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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Substructures of reduced powers

by

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Abstract. THEOREM. *If $\mathcal{D}$ and $\mathcal{E}$ are filters on sets I and J respectively then for every structure $\mathfrak{A}$ we have $(\mathfrak{A}^I_{\mathcal{D}})^J_{\mathcal{E}} \equiv (\mathfrak{A}^I_{\mathcal{E}})^J_{\mathcal{D}}$.*

In this paper we prove the following theorem due to F. Galvin:

THEOREM. *For any filters $\mathcal{D}$ and $\mathcal{E}$ on I and J respectively, and for any structure $\mathfrak{A}$, we have*:

$$(\mathfrak{A}^I_{\mathcal{D}})^J_{\mathcal{E}} \equiv (\mathfrak{A}^J_{\mathcal{E}})^I_{\mathcal{D}}\,.$$

This theorem has been announced without proof by Galvin [1]. In his subsequent paper [2], Galvin omits the proof of this and some other theorems, but promises: *if they are still true, their proofs will appear elsewhere*. The double omission of the proof of this theorem inspired the author to find his own. Galvin's proof, as privately communicated to the author, is longer and also our Theorem 4, § 3, is stronger than the theorem above.

We shall also apply our technique to prove that free products of Boolean algebras preserve the elementary equivalence (see § 4). The author believes that this result is known, but he is unable to find it in the literature.

In § 1, we recall some basic definitions and theorems concerning limit reduced powers. Then, in § 2, we introduce some product operations on filters, which correspond to iteration of limit powers. Applications of the results from § 2 to Model Theory (with the proof of Galvin's Theorem mentioned above) are contained in § 3. Finally, § 4 contains applications of our technique to Boolean algebras.

§ 1. Limit reduced powers. We shall use standard model-theoretical notations and terminology, but for the readers' convenience we recall some definitions and facts concerning limit reduced powers.

Let $f \in A^I$. Then by $eq(f)$ we denote the following equivalence relation over I: $\{\langle i,j\rangle \in I^2\colon f(i) = f(j)\}$. Suppose $\mathcal{F}$ is a filter on $I\times I$. By $\mathfrak{A}^I|\mathcal{F}$ we denote the limit power of $\mathfrak{A}$, i.e., the substructure of the direct power $\mathfrak{A}^I$ with the universe

$$A^I|\mathcal{F} = \{f \in A^I\colon\ eq(f) \in \mathcal{F}\}\,.$$

If moreover $\mathfrak{D}$ is a filter on I, then by $\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F}$ we denote the limit reduced power of $\mathfrak{A}$, i.e., the substructure of the reduced power $\mathfrak{A}^I_{\mathfrak{D}}$ with the universe $A^I_{\mathfrak{D}}|\mathcal{F}$ which is the corresponding quotient of $A^I|\mathcal{F}$.

Finally, if $\boldsymbol{T}$ is a topological space, then $\mathfrak{A}^{\boldsymbol{T}}$ denotes the substructure of $\mathfrak{A}^T$, the universe of which consists of all continuous functions from $\boldsymbol{T}$ into the universe of $\mathfrak{A}$ regarded as a discrete space (T denotes the carrier of $\boldsymbol{T}$). It is easy to see that $\mathfrak{A}^{\boldsymbol{T}} = \mathfrak{A}^T|\mathcal{F}$, where $\mathcal{F}$ is the filter over T^2 generated by all closed-and-open decompositions of $\boldsymbol{T}$ (here we consider decompositions as equivalence relations). For more information see [3], [4] and [5]. $\mathbf{2}$ always denotes the two-element Boolean algebra.

We shall use the following fact:

THEOREM. (a) *If* $\mathbf{2}^I_{\mathfrak{D}}|\mathcal{F} \equiv \mathbf{2}^J_{\mathfrak{E}}|\mathfrak{G}$ *then* $\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F} \equiv \mathfrak{A}^J_{\mathfrak{E}}|\mathfrak{G}$.

(b) *If* $\mathbf{2}^I_{\mathfrak{D}}|\mathcal{F} \precsim \mathbf{2}^I_{\mathfrak{E}}|\mathfrak{G}$ *and* $\mathcal{F} \subseteq \mathfrak{G}$ *then* $\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F} \precsim \mathfrak{A}^I_{\mathfrak{E}}|\mathfrak{G}$.

This fact is an easy refinement of a theorem of Galvin ([2], Theorem 4.4), see also [4].

§ 2. Products of filters. By a partition of a given set $J \neq 0$ we mean a function $G\colon \lambda \to P(J)$ (where λ is a cardinal) which satisfies the following three conditions:

(i) *For each* $\xi < \lambda$, $G(\xi) \neq 0$.

(ii) $\bigcup_{\xi<\lambda} G(\xi) = J$.

(iii) *For any* $\xi < \eta < \lambda$, $G(\xi) \cap G(\eta) = 0$.

Let $D(J)$ denote the set of all partitions of a set J. With any $G \in D(J)$ we correlate the family

$$[G] = \{G(\xi)\colon\ \xi \in \mathrm{dom}(G)\}\,,$$

and we call it a *decomposition* of J. The family $[G]$ will also be called the *type of the partition* G. The set of all decompositions of J (i.e., the set of all types of partitions of J) will be denoted by $[D(J)]$.

Now, let us suppose that two non-void sets I and J are given. Using $D(I)$ and $D(J)$ we shall define some special types of partitions of $I\times J$.

Let $G \in D(J)$, $\mathrm{dom}(G) = \lambda$, and $\boldsymbol{F}\colon \lambda \to D(I)$. Let $(\boldsymbol{F}, G) = \{(\boldsymbol{F}(\xi))(\eta)\times G(\xi)\colon\ \xi < \lambda$ and $\eta \in \mathrm{dom}(\boldsymbol{F}(\xi))\}$. It is easy to see that $(\boldsymbol{F}, G)$ is a decomposition of $I\times J$. If $F \in D(I)$, $\mathrm{dom}(F) = \lambda$, and $\boldsymbol{G}\colon \lambda \to D(J)$, we put $(F, \boldsymbol{G}) = \{F(\xi)\times(\boldsymbol{G}(\xi))(\eta)\colon\ \xi < \lambda$ and $\eta \in \mathrm{dom}(\boldsymbol{G}(\xi))\}$. $(F, \boldsymbol{G})$ is also a decomposition of $I\times J$. Finally, let $\boldsymbol{F}$ be a constant function; say $\boldsymbol{F}(\xi) = F \in D(I)$, for $\xi \in \mathrm{dom}(\boldsymbol{F})$, then $(F, G) = (\boldsymbol{F}, G)$.

DEFINITION. Let H be a partition of the set $I\times J$. Then H is called:

(1) *regular* if there are $G \in D(J)$ and $\boldsymbol{F}\colon \mathrm{dom}(G) \to D(I)$ such that $[H] = (\boldsymbol{F}, G)$;

(2) *transposed* if there are $F \in D(I)$ and $\boldsymbol{G}\colon \mathrm{dom}(F) \to D(J)$ such that $[H] = (F, \boldsymbol{G})$;

(3) *symmetric* if there are $F \in D(I)$ and $G \in D(J)$ such that $[H] = (F, G)$.

PROPOSITION 1. *A partition H of $I\times J$ is symmetric iff it is regular and transposed.*

Let $\mathcal{A} \subseteq D(I)$ and $\mathcal{B} \subseteq D(J)$. The set $\mathrm{Reg}(\mathcal{A}, \mathcal{B})$ of all regular partitions of $I\times J$ determined by $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

$$H \in \mathrm{Reg}(\mathcal{A}, \mathcal{B}) \quad \text{iff} \quad [H] = (\boldsymbol{F}, G) \quad \text{for some } G \in \mathcal{B} \text{ and } \boldsymbol{F}\colon \mathrm{dom}(G) \to \mathcal{A}.$$

(The sets $\mathrm{Trans}(\mathcal{A}, \mathcal{B})$ of all transposed partitions of $I\times J$ and $\mathrm{Sym}(\mathcal{A}, \mathcal{B})$ of all symmetric partitions of $I\times J$ are defined in a similar way).

Let $\mathcal{F}$ be a filter on $I\times I$. Let us define

$$\overline{\mathcal{F}} = \{X \in \mathcal{F}\colon \text{ there is an equivalence } Y \in \mathcal{F} \text{ such that } Y \subseteq X\}\,.$$

It is easy to see that:

(1) $\overline{\mathcal{F}}$ is a filter on $I\times I$;

(2) $\overline{\mathcal{F}} \subseteq \mathcal{F}$ and $\overline{\overline{\mathcal{F}}} = \overline{\mathcal{F}}$;

(3) for any set A, we have $A^I|\mathcal{F} = A^I|\overline{\mathcal{F}}$.

By those facts, in investigations of limit powers we can assume (without loss of generality) that all filters under consideration have the property $\mathcal{F} = \overline{\mathcal{F}}$. Consequently a filter $\mathcal{F}$ is characterized by all equivalence relations in $\mathcal{F}$.

With any filter $\mathcal{F}$ on $I\times I$ we shall correlate a set $\hat{\mathcal{F}} \subseteq D(I)$ of partitions such that $H \in \hat{\mathcal{F}}$ iff $H \in D(I)$ and there is an equivalence relation $\varrho \in \mathcal{F}$ such that $[H] = I/\varrho$. On the other hand, if $\mathcal{A} \subseteq D(I)$, then the smallest filter $\mathcal{F}$ on $I\times I$ with the property $\mathcal{A} \subseteq \hat{\mathcal{F}}$ will be called the *filter generated* by $\mathcal{A}$.

If $f \in A^{I\times J}$ then by f_T we denote a function from $J\times I$ into A defined by $f_T(j, i) = f(i, j)$ for all $(j, i) \in J\times I$. Obviously $f_{TT} = f$. If $Z \subseteq A^{I\times J}$ then we put $Z_T = \{f_T\colon f \in Z\}$. We also identify the set $(A^I)^J$ with $A^{I\times J}$.

DEFINITION. Let $\mathcal{F}$ be a filter on $I\times I$ and $\mathfrak{G}$ a filter on $J\times J$. Then:

(1) $\mathcal{F} \otimes \mathfrak{G}$ denotes the filter on $(I\times J)\times(I\times J)$ generated by $\mathrm{Reg}(\hat{\mathcal{F}}, \hat{\mathfrak{G}})$.

(2) $\mathcal{F} \otimes_T \mathfrak{G}$ denotes the filter on $(I\times J)\times(I\times J)$ generated by $\mathrm{Trans}(\hat{\mathcal{F}}, \hat{\mathfrak{G}})$.

(3) $\mathcal{F} \otimes_S \mathcal{G}$ denotes the filter on $(I\times J)\times(I\times J)$ generated by $\mathrm{Sym}(\hat{\mathcal{F}}, \hat{\mathcal{G}})$.

Of course $\mathcal{F}\otimes_S \mathcal{G} \subseteq \mathcal{F}\otimes\mathcal{G}$ and $\mathcal{F}\otimes_S\mathcal{G}\subseteq\mathcal{F}\otimes_T\mathcal{G}$.

Immediately from the definition we have.

PROPOSITION 2. *If $\mathcal{F}$ is a filter on $I\times I$ and $\mathcal{G}$ a filter on $J\times J$, then for any set A we have:*

(i) $(A^I|\mathcal{F})^J|\mathcal{G} = A^{I\times J}|\mathcal{F}\otimes\mathcal{G}$,

(ii) $((A^J|\mathcal{G})^I|\mathcal{F})_T = A^{I\times J}|\mathcal{F}\otimes_T\mathcal{G}$.

A filter $\mathcal{F}$ on $I\times I$ is finitary iff for each equivalence relation $\varrho\in\mathcal{F}$, the set I/ϱ is finite. Let $\mathcal{S}_I$ be the greatest finitary filter on $I\times I$. For any filter $\mathcal{F}$ on $I\times I$, we put $\mathcal{F}_{\mathrm{fin}} = \mathcal{F}\cap\mathcal{S}_I$. We call $\mathcal{F}_{\mathrm{fin}}$ the *finitary part* of $\mathcal{F}$. Of course $\mathcal{F}_{\mathrm{fin}}\subseteq\mathcal{F}$, also $\mathcal{F}_{\mathrm{fin}}=\mathcal{F}$ iff $\mathcal{F}$ is finitary.

COROLLARY 3. *For any structure $\mathfrak{A}$ we have*

$$\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F}_{\mathrm{fin}} \prec \mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F}\,.$$

Proof. Indeed, since $\mathbf{2}$ is finite, we have $\mathbf{2}^I_{\mathfrak{D}}|\mathcal{F}_{\mathrm{fin}} = \mathbf{2}^I_{\mathfrak{D}}|\mathcal{F}$, also $\mathcal{F}_{\mathrm{fin}}\subseteq\mathcal{F}$; thus Theorem (b) from § 1 yields the corollary.

LEMMA 4. (i) *If $\mathcal{G}$ is a finitary filter on $J\times J$ then $\mathcal{F}\otimes\mathcal{G} = \mathcal{F}\otimes_S\mathcal{G}$, for any filter $\mathcal{F}$ on $I\times I$.*

(ii) *If $\mathcal{F}$ is a finitary filter on $I\times I$, then $\mathcal{F}\otimes_T\mathcal{G}=\mathcal{F}\otimes_S\mathcal{G}$, for any filter $\mathcal{G}$ on $J\times J$.*

(iii) *If both $\mathcal{F}$ and $\mathcal{G}$ are finitary then $\mathcal{F}\otimes\mathcal{G}=\mathcal{F}\otimes_T\mathcal{G}=\mathcal{F}\otimes_S\mathcal{G}$.*

(iv) $(\mathcal{F}\otimes\mathcal{G})\cap(\mathcal{F}\otimes_T\mathcal{G}) = \mathcal{F}\otimes_S\mathcal{G}$.

Proof. (i) The inclusion $\mathcal{F}\otimes_S\mathcal{G}\subseteq\mathcal{F}\otimes\mathcal{G}$ always holds. We shall prove that $\mathcal{F}\otimes\mathcal{G}\subseteq\mathcal{F}\otimes_S\mathcal{G}$.

Let us take an equivalence relation $\varrho\in\mathcal{F}\otimes\mathcal{G}$. Then there is a partition $H\in\mathrm{Reg}(\hat{\mathcal{F}},\hat{\mathcal{G}})$ and an equivalence relation ϱ' on $I\times J$ such that $[H]=(I\times J)/\varrho'$ and $\varrho'\subseteq\varrho$. Since $H\in\mathrm{Reg}(\hat{\mathcal{F}},\hat{\mathcal{G}})$, there are $G\in\hat{\mathcal{G}}$ and $\boldsymbol{F}$: $\mathrm{dom}(G)\to\hat{\mathcal{F}}$ such that $[H]=(\boldsymbol{F}, G)$. Now, since G is finitary, $\boldsymbol{F} = \langle \boldsymbol{F}(0), \ldots, \boldsymbol{F}(n-1)\rangle$, where $n=\mathrm{dom}(G)$ and $\boldsymbol{F}(0),\ldots,\boldsymbol{F}(n-1)\in\hat{\mathcal{F}}$. Let for each $i<n$, $\varrho_i\in\mathcal{F}$ be an equivalence relation over I such that $[\boldsymbol{F}(i)] = I/\varrho_i$. We have $\varrho''=\varrho_0\cap\ldots\cap\varrho_{n-1}\in\mathcal{F}$. Let $F\in D(I)$ be such that $[F]=I/\varrho''$; then $F\in\hat{\mathcal{F}}$. Let ϱ^* be an equivalence relation over $I\times J$ with the property that $(I\times J)/\varrho^* = (F, G)$. Then $\varrho^*\subseteq\varrho'\subseteq\varrho$ and $\varrho^*\in\mathcal{F}\otimes_S\mathcal{G}$, and thus $\varrho\in\mathcal{F}\otimes_S\mathcal{G}$. Consequently $\mathcal{F}\otimes\mathcal{G}\subseteq\mathcal{F}\otimes_S\mathcal{G}$. Q.E.D.

(ii) The proof of (ii) proceeds in the same way as the proof of (i).

(iii) It is an immediate corollary from (i) and (ii).

(iv) The inclusion $\mathcal{F}\otimes_S\mathcal{G}\subseteq(\mathcal{F}\otimes\mathcal{G})\cap(\mathcal{F}\otimes_T\mathcal{G})$ is always true, and so we shall prove that the converse inclusion holds. Let $\varrho\in(\mathcal{F}\otimes\mathcal{G})\cap(\mathcal{F}\otimes_T\mathcal{G})$. Take any $\varrho_1\in(\mathcal{F}\otimes\mathcal{G})$ and $\varrho_2\in(\mathcal{F}\otimes_T\mathcal{G})$ such that $\varrho_1\subseteq\varrho$ and $\varrho_2\subseteq\varrho$. Without loss of generality, we can assume that there are partitions $H_1\in\mathrm{Reg}(\hat{\mathcal{F}},\hat{\mathcal{G}})$ and $H_2\in\mathrm{Trans}(\hat{\mathcal{F}},\hat{\mathcal{G}})$ such that $[H_1]=(I\times J)/\varrho_1$ and $H_2=(I\times J)/\varrho_2$. Since $H_1\in\mathrm{Reg}(\hat{\mathcal{F}},\hat{\mathcal{G}})$, there are $G_1\in\hat{\mathcal{G}}$ and $\boldsymbol{F}_1$: $\mathrm{dom}(G_1)\to\hat{\mathcal{F}}$ such that $[H_1]=(\boldsymbol{F}_1, G_1)$. Similarly, $[H_2]=(F_2, \boldsymbol{G}_2)$ for some $F_2\in\hat{\mathcal{F}}$ and some $\boldsymbol{G}_2$: $\mathrm{dom}(F_2)\to\hat{\mathcal{G}}$. Take $H\in\mathrm{Sym}(\hat{\mathcal{F}},\hat{\mathcal{G}})$ such that $[H]=(F_2, G_1)$ and let $\varrho^*\in\mathcal{F}\otimes_S\mathcal{G}$ be an equivalence relation with the property that $(I\times J)/\varrho^* = [H]$. Now, it is easy to see that for any equivalence relation ϱ' over $I\times J$, if $\varrho_1\subseteq\varrho'$ and $\varrho_2\subseteq\varrho'$ then $\varrho^*\subseteq\varrho'$. (Let us remark that this does not mean that $\varrho_1\subseteq\varrho^*$ or $\varrho_2\subseteq\varrho^*$!). Consequently $\varrho^*\subseteq\varrho$ and therefore $\varrho\in\mathcal{F}\otimes_S\mathcal{G}$. Q.E.D.

§ 3. Applications to iterated powers of models. Let $\mathfrak{B}=\langle B, \ldots\rangle \subseteq \mathfrak{A}^{I\times J}$. By $\mathfrak{B}_T$ we denote the isomorphic copy of $\mathfrak{B}$ with the universe B_T. Of course $\mathfrak{B}_T\subseteq\mathfrak{A}^{J\times I}$ and $\mathfrak{B}_{TT}=\mathfrak{B}$.

THEOREM. 1. *If $\mathcal{F}$ and $\mathcal{G}$ are finitary filters, then for any model $\mathfrak{A}$ we have*

$$(\mathfrak{A}^I|\mathcal{F})^J|\mathcal{G} \cong (\mathfrak{A}^J|\mathcal{G})^I|\mathcal{F}\,.$$

Moreover $(\mathfrak{A}^I|\mathcal{F})^J|\mathcal{G} = ((\mathfrak{A}^J|\mathcal{G})^I|\mathcal{F})_T$.

Proof. By Proposition 2(i), § 2, we have $(A^I|\mathcal{F})^J|\mathcal{G} = A^{I\times J}|(\mathcal{F}\otimes\mathcal{G})$. Similarly $((A^J|\mathcal{G})^I|\mathcal{F})_T = A^{I\times J}|(\mathcal{F}\otimes_T\mathcal{G})$. But, by Lemma 4(iii), § 2, we have $\mathcal{F}\otimes\mathcal{G}=\mathcal{F}\otimes_T\mathcal{G}$. Thus our theorem follows.

THEOREM 2. *Let $\mathfrak{B}_1=(\mathfrak{A}^I|\mathcal{F})^J|\mathcal{G}$ and $\mathfrak{B}_2=(\mathfrak{A}^J|\mathcal{G})^I|\mathcal{F}$. Then there is a model $\mathfrak{C}$ such that $\mathfrak{C}\prec\mathfrak{B}_1$ and $\mathfrak{C}_T\prec\mathfrak{B}_2$. Consequently $\mathfrak{B}_1\equiv\mathfrak{B}_2$.*

Proof. By Corollary 3, § 2, we have

$$\mathfrak{B}_1 \succ (\mathfrak{A}^I|\mathcal{F}_{\mathrm{fin}})^J|\mathcal{G} \succ (\mathfrak{A}^I|\mathcal{F}_{\mathrm{fin}})^J|\mathcal{G}_{\mathrm{fin}}$$

and

$$\mathfrak{B}_2 \succ (\mathfrak{A}^J|\mathcal{G}_{\mathrm{fin}})^I|\mathcal{F} \succ (\mathfrak{A}^J|\mathcal{G}_{\mathrm{fin}})^I|\mathcal{F}_{\mathrm{fin}}\,.$$

But the filters $\mathcal{F}_{\mathrm{fin}}$ and $\mathcal{G}_{\mathrm{fin}}$ satisfy the hypothesis of Theorem 1. Consequently, putting $\mathfrak{C}=(\mathfrak{A}^I|\mathcal{F}_{\mathrm{fin}})^J|\mathcal{G}_{\mathrm{fin}}$ we get the required model.

THEOREM 3 (Galvin). *For any model $\mathfrak{A}$, we have $(\mathfrak{A}^I_{\mathcal{F}})^J_{\mathcal{G}}\equiv(\mathfrak{A}^J_{\mathcal{G}})^I_{\mathcal{F}}$.*

Proof. Let us consider the Boolean algebras $\mathcal{B}_1=\mathbf{2}^I_{\mathcal{F}}$ and $\mathcal{B}_2=\mathbf{2}^J_{\mathcal{G}}$. By the Stone Representation Theorem there are sets I_1 and J_1 and filters $\mathcal{F}_1$ on $I_1\times I_1$ and $\mathcal{G}_1$ on $J_1\times J_1$ such that $\mathcal{B}_1\cong\mathbf{2}^{I_1}|\mathcal{F}_1$ and $\mathcal{B}_2\cong\mathbf{2}^{J_1}|\mathcal{G}_1$. Now, by Theorem (a), § 1, we have $\mathfrak{A}^I_{\mathcal{F}}\equiv\mathfrak{A}^{I_1}|\mathcal{F}_1$ and $\mathfrak{A}^J_{\mathcal{G}}\equiv\mathfrak{A}^{J_1}|\mathcal{G}_1$, for any model $\mathfrak{A}$. Consequently $(\mathfrak{A}^I_{\mathcal{F}})^J_{\mathcal{G}}\equiv(\mathfrak{A}^{I_1}|\mathcal{F}_1)^J_{\mathcal{G}}\equiv(\mathfrak{A}^{I_1}|\mathcal{F}_1)^{J_1}|\mathcal{G}_1$ and $(\mathfrak{A}^J_{\mathcal{G}})^I_{\mathcal{F}}\equiv(\mathfrak{A}^{J_1}|\mathcal{G}_1)|^{I_1}\mathcal{F}_1$. Thus the result follows from Theorem 2.

It is possible to generalize Theorem 3 in the direction suggested by Theorem 2.

THEOREM 4. *Let $\mathfrak{B}_1 = (\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{F})^J_{\mathcal{E}}|\mathcal{G}$ and $\mathfrak{B}_2 = (\mathfrak{A}^J_{\mathcal{E}}|\mathcal{G})^I_{\mathfrak{D}}|\mathcal{F}$. Then there is a model $\mathfrak{C}$ such that $\mathfrak{C} \prec \mathfrak{B}_1$ and $\mathfrak{B}_2$ contains an elementary submodel which is an isomorphic copy of $\mathfrak{C}$.*

We shall give a proof of a weaker result. The proof of Theorem 4 can be obtained by some inessential modifications of the proof of Theorem 4′.

THEOREM 4′. *Let $\mathfrak{B}_1 = (\mathfrak{A}^I_{\mathfrak{D}})^J_{\mathcal{E}}$ and $\mathfrak{B}_2 = (\mathfrak{A}^J_{\mathcal{E}})^I_{\mathfrak{D}}$. Then there is a model $\mathfrak{C}$ such that $\mathfrak{C} \prec \mathfrak{B}_1$ and $\mathfrak{B}_2$ contains an elementary submodel which is an isomorphic copy of $\mathfrak{C}$.*

Proof. Let $\mathfrak{C}_1 = (\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{S}_I)^J_{\mathcal{E}}|\mathcal{S}_J$ and $\mathfrak{C}_2 = (\mathfrak{A}^J_{\mathcal{E}}|\mathcal{S}_J)^I_{\mathfrak{D}}|\mathcal{S}_I$. Obviously, by Corollary 3, § 2, we have $\mathfrak{C}_1 \prec \mathfrak{B}_1$ and $\mathfrak{C}_2 \prec \mathfrak{B}_2$. We are going to show that $\mathfrak{C}_1 \cong \mathfrak{C}_2$.

Let $K = \beta I$ and $L = \beta J$ (βX denotes the Cech–Stone compactification of a discrete topological space X). Let us remark that for any structure $\mathfrak{A}$ we have $\mathfrak{A}^I|\mathcal{S}_I \cong \mathfrak{A}^K$ and similarly $\mathfrak{A}_J|\mathcal{S}_J \cong \mathfrak{A}^L$. Let $\mathfrak{D}'$ be the filter closed-and-open subsets of K defined by $X \in \mathfrak{D}'$ iff there is $Y \in \mathfrak{D}$ such that $X = \overline{Y}^K$. Let $D = \bigcap \mathfrak{D}'$. In the same way we define $\mathcal{E}'$ and E. Then D and E are closed subspaces of K and L respectively. Moreover, by the definition of reduced powers, for any structure $\mathfrak{A}$ we have $\mathfrak{A}^I_{\mathfrak{D}}|\mathcal{S}_I \cong \mathfrak{A}^D$ and similarly $\mathfrak{A}^J_{\mathcal{E}}|\mathcal{S}_J \cong \mathfrak{A}^E$. Consequently $\mathfrak{C}_1 \cong (\mathfrak{A}^D)^E$ and $\mathfrak{C}_2 \cong (\mathfrak{A}^E)^D$. Therefore $\mathfrak{C}_1 \cong \mathfrak{C}_2$. Thus putting $\mathfrak{C} = \mathfrak{C}_1$ we get Theorem 4′.

§ 4. Applications to Boolean algebras.

THEOREM 1. *Let $\mathcal{F}$ be a finitary filter on $I \times I$ and $\mathcal{G}$ a finitary filter on $J \times J$. Then*

$$(2^I|\mathcal{F})^J|\mathcal{G} \cong (2^I|\mathcal{F}) * (2^J|\mathcal{G}),$$

where $*$ denotes the free product of Boolean algebras.

Proof. Let $\mathcal{B}_1 = 2^I|\mathcal{F}$, $\mathcal{B}_2 = 2^J|\mathcal{G}$ and $\mathcal{B} = 2^{I\times J}|\mathcal{F} \otimes \mathcal{G}$. We shall construct isomorphisms $h_1\colon \mathcal{B}_1 \to \mathcal{B}$, $h_2\colon \mathcal{B}_2 \to \mathcal{B}$ such that $\mathcal{B} = h_1(\mathcal{B}_1) * h_2(\mathcal{B}_2)$.

Let us define $h_1(X) = X \times J$ for each $X \in \mathcal{B}_1$ and $h_2(X) = I \times X$ for each $X \in \mathcal{B}_2$. Then it is easy to see that h_1 and h_2 are isomorphisms of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively into $\mathcal{B}$ and $h_1(\mathcal{B}_1) \cup h_2(\mathcal{B}_2)$ generates $\mathcal{B}$. Moreover, for any $Y_1 \in h_1(\mathcal{B}_1)$ and $Y_2 \in h_2(\mathcal{B}_2)$, if $Y_1 \neq 0 \neq Y_2$ then $Y_1 \cap Y_2 \neq 0$. Thus $\mathcal{B}$ is the free product of $h_1(\mathcal{B}_1)$ and $h_2(\mathcal{B}_2)$.

COROLLARY 2. *Let $\mathcal{B}$ be a Boolean algebra. Then for any filter $\mathcal{F}$ over I we have $\mathcal{B}^I_{\mathcal{F}} \equiv \mathcal{B} * 2^I_{\mathcal{F}}$.*

COROLLARY 3. *If $\mathcal{B}_1 \equiv \mathcal{B}_2$ and $\mathcal{B}'_1 \equiv \mathcal{B}'_2$ then $\mathcal{B}_1 * \mathcal{B}'_1 \equiv \mathcal{B}_2 * \mathcal{B}'_2$.*

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