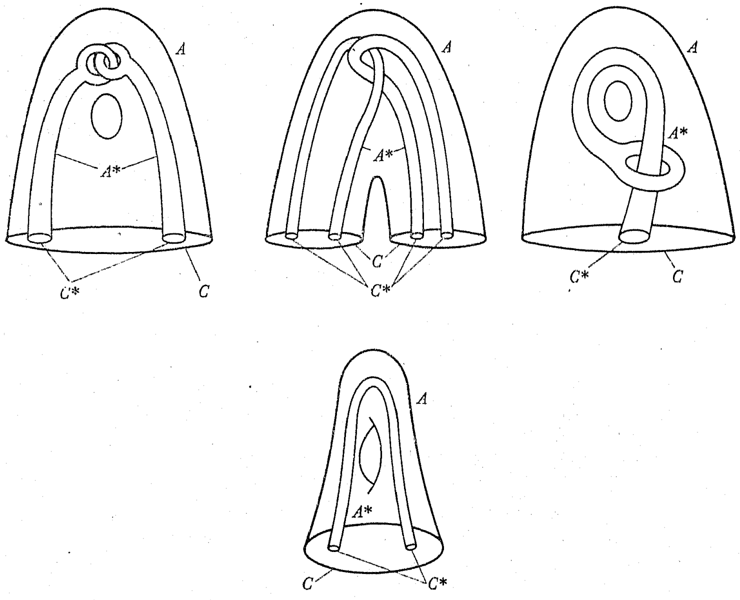


Fig. 6

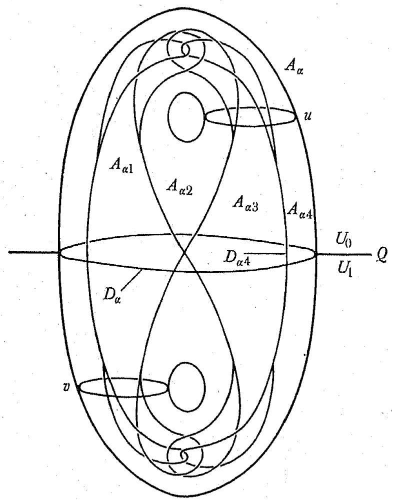


Fig. 7. R. H. Bing's dog bone decomposition [5]

LEMMA 3.0. *If f is a map from a disk D into E^3 such that f is regular with respect to (A, C), and $f(D)$ and* $\mathrm{Bd}\,A^*$ *are in general position, then there exists a subdisk D^* of D such that $f|D^*$ is regular with respect to (A^*, C^*).*

Proof. Since for each of the manifolds in Figure 6 the techniques of [6; Theorem 9, Theorem 11] are applicable, we only give an outline of the main steps in a proof. Step 1. Show that there exists some simple closed curve J in D such that $f|J$ is non-trivial in $(\mathrm{Bd}\,A^*)-C^*$. Step 2. Let D^* be the disk bounded by an inner most curve from Step 1. Show that there exists a collar B^* of $\mathrm{Bd}\,D^*$ in D^* such that $f(B^*)\subset A^*$.

3.1. The dog bone decomposition. Figure 7 represents the iterative step in R. H. Bing's [5] dog bone decomposition G of E^3. We use our techniques to show that E^3/G is not E^3. The proof depends on a solution to the following combinatorial selection problem.

SELECTION PROBLEM. Is it possible to select a point from each of the eight pairs of points $\{a_1^0, a_3^0\}$, $\{a_1^0, a_4^0\}$, $\{a_2^0, a_3^0\}$, $\{a_2^0, a_4^0\}$, $\{a_1^1, a_2^1\}$, $\{a_1^1, a_4^1\}$, $\{a_2^1, a_3^1\}$, and $\{a_3^1, a_4^1\}$ without selecting both end points from one of the four arcs $a_1, \ldots, a_4$ in Figure 7a?

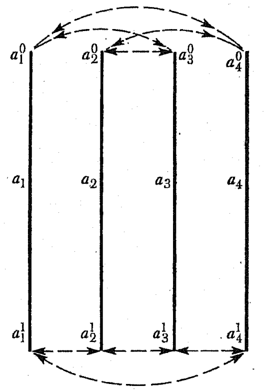


Fig. 7a. Selection problem

As the reader may easily determine the solution is negative. That is, there exists an arc a_j such that both end points a_j^0, a_j^1 of a_j are selected. We now give an interpretation of the selection problem in the dog bone decomposition.

If f_i $(i = 0, 1)$ is a map from a disk D_i into U_i such that f_i is regular with respect to (A_α^i, D_α) of Figure 7, and $f_i(D_i)$ and $\bigcup_j \mathrm{Bd}\,A_{\alpha j}$ are in general position, then, by Lemma 3.0, either there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{\alpha 1}^0, D_{\alpha 1})$ or there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{\alpha 3}^0, D_{\alpha 3})$. Also, there exist seven other linked pairs $\{A_{\alpha 1}^0, A_{\alpha 4}^0\}$, $\{A_2^0, A_3^0\}$, ..., $\{A_3^1, A_4^1\}$ in Figure 7 for which Lemma 3.0 is applicable. In the selection problem interpret the phrase "point a_j^i is selected" to mean "there exists a subdisk D_i^* of D_i such that $f_i|D_i^*$ is regular with respect to $(A_{\alpha j}^i, D_{\alpha j})$" and interpret arc a_j as dog bone $A_{\alpha j}$. The negative solution to the selection problem may then be interpreted as a proof to the following lemma.

LEMMA 3.1. *If f_i $(i = 0, 1)$ is a map from a disk D_i into U_i such that f_i is regular with respect to (A_α^i, D_α) of Figure 7, and $f_i(D_i)$ and $\bigcup_j \mathrm{Bd}\, A_{\alpha j}$ are in general position, then there exists an integer k and subdisks $D_i^* \subset D_i$ such that $f_i|D_i^*$ is regular with respect to $(A_{\alpha k}^i, D_{\alpha k})$ $(i = 0, 1)$.*

Suppose $E^3/G \approx E^3$. We show that Theorem 2 and Lemma 3.1 can not hold simultaneously and thus reach a contradiction. By Theorem 2, there exist maps f_i $(i = 0, 1)$ from a disk D_i into E^3 such that (1) $f_i(D_i) \subset U_i$, $f_0|\mathrm{Bd}\, D_0 = u$ and $f_1|\mathrm{Bd}\, D_1 = v$ of Figure 7, (2) f_i is regular with respect to A_α^i, (3) f_i and $\mathrm{Bd}\, A_{\alpha\beta}$ are in general position for each β, and (4) there exists an integer j such that $|f_i| \cap A_{\alpha m_1 \ldots m_j} = \emptyset$ if $|f_{1-i}| \cap A_{\alpha m_1 \ldots m_j} \neq \emptyset$. But, by applying Lemma 3.1 repeatedly, we find that there also exist nests $A_\alpha \supset A_{\alpha m_1} \supset A_{\alpha m_1 m_2} \supset \ldots \supset A_{\alpha m_1 m_2 \ldots m_j}$ and $D_i \supset D_i^1 \supset \ldots \supset D_i^j$ such that $f_i|D_i^k$ is regular with respect to $(A_{\alpha m_1 \ldots m_k}^i, D_{\alpha m_1 \ldots m_k})$ $(i = 0, 1$; $k = 1, \ldots, j)$. Thus, $|f_i| \cap A_{\alpha m_1 \ldots m_j} \neq \emptyset$ for $i = 0, 1$ contradicting (4).

Only the linking of the $A_{\alpha j}$'s in Figure 8a was used in the above analysis. Thus, the "looser" dog bone decomposition G indicated in Figure 8a has a decomposition space different from E^3. Lemma 3.1 is also true for the modifications of the dog bone decomposition given by E. H. Anderson [1] and indicated in Figures 8b and 8c (use Lemma 3.0 and a different selection problem for a proof). Thus, by the argument

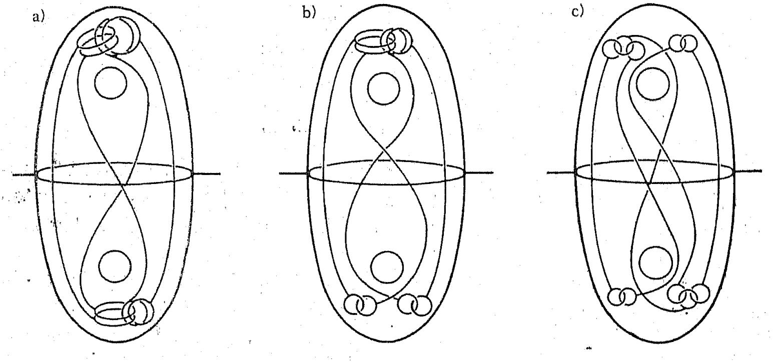


Fig. 8

above, neither of these decompositions has a decomposition space equal to E^3.

3.2. A decomposition of C. D. Bass and R. J. Daverman. Figure 9 represents the iterative step in the Bass-Daverman [3] decomposition G of E^3 used to show that self-universal crumpled cubes are not necessarily universal. We also show that E^3/G is not E^3. The proof depends on the following lemma concerning the embedding of $A_{\alpha 1}$ and $A_{\alpha 2}$ and A_α.

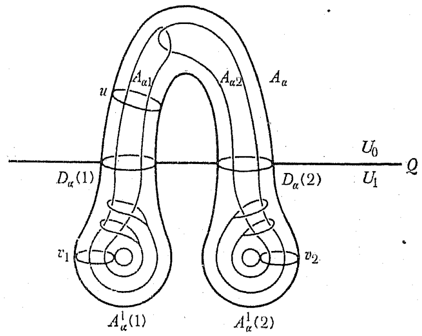


Fig. 9. A C. D. Bass - R. J. Daverman decomposition [3]

LEMMA 3.2. *If f_0 is a map from a disk D_0 into U_0, f_1 is a map from the union of a pair of disk $D_1(1)$ and $D_1(2)$ into U_1 such that $f_0, f_1|D_1(1)$, and $f_2|D_1(2)$ are regular with respect to $(A_\alpha^0, D_\alpha(1) \cup D_\alpha(2))$, $(A_\alpha^1(1), D_\alpha(1))$, and $(A_\alpha^1(2), D_\alpha(2))$, respectively, and $\bigcup_j \mathrm{Bd}\, A_{\alpha j}$ and $|f_0| \cup |f_1|$ are in general position, then there exists an integer $k \in \{1, 2\}$ and subdisks D_0^* of D_0 and $D_1^*(1)$, $D_1^*(2)$ of $D_1(1) \cup D_1(2)$ such that $f_0|D_0^*, f_1|D_1^*(1)$, and $f_1|D_1^*(2)$ are regular with respect to $(A_{\alpha k}^0, D_{\alpha k}(1) \cup D_{\alpha k}(2))$, $(A_{\alpha k}^1(1), D_{\alpha k}(1))$, and $(A_{\alpha k}^1(2), D_{\alpha k}(2))$, respectively.*

Proof. By Lemma 3.0 applied to $A_{\alpha 1}^0 \cup A_{\alpha 2}^0$, there exists an integer $k \in \{1, 2\}$ and subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{\alpha k}^0, D_{\alpha k}(1) \cup D_{\alpha k}(2))$. By Lemma 3.0 applied twice (once to $A_{\alpha k}^1(1)$ and once to $A_{\alpha k}^1(2)$) using regular map $f_1|D_1(k)$, there exists subdisks $D_1^*(1)$ and $D_1^*(2)$ of $D_1(k)$ such that $f_1|D_1^*(1)$ and $f_1|D_1^*(2)$ are regular with respect to $(A_{\alpha k}^1(1), D_{\alpha k}(1))$ and $(A_{\alpha k}^1(2), D_{\alpha k}(2))$, respectively.

The proof that E^3/G is different from E^3 is similar to our proof that the dog bone space is different from E^3. That is, we suppose $E^3/G \approx E^3$ and show that Theorem 2 and Lemma 3.2 can not hold simultaneously. By Theorem 2, there exist maps f_0 and f_1 satisfying the hypothesis of Lemma 3.2 with the additional requirements that $|f_0| \cup |f_1|$ and $\mathrm{Bd}\, A_{\alpha\beta}$ are in general position for all β, $f_0|\mathrm{Bd}\, D_0 = u$, $f_1|\mathrm{Bd}\, D_1(1) = v_1$, and

$f_1|\mathrm{Bd}\,D_1(2) = v_2$ of Figure 9, and there exists an integer j such that $|f_i| \cap \cap A_{am_1\ldots m_j} \neq \emptyset$ if $|f_{1-i}| \cap A_{am_1\ldots m_j} \neq \emptyset$. But, by applying Lemma 3.2 repeatedly, we find there also exist nests

$$A_a \supset A_{am_1} \supset A_{am_1m_2} \supset \ldots \supset A_{am_1m_2\ldots m_j}, \quad D_0 \supset D_0^1 \supset D_0^2 \supset \ldots \supset D_0^j,$$

and

$$D_1(1) \cup D_1(2) \supset D_1^1(1) \cup D_1^1(2) \supset D_1^2(1) \cup D_1^2(2) \supset \ldots \supset D_1^j(1) \cup D_1^j(2)$$

such that $f_0|D_0^k$, $f_1|D_1^k(1)$, and $f_1|D_1^k(2)$ are regular with respect to

$$(A^0_{am_1\ldots m_k}, D_{am_1^{(1)}\ldots m_k} \cup D_{am_1^{(2)}\ldots m_k}),$$

$$(A^1_{am_1^{(1)}\ldots m_k}, D_{am_1^{(1)}\ldots m_k}) \quad \text{and} \quad (A^1_{am_1^{(2)}\ldots m_k}, D_{am_1^{(2)}\ldots m_k}),$$

respectively ($k = 1, \ldots, j$). Thus, $|f_i| \cap A_{am_2\ldots m_j} \neq \emptyset$ for $i = 0, 1$ which contradicts the property of the integer j from Theorem 2.

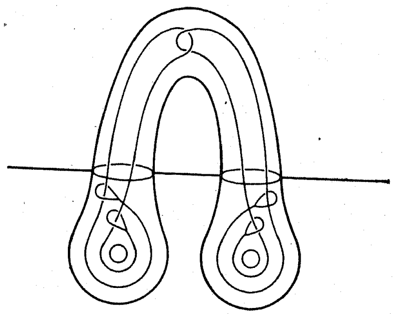


Fig. 10

Note that only the linking in Figure 10 was used in our analysis; hence, the decomposition indicated in Figure 10 has a decomposition space different from E^3.

3.3. R. H. Bing's straight line intervals decomposition. In [7] and [8], Bing describes an upper semi-continuous decomposition G of E^3 into points and straight line intervals and conjectures that the decomposition space is not E^3. We use our techniques to show that E^3/G is not E^3 and thus establish his conjecture.

The topological embeddings (see [7], [8] for the geometrical embeddings) of the A_{aj}'s in A_a are indicated in Figure 11. Since in our analysis we only use an appropriate four of the eight A_{aj}'s in A_a and all eight A_{aj}'s is somewhat difficult to sketch in A_a, Figure 11 shows only A_{a1}, A_{a2}, A_{a5}, and A_{a6} among $A_{a1}, \ldots, A_{a8}$. The linking and embeddings of the other

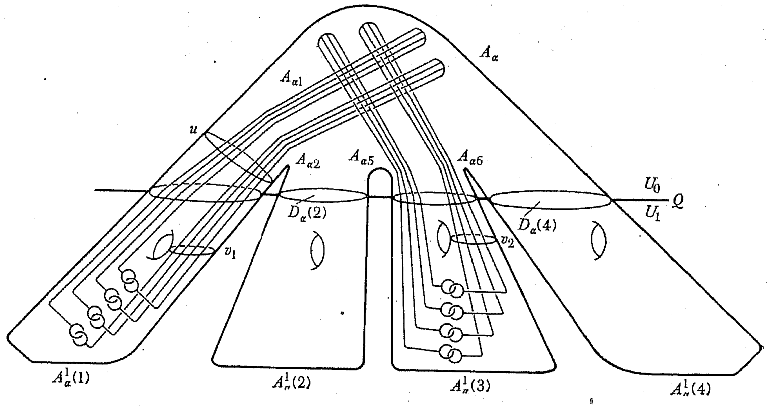


Fig. 11. R. H. Bing's straight line interval decomposition [7], [8]

four A_{aj}'s is symmetrical to the ones shown. In fact, the decomposition may be viewed as the four-fingered version of Bing's three-fingered decomposition indicated in Figure 5. If for $j = 1, \ldots, 8$ and $k, k' = 1, \ldots, 4$, $a(j, k, k')$ represents an arc in A^0_{aj} from $D_{aj}(k)$ to $D_{aj}(k')$ then the table in Figure 12a describes all the linking of the A^0_{aj}'s in A^0_a, that is, $a(j, k_1, k_1')$ links $a(j', k_2, k_2')$ if there exists an "*" in the (j, k_1, k_1') row and the (j', k_2, k_2') column.

The proof that E^3/G is not E^3 depends on a solution to the following combinatorial selection problem.

	(j',1,2)	(j',1,3)	(j',1,4)	(j',2,3)	(j',2,4)	(j',3,4)
(j,1,2)	*	*	*			
(j,1,3)		*	*	*	*	
(j,1,4)			*		*	*
(j,2,3)	*			*	*	
(j,2,4)	*	*			*	*
(j,3,4)		*		*		*

$j = 1, 3, 5$, and $j' = j+2, j+3, \ldots, 8$
or
$j = 2, 4, 6$, and $j' = j+1, j+2, \ldots, 8$

Fig. 12a

Selection problem. Is is possible to select an arc from each of the eighty pairs of arcs $\{\{a(j, k_1, k_1'), a(j', k_2, k_2')\} | j = 1, 2, j' = 5, 6$ and there exists an "*" in the (j, k_1, k_1') row and (j', k_2, k_2') column of the table in Figure 12a$\}$ and to select a point from each of eight pairs of points $\{\{a_1(k), a_2(k)\} | \; k = 1, \ldots, 4\} \cup \{\{a_5(k), a_6(k)\} | \; k = 1, \ldots, 4\}$ so that no arc in Figure 12b is selected if both of its end points are also selected?

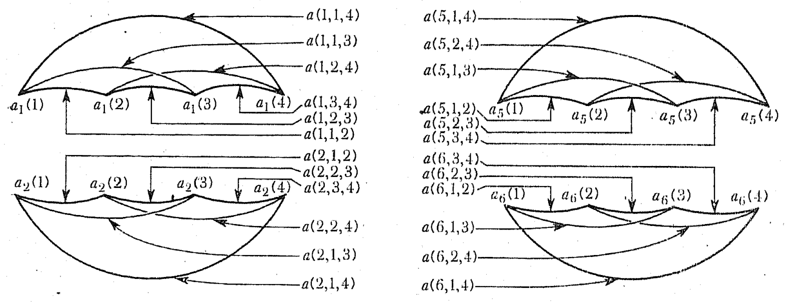


Fig. 12b. Selection problem

By arguing cases it is straight forward to check that the solution is negative. That is, an arc $a(j, k_1, k_1')$ must be selected so that both of its end points $a_j(k_1)$ and $a_j(k_1')$ are also selected. The selection problem we've isolated is just one of six equivalent problems associated with the table in Figure 12a. For example, to obtain an equivalent selection problem take $j = 3, 4$ and $j' = 7, 8$.

Definition. The pair (f_0, f_1) of maps satisfies *property* P with respect to A_a if there exist distinct integers k and k' in $\{1, 2, 3, 4\}$ such that

(1) f_0 is a PL map from a disk D_0 into U_0 such that f_0 is regular with respect to $(A_a^0, D_a(k) \cup D_a(k'))$, and

(2) f_1 is a PL map from the union of a pair of disks $D_1(1)$ and $D_1(2)$ into U_1 such that $f_1|D_1(1)$ and $f_1|D_1(2)$ are regular with respect to $(A_a^1(k), D_a(k))$ and $(A_a^1(k'), D_a(k'))$, respectively.

Lemma 3.3. *If the pair (f_0, f_1) of maps satisfies property P with respect to A_a and $|f_0| \cup |f_1|$ and $\bigcup_{j=1}^{8} \mathrm{Bd}\, A_{aj}$ are in general position then there exists an integer $j \in \{1, 2, \ldots, 8\}$ and subdisks D_0^* and D_0 and $D_1^*(1)$, $D_1^*(2)$ of $D_1(1) \cup D_1(2)$ such that $(f_0|D_0^*, f_1|D_1^*(1) \cup D_1^*(2))$ satisfies property P with respect to A_{aj}.*

Proof. As in the dog bone decomposition, the proof is just an interpretation of the selection problem using Lemma 3.0. Without loss of generality we assume that the pair of integers (k, k') in the definition of property P is $(1, 3)$. The required integer j is then one of the four integers $\{1, 2, 5, 6\}$. By Lemma 3.0 (push $D_{a1}(1)$, $D_{a1}(3)$, $D_{a6}(1)$, and $D_{a6}(2)$ very slightly into $\mathrm{Int}\, A_a^0$), either there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{a1}^0, D_{a1}(2) \cup D_{a1}(4))$ or there exists a subdisk D_0^* of D_0 such that $f_0|D_0^0$ is regular with respect to $(A_{a6}^0, D_{a6}(3) \cup D_{a6}(4))$. Also, there exist seventy-nine other linked pairs in Figure 11 and indicated in the table in Figure 12a for which Lemma 3.0 is applicable. In the selection problem interpret the phrase "arc $a(j, k_1, k_2')$ is selected "to mean" there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{aj}^0, D_{aj}(k_1) \cup D_{aj}(k_2'))$." By Lemma 3.0, either there exists a subdisk $D_1^*(1)$ of $D_1(1)$ such that $f_1|D_1^*(1)$ is regular with respect to $(A_{a5}^1(2), D_{a5}(2))$ or there exists a subdisk $D_1^*(1)$ of $D_1(1)$ such that $f_1|D_1^*(1)$ is regular with respect to $(A_{a6}^1(2), D_{a6}(2))$. Also, there exist seven other pairs $\{A_{a1}^1(1), A_{a3}^1(1)\}, \ldots, \{A_{a5}^1(4), A_{a6}^1(4)\}$ for which Lemma 3.0 is applicable. In the selection problem interpret the phrase "point $a_j(k)$ is selected" to mean "there exists a subdisk D_1^* of $D_1(1) \cup D_1(2)$ such that $f_1|D_1^*$ is regular with respect to $(A_{aj}^1(k), D_{aj}(k))$". The negative solution to the selection problem may then be interpreted to mean that exists an integer j and subdisks D_0^* of D_0 and $D_1^*(1)$, $D_1^*(2)$ of $D_1(1) \cup D_1(2)$ such that $(f_0|D_0^*, f_1|D_1^*(1) \cup D_1^*(2))$ satisfies property P with respect to A_{aj}.

The proof that E^3/G is different from E^3 is similar to the proofs in (3.1) and (3.2). That is, we suppose $E^3/G \approx E^3$ and show that Theorem 2 and Lemma 3.3 can not hold simultaneously. By Theorem 2, there exist a pair (f_0, f_1) of maps satisfying property P with respect to A_a such that $|f_0| \cup |f_1|$ and $\mathrm{Bd}\, A_{a\beta}$ are in general position for all β, $f_0|\mathrm{Bd}\, D_0 = u$, $f_1|\mathrm{Bd}\, D_1(1) = v_1$, and $f_1|\mathrm{Bd}\, D_1(2) = v_2$ of Figure 11, and there exists an integer j such that $|f_i| \cap A_{am_1 \ldots m_j} = \emptyset$ if $|f_{1-i}| \cap A_{am_1 \ldots m_j} \neq \emptyset$. But, by applying Lemma 3.3 repeatedly, we find that there also exist nests

$$A_a \supset A_{am_1} \supset A_{am_1m_2} \supset \ldots \supset A_{am_1m_2 \ldots m_j}, \qquad D_0 \supset D_0^1 \supset D_0^2 \supset \ldots \supset D_0^j,$$

and

$$D_1(1) \cup D_1(2) \supset D_1^1(1) \cup D_1^1(2) \supset D_1^2(1) \cup D_1^2(2) \supset \ldots \supset D_1^j(1) \cup D_1^j(2)$$

such that $f_0|D_0^k$, $f_1|D_1^k(1)$, and $f_1|D_1^k(2)$ are regular with respect to

$$(A^0_{am_1 \ldots m_k}, D_{am_1^{(ik)} \ldots m_k} \cup D_{am_1^{(jk)} \ldots m_k}),$$

$$(A^1_{am_1^{(ik)} \ldots m_k}, D_{am_1^{(ik)} \ldots m_k}) \quad \text{and} \quad (A^1_{am_1^{(jk)} \ldots m_k}, D_{am_1^{(jk)} \ldots m_k}),$$

respectively $(k = 1, \ldots, j)$. Thus, $|f_i| \cap A_{am_1 \ldots m_k} \neq \emptyset$ for $i = 0, 1$ which contradicts the property of the integer j from Theorem 2.

Steve Armentrout [2] was the first to show that there exists an upper semi-continuous decomposition G of E^3 into points and straight line intervals such that E^3/G is not topologically E^3. Since his decomposition is like Bing's except for many more fingers in the A_α's, the above analysis also shows that E^3/G is not E^3 for Armentrout's decomposition by ignoring redundant fingers.

3.4. A simpler straight line interval example. We give another example of an upper semi-continuous decomposition G of E^3 into straight line intervals and singletons such that the associated decomposition space

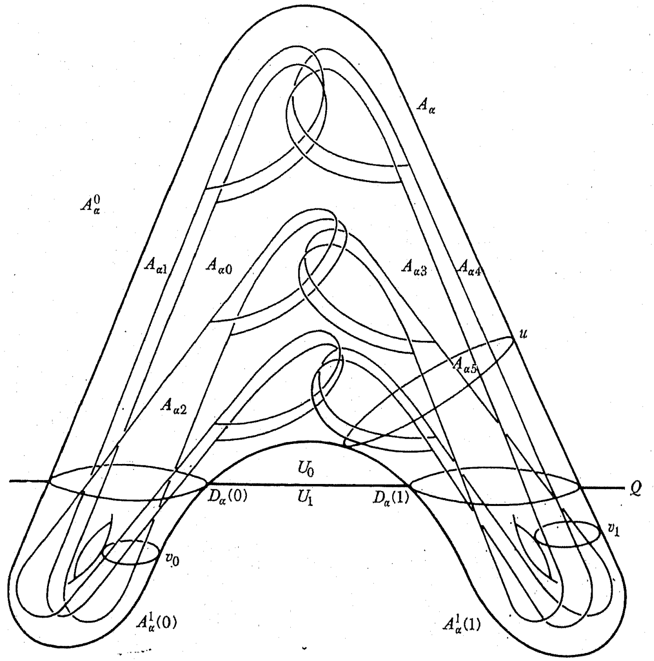


Fig. 13

is topologically different from E^3. The example is simple compared with the first examples given by Armentrout [2] and Bing [1]. Unlike these examples, our intervals are describable by cubes-with-two-handles and we require only six of these dog bones in the iterative step (see Fig. 13). R. B. Sher has shown [17, Theorem 6] that each toroidal decomposition of E^3 into intervals has a decomposition space equal to E^3. Thus, our example shows that Sher's result is best possible in the sense that no theorem like his exists for double toroidal decompositions.

3.4.1. A description of G. Let θ_0 and θ_1 be two horizontal planes in E^3 with θ_1 below θ_0. As in the examples of Armentrout and Bing, each non-degenerate element of G is a straight segment with one end in θ_0 and the other in θ_1, and each horizontal plane that intersects one of the segments will intersect the union of the segments in a topological Cantor set.

The union H^* of the collection H of nondegenerate elements of G is the intersection of a nested sequence of open sets $U_1 \supset U_2 \supset U_3 \supset \ldots$ Each U_i is a thin tubular neighborhood of a finite graph G_i. Figure 14 shows G_1 and we describe it as follows.

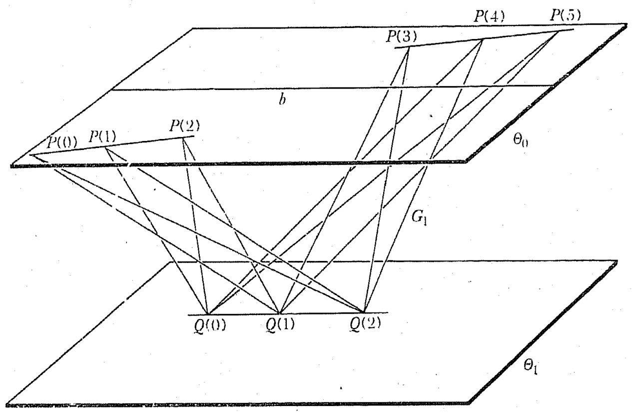


Fig. 14.

When A and B are two points in E^3, we use $[A, B]$ to denote the straight segment from A to B. Let $Q(0)$, $Q(1)$, $Q(2)$ be three colinear points in θ_1 with $Q(1)$ between $Q(0)$ and $Q(2)$. Let b be a line in θ_0 parallel to the line containing $Q(0)$, $Q(1)$, and $Q(2)$ and let $P(1)$ and $P(4)$ be points in θ_0 on opposite sides of b. Let $P(0)$, $P(2)$, $P(3)$, $P(5)$ be four points in θ_0 such that $P(1)$ and $P(4)$ are interior points of $[P(0), P(2)]$ and $[P(3), P(5)]$, respectively, and $[P(0), P(2)]$ and $[P(3), P(5)]$ lie on opposite sides of b. We choose $[P(0), P(2)]$ and $[P(3), P(5)]$ to be parallel and each to be approximately parallel (but not parallel) to b. $[P(3), P(5)]$ to be parallel and each to be approximately parallel to b. The finite graph G_1 is the union of the twelve segments $\{[P(i), Q(j)] \mid i = 0, 1, 2, 3, 4, 5;\ j = 0, 1, 2 \text{ and } j \neq i \bmod 3\}$ as shown in Figure 14. U_1 is a thin tubular neighborhood of G_1. We refer to the segment $[Q(0), Q(2)]$ as the *bend* in G_1, and to $[P(i), Q(i+1 \bmod 3)] \cup [P(i), Q(i-1 \bmod 3)]$ $(i = 0, \ldots, 5)$ as the *hair-pin* with bend at $P(i)$. Then G_1 is the union of six hair-pins with bends in θ_0.

We obtain G_2 by replacing each of the six hair-pins of G_1 by a copy of G_1. Each of the six copies of G_1 in G_2 will have its bend in θ_0 and near the bend of the hair-pin it replaces. The four copies of G_1 in G_2 that are near $Q(i)$ $(i = 0, 1, 2)$ entwine in a manner to be described presently. Before describing the six copies of G_1 in G_2 in detail, we remark that their union lies in U_1. We obtain G_3 by replacing each of the 36 hair-pins of G_2 by a copy of G_1 with the bend of the copy near the bend of the hair-pin it replaces. Continuing in this fashion we get a sequence of finite graphs $G_1, G_2, G_3, \ldots$ and a sequence of tubular neighborhoods $U_1, U_2, U_3, \ldots$ of these graphs such that $G_{i+1} \subset U_{i+1} \subset \operatorname{Cl} U_{i+1} \subset U_i$. Since for $j = 0$ or 1 the bends of the hair-pins of G_{i+1} lie in θ_{1-j} if the bends of the hair-pins of G_i lie in θ_j, and since the tubular neighborhoods U_1 get progressively thinner as i increases, the intersection of the U_i's is the union of a Cantor set of segments each with one end on θ_0 and the other on θ_1. These segments are the nondegenerate elements of the decomposition G.

The copy of G_1 in G_2 replacing the hair-pin $[P(i), Q(i+1 \bmod 3)] \cup \cup [P(i), Q(i-1 \bmod 3)]$ $(i = 0, 1, 2, 3, 4, 5)$ is denoted by $G(i)$. It is the union of the twelve segments $\{[P(i,j), Q(i,k)] \mid j = 0, 1, 2, 3, 4, 5;\ k = 0, 1, 2 \text{ and } k \neq j \bmod 3\}$ where the $Q(i,k)$'s are points in θ_0 near $P(i)$ and the $P(i,j)$'s are points in θ_1 with $P(i,0)$, $P(i,1)$, and $P(i,2)$ near $Q(i-1 \bmod 3)$ and $P(i,3)$, $P(i,4)$, and $P(i,5)$ near $Q(i+1 \bmod 3)$. We let $Q(i,1) = P(i)$ and require that $Q(i,0)$ and $Q(i,2)$ lie on opposite sides of $Q(i,1)$ on the line containing $P(0)$, $P(1)$, and $P(2)$ if $i = 0, 1, 2$ or on the line containing $P(3)$, $P(4)$, and $P(5)$ if $i = 3, 4, 5$. Before we describe in detail the location of the $P(i,j)$'s, i.e., the location of the bends of the 36 hair-pins of G_2, we remark that a pair of hair-pins of G_2 will link if and only if both "endpoints" of one lie on one side of the line b, and both "endpoints" of the other lie on the other side of b, and their bends are near the same $Q(i)$.

Let $Q'(i)$ $(i = 0, 1, 2)$ be a point slightly above $Q(i)$ and consider the pair of planes $\pi_0(i)$ and $\pi_1(i)$ $(i = 0, 1, 2)$ determined by

$$\{Q'(i), P(i-1 \bmod 3), P(i+1 \bmod 3)\}$$

and

$$\{Q'(i), P((i-1 \bmod 3)+3), P((i+1 \bmod 3)+3)\},$$

respectively, as shown in Figure 15. The first approximations to the twelve hair-pins of G_2 with bends near $Q(i)$ are shown in Figure 15 and are described as follows.

Let $m = i-1 \bmod 3$ or $i+1 \bmod 3$. Let $P'(m,0)$ be the point of intersection of the line through the points $Q(m,1)$ and $Q'(i)$ and the plane θ_1, and let $P'(m,1) = P'(m,2)$ be the point of intersection of the line through $Q(m,0)$ and $Q'(i)$, and θ_1. The points $P'(m,0)$, $P'(m,1)$, $P'(m,2)$ are first approximations to $P(m,0)$, $P(m,1)$, $P(m,2)$ (or $P(m,3)$, $P(m,4)$, $P(m,5)$ depending on the value of i and m). Let $n = (i-1 \bmod 3)+3$ or $(i+1 \bmod 3)+3$. Let $P'(n,2)$ be the point of intersection of the line through the points $Q(n,1)$ and $Q'(i)$ and the plane θ_1, and let $P'(n,0) = P'(n,1)$ be the point of intersection of the line through $Q(n,2)$ and

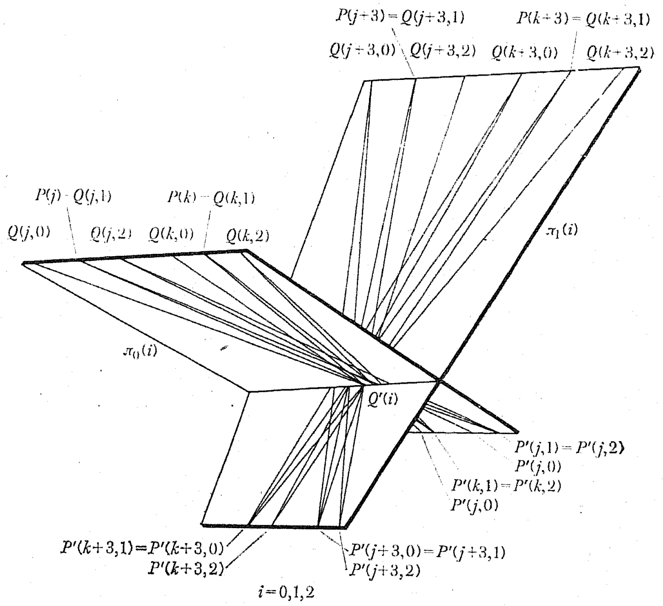


Fig. 15. j is the smaller of $i-1 \bmod 3$, $i+1 \bmod 3$ and k is the larger

$Q'(i)$, and θ_1. The points $P'(n,0)$, $P'(n,1)$, $P'(n,2)$ are first approximations to $P(n,0)$, $P(n,1)$, $P(n,2)$ (or $P(n,3)$, $P(n,4)$, $P(n,5)$ depending on the value of i and n). As we have indicated in Figure 15, it is convenient to let j be the smaller of $i-1 \bmod 3$, $i+1 \bmod 3$ and k be the larger.

The purpose of the next adjustment is to insure that each of the six relevant hair-pins in $\pi_0(i)$ link each of the six hair-pins in $\pi_1(i)$. We do this by pushing each of $P'(k,0)$, $P'(k,1) = P'(k,2)$, $P'(j,0)$, $P'(j,1) = P'(j,2)$ slightly to the left, (as in Figure 15) along the line $\theta_1 \cap \pi_0(i)$ or by pushing each of $P'(k+3,0) = P'(k+3,1)$, $P'(k+3,2)$, $P'(j+3,0) = P'(j+3,1)$, $P'(j+3,2)$ slightly to the right along the line $\theta_1 \cap \pi_1(i)$ or both. We denote the adjusted P''s with P''''s respectively. The remainder of the adjustments move the P''''s so little that the linking of the hair-pins we've established is unaltered.

We next push $P''(k+3,0) = P''(k+3,1)$ and $P''(k+3,2)$ slightly in front (as in Figure 15) of $\pi_1(i)$. Now, the hair-pins with bends at $P''(k+3,r)$ do not intersect the hair-pins with bends at $P''(j+3,s)$. Similarly, we push $P''(k,0)$, $P''(k,1) = P''(k,2)$ slightly in front of $\pi_0(i)$. We denote the adjusted P'''s by P''''s, respectively.

The P''''s are now moved by twisting each of the segments

$$[P'''(k+3,0), P'''(k+3,2)], \quad [P'''(j+3,0), P'''(j+3,2)],$$
$$[P'''(k,0), P'''(k,2)], \quad [P'''(j,0), P'''(j,2)]$$

slightly in θ_1, and pushing each of the points $P'''(k+3,1)$, $P'''(j+3,1)$, $P'''(k,1)$, $P'''(j,1)$ from $P'''(k+3,0)$, $P'''(j+3,0)$, $P'''(k,2)$, $P'''(j,2)$ respectively, to a point in the interior of $[P'''(k+3,0), P'''(k+3,2)]$, $[P'''(j+3,0), P'''(j+3,2)]$, $[P'''(k,0), P'''(k,2)]$, $[P'''(j,0), P'''(j,2)]$, respectively. The adjusted P''' are the required P's. This final adjustment is carried out in coordination with the adjustments near the other $Q(i)$'s, so that $[P(i,0), P(i,2)]$ and $[P(i,3), P(i,5)]$ are parallel for all i.

3.4.2. An intermediate step. To obtain a dog bone description of the nondegenerate elements of G we may thicken each hair-pin of the above description and drill holes at an appropriate spot in each end. In Figure 13 we do not emphasize straightness, but the topological embedding of the six solid double tori in a solid double torus is correctly illustrated. We use this dog bone description in this section to show that E^3/G is different from E^3.

The proof of the next lemma depends on a solution to the following combinatorial selection problem.

SELECTION PROBLEM. Is it possible to select a point from each of the twelve pairs of points $\{a_1^0, a_4^0\}$, $\{a_1^0, a_3^0\}$, $\{a_0^0, a_4^0\}$, $\{a_0^0, a_3^0\}$, $\{a_0^1, a_3^1\}$,

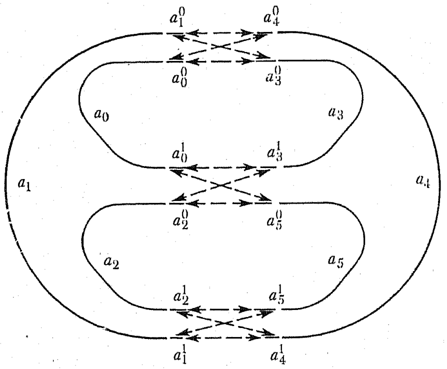


Fig. 16. Selection problem

$\{a_0^1, a_5^0\}$, $\{a_2^0, a_3^1\}$, $\{a_2^0, a_5^0\}$, $\{a_2^1, a_5^1\}$, $\{a_2^1, a_4^1\}$, $\{a_1^1, a_5^1\}$, and $\{a_1^1, a_4^1\}$ without selecting both end points from one of the six arcs $a_0, \ldots, a_5$ in Figure 16.

As the reader may easily determine the solution is negative. That is, there exists an arc a_j such that both end points a_j^0, a_j^1 of a_j are selected.

The iterated step in the description of the non-degenerate elements of G is indicated in Figure 13. The horizontal plane Q intersects the dog bone A_α in two disks $D_\alpha(0)$ and $D_\alpha(1)$. The components of $E^3 - Q$ are U_0 and U_1. The six dog bones in $\operatorname{Int} A_\alpha$ are denoted by $A_{\alpha 0}, \ldots, A_{\alpha 5}$ and in general the six dog bones in A_β are denoted by $A_{\beta 0}, \ldots, A_{\beta 5}$. The plane Q intersects each dog bones A_β in two disjoint disk $D_\beta(0)$ and $D_\beta(1)$. If A_β is any dog bone then $A_\beta \cap \operatorname{Cl} U_i$ is denoted by A_β^i, and if A_β^i has two components then we denote these components by $A_\beta^i(0)$ and $A_\beta^i(1)$ so that $A_\beta^i(j) \cap Q = D_\beta(j)$.

LEMMA 2. *If f_0 is a map from a disk D_0 into U_0, f_1 is a map from the union of a pair of disks $D_1(0)$ and $D_1(1)$ into U_1 such that $f_0, f_1|D_1(0)$, and $f_1|D_1(1)$ are regular with respect to $(A_\alpha^0, D_\alpha(0) \cup D_\alpha(1))$, $(A_\alpha^1(0), D_\alpha(0))$, and $(A_\alpha^1(1), D_\alpha(1))$, respectively, and $\bigcup_j \operatorname{Bd} A_{\alpha j}$ and $f_0(D_0) \cup f_1(D_1(0) \cup D_1(1))$ are in general position, then there exists an integer $k \in \{0, \ldots, 5\}$ and subdisks D_1^* of $D_1(0) \cup D_1(1)$, and $D_0^*(0)$, $D_0^*(1)$ of D_0 such that $f_0|D_0^*(0)$, $f_0|D_0^*(1)$, and $f_1|D_1^*$ are regular with respect to $(A_{\alpha k}^0(0), D_{\alpha k}(0))$, $(A_{\alpha k}^0(1), D_{\alpha k}(1))$, and $(A_{\alpha k}^1, D_{\alpha k}(0) \cup D_{\alpha k}(1))$, respectively.*

Proof. By Lemma 0, for each $k \in \{0, \ldots, 5\}$ there exists a subdisk D_1^* of $D_1(0) \cup D_1(1)$ such that $f_1|D_1^*$ is regular with respect to $(A_{\alpha k}^1, D_{\alpha k}(0) \cup D_{\alpha k}(1))$. By Lemma 0, either there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{\alpha 1}^0(0), D_{\alpha 1}(0))$ or there exists a subdisk D_0^* of D_0 such that $f_0|D_0^*$ is regular with respect to $(A_{\alpha 4}^0(0), D_{\alpha 4}(0))$. Also, there exist eleven other linked pairs $\{A_{\alpha 1}^0(0), A_{\alpha 3}^0(0)\}, \ldots, \{A_{\alpha 1}^0(1), A_{\alpha 4}^0(1)\}$ in Figure 13 for which Lemma 0 is applicable. In the selection problem interpret the phrase "point a_j^i is selected" to mean "there exists a subdisk $D_0^*(i)$ of D_0 such that $f_0|D_0^*(i)$ is regular with respect to $(A_{\alpha j}^0(i), D_{\alpha j}(i))$" and interpret "arc a_j" as "dog bone $A_{\alpha j}$". The negative solution to the selection problem completes the proof to the lemma.

We now use Theorem 2 to show that E^3/G is not E^3. Suppose $E^3/G \approx E^3$. By Teorem 2, there exist a map f_0 from a disk D_0 and a map f_1, from the union of disks $D_1(0)$ and $D_1(1)$ such that (1) $f_0(D_0) \subset U_0$, $f_1(D_1(0) \cup D_1(1)) \subset U_1$, $f_0|\operatorname{Bd} D_0 = u$, and $f_1|\operatorname{Bd} D_1(j) = v_j$ $(j = 0, 1)$ of Figure 13 (2) f_0, $f_1|D_1(0)$, and $f_1|D_1(1)$ are regular with respect to

$$(A_\alpha^0, D_\alpha(0) \cup D_\alpha(1)), \quad (A_\alpha^1(0), D_\alpha(0)) \quad \text{and} \quad (A_\alpha^1(1), D_\alpha(1)),$$

respectively, (3) $f_0(D_0) \cup f_1(D_1(0) \cup D_1(1))$ and $\operatorname{Bd} A_{\alpha\beta}$ are in general position for each β, and (4) there exists an even integer j such that $|f_i| \cap A_{\alpha m_1 \ldots m_j} = \emptyset$ if $|f_{1-i}| \cap A_{\alpha m_1 \ldots m_j} \neq \emptyset$ where $|f_0| = f_0(D_0)$ and $|f_1|$

$= f_1(D_1(0) \cup D_1(1))$. But, by applying Lemma 2 repeatedly, we find there also exist nests

$$A_a \supset A_{am_1} \supset A_{am_1m_2} \supset \ldots \supset A_{am_1m_2\ldots m_j}, \qquad D_0 \supset D_0^1(0) \cup D_0^1(1) \supset D_0^2 \supset \ldots \supset D_0^j,$$

and

$$D_1(0) \cup D_1(1) \supset D_1^1 \supset D_1^2(0) \cup D_1^2(1) \supset \ldots \supset D_1^j(0) \cup D_1^j(1)$$

such that $f_0(D_0(k))$, $f_1|D_1^k(0)$, and $f_1|D_1^k(1)$ are regular with respect to $(A^0_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k} \cup D_{am_1^{(1)}\ldots m_k})$, $(A^1_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k})$ and $(A^1_{am_1^{(1)}\ldots m_k}, D_{am_1^{(1)}\ldots m_k})$ respectively, if $k = 2, 4, \ldots, j$, and $f_0|D_0^k(0)$, $f_0|D_0^k(1)$, and $f_1|D_1^k$ are regular with respect to $(A^0_{am_1^{(0)}\ldots m_k}, D_{am_1^{(0)}\ldots m_k})$, $(A^0_{am_1^{(1)}\ldots m_k}, D_{am_1^{(1)}\ldots m_k})$, and $(A^1_{am_1\ldots m_k}, D_{am_1^{(0)}\ldots m_k} \cup D_{am_1^{(1)}\ldots m_k})$, respectively, if $k = 1, 3, \ldots, j-1$. Thus, $|f_i| \cap A_{am_1\ldots m_j} \neq \emptyset$ for $i = 0, 1$ and we have a contradiction to (4).

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Examples of statisch and finite-statisch AC-lattices

by

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Abstract. The purpose of this paper is to introduce a class of examples of statisch and finite-statisch atomistic lattices having the covering property. It will follow that any weakly modular atomistic lattice with the covering property is statisch, hence *M*-symmetric.

1. Basic terminology. Though the terminology will essentially follow that of [2], we introduce its more salient features here. An *AC-lattice* is an atomistic lattice with the covering property:

$$p \text{ an atom, } p \not\leqslant a \text{ implies } p \vee a \text{ covers } a.$$

An element of a lattice with 0 is called a *finite* element if it is either zero or the join of a finite number of atoms; an *infinite* element is simply an element that is not finite.

For complete atomistic lattices the notion of *statisch* was introduced by Wille in [3] and extended by the author in [1] to the more general concept of a *finite-statisch* lattice. In [2], p. 65, S. Maeda shows how these ideas may be generalized to an arbitrary atomistic lattice, and it is his idea that leads us to adopt the following definition:

Definition 1. Let L be an atomistic lattice. Then L is called *statisch* if p an atom, $p \leqslant a \vee b$ implies the existence of finite elements a_1 and b_1 such that $p \leqslant a_1 \vee b_1$, $a_1 \leqslant a$ and $b_1 \leqslant b$; it is called *finite-statisch* if p, q atoms with $p \leqslant q \vee a$ implies $p \leqslant q \vee a_1$ for some finite element $a_1 \leqslant a$.

It should be noted that any modular atomistic lattice as well as any compactly generated atomistic lattice is statisch, and any finite-modular AC-lattice ([2], Lemma 15.11, p. 65) is finite-statisch.

2. The examples. We now present a pair of theorems that provide a large number of examples of statisch and finite-statisch atomistic lattices. In connection with this we shall write $[a, \rightarrow]$ for an element a of