THEOREM 7. *Let $X$ be a $T_1$ space, and let $\mathfrak{F} = \{F(\alpha)|\ \alpha \in A\}$ be a closure-preserving cover of $X$ such that for each $\alpha \in A$, $F(\alpha)$ is finite. Then $X$ is $\sigma$-metacompact.*

Proof. For each $n$, let $X(n) = \bigcup \{F(\alpha)|\ |F(\alpha)| \leqslant n\}$. Since $\mathfrak{F}$ is closure-preserving, each set $X(n)$ is closed. By Theorem 7, each $X(n)$ is metacompact, hence $X$ is $\sigma$-metacompact.

COROLLARY 8. *Let $X$ be collectionwise-normal and $T_1$. Suppose $X$ has a closure-preserving cover consisting of finite sets. Then $X$ is paracompact.*

Proof. By Theorem 7, $X$ is $\sigma$-metacompact. By Corollary 3, $X$ is $\theta$-refinable. By Theorem (iii) of [10], $X$ is paracompact.

**References**

[1] E. Michael, *Point-finite and locally-finite coverings*, Can. J. Math. 7 (1955), pp. 275–279.
[2] H. Potoczny, *Closure-preserving families of compact sets*, General Topology and Its Applications (3) (1973), pp. 243–248.
[3] — *On a problem of Tamano*, Fund. Math. 75 (1972), pp. 29–31.
[4] — *A non-paracompact space which admits a closure-preserving cover of compact sets*, Proc. Amer. Math. Soc. 32 (1972), pp. 309–311.
[5] H. Tamano, *A characterization of paracompactness*, Fund. Math. 72 (1972), pp. 189–311.
[6] R. Telgársky, *Concerning two covering properties*, Proc. Japan Acad. (in press).
[7] — *Closure-preserving covers*, Fund. Math. 85 (1974), pp. 165–175.
[8] — *On topological properties defined by games*, Proc. Topol. Colloq. held in Keszthely, June 19–23, 1972.
[9] — *Spaces defined by topological games*, Fund. Math. 88 (1975), pp. 193–223.
[10] J. N. Worrell and H. H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), pp. 820–830.

UNIVERSITY OF DAYTON

*Accepté par la Rédaction le 21. 11. 1973*

# Proper shape retracts

by

**B. J. Ball** (Athens, Ga.)

**Abstract.** Absolute shape retracts, introduced by K. Borsuk, have been studied by a number of authors. The present paper is concerned with an analogous notion in the "proper shape" theory of the author and R. B. Sher, with particular attention being given to the relationship between these spaces and the absolute proper retracts recently considered by Sher.

The notion of a fundamental absolute retract (FAR) was introduced by K. Borsuk [6] for compact metric spaces, and extended to compact Hausdorff spaces by S. Mardešić, who used the term absolute shape retract (ASR) for the extended notion. These spaces are the natural analogs of compact absolute retracts in the shape theory of compacta.

In [3], the author and R. B. Sher began a study of a theory of shape for locally compact metric spaces based on proper mappings, and in [25], Sher introduced the notion of absolute *proper* retract. In the present paper, absolute *proper shape* retracts are defined in a manner entirely analogous — in the frame-work of proper shape theory — to Borsuk's definition of fundamental absolute retracts, and it is shown that these spaces are related, in proper shape theory, to absolute proper retracts in a way quite parallel to the relationship, in compact shape theory, between absolute shape retracts and absolute retracts.

**§ 1. Definitions and notations.** A familiarity with the basic terminology and notations of Borsuk's shape theory for compacta [4, 5] and of proper shape theory for local compacta [3] is assumed. A number of other technical or specialized definitions are given in the text, but the remaining terminology is largely standard.

The term *absolute retract*, and the notation AR, will be used to mean absolute retract for metrizable spaces, and similarly for *absolute neighborhood retract* and ANR.

By an *embedding* of a space $X$ into a space $Y$, we always understand a *closed* embedding; i.e., a map $f\colon X\to Y$ such that $f$ induces a homeomorphism of $X$ onto $f(X)$, and such that $f(X)$ is closed in $Y$.

A *compactification* of a space $X$ is a compact Hausdorff space $Y$ containing $X$ as a dense subset. Two compactifications of $X$ are equivalent if there is a homeomorphism of one onto the other which reduces to the identity on $X$.

A map $f\colon X\to Y$ is said to be *proper* if $f^{-1}(C)$ is compact for every compact subset $C$ of $Y$. A *proper homotopy* of $X$ into $Y$ is a proper map $\varphi\colon X\times[0,1]\to Y$. The symbols $\simeq$, $\underset{p}{\simeq}$, and $\approx$ denote, respectively, the relations of homotopy, proper homotopy, and homeomorphism.

If $A\subset X$, then $\mathrm{Cl}_X A$ and $\mathrm{Bd}_X A$ will denote the closure and the (point set) boundary of $A$ relative to $X$. The identity map on a set $X$ will be denoted by $i_X$.

**§ 2. The Freudenthal compactification.** A space $X$ is said to be *rim compact* if each point of $X$ has a neighborhood base consisting of open sets with compact boundaries. It was shown by H. Freudenthal [10, 11] and K. Morita [18] that every rim compact Hausdorff space $X$ has a compactification $FX$, obtained as the disjoint union of $X$ and a set $EX$ of "ends" of $X$, which is maximal among all compactifications $Y$ of $X$ with small inductive dimension $ind(Y-X)=0$. The following useful characterization of $FX$ is proved in [10] for the case in which $FX$ is metrizable, and follows in the general case from [18, Theorem 2].

2.1. THEOREM. *If $FX=X\cup EX$ is the Freudenthal compactification of a rim compact Hausdorff space $X$, then*

(a) *each point of $EX$ has a neighborhood basis of open sets in $FX$ whose boundaries* (*in $FX$*) *are compact subsets of $X$, and*

(b) *no open neighborhood of a point $e\in EX$ is separated by $EX$ into two disjoint sets, each open in $X$ and each having $e$ as a limit point.*

*Moreover, if $F'X=X\cup E'X$ is a compactification of $X$ satisfying* (a) *and* (b), *then $F'X$ is equivalent to $FX$.*

It is shown in [10] that if $X$ is a rim compact separable metric space, then $FX$ is metrizable if and only if the space $QX$ of quasicomponents of $X$ is compact, and that compactness of $QX$ is equivalent to the condition that every decreasing sequence of non-empty open and closed subsets of $X$ have a non-empty intersection. We will be concerned here primarily with those locally compact (not just rim compact) separable metric spaces $X$ for which $FX$ is metrizable, and for convenience, we let $\Sigma$ denote the class of all such spaces; thus $X\in\Sigma$ if and only if $X$ is a locally compact separable metric space and $QX$ is compact. For spaces in $\Sigma$ it is possible, and sometimes convenient, to identify their Freudenthal "ends" with certain equivalence classes of sequences of points of the space, as described below.



Suppose $X\in\Sigma$. A sequence $\alpha$ of points of $X$ will be said to be *admissible in $X$* provided that (1) no subsequence of $\alpha$ converges to a point of $X$ and (2) no compact subset of $X$ separates (in $X$) two infinite subsequences of $\alpha$. Let $\mathcal{A}_X$ denote the set of all admissible sequences in $X$, and for $\alpha,\beta\in\mathcal{A}_X$, let $\alpha\sim\beta$ mean that no compact subset of $X$ separates an infinite subsequence of $\alpha$ from an infinite subsequence of $\beta$. Clearly, $\alpha\sim\beta$ holds if and only if there is an admissible sequence $\gamma$ such that each of $\alpha$ and $\beta$ is a subsequence of $\gamma$. The relation $\sim$ is an equivalence relation on $\mathcal{A}_X$, and for $\alpha\in\mathcal{A}_X$, $[\alpha]$ will denote the equivalence class containing $\alpha$.

2.2. LEMMA. *If $X\in\Sigma$ and $\alpha$ is a sequence of points of $X$, then $\alpha\in\mathcal{A}_X$ if and only if $\alpha$ converges in $FX$ to a point of $EX$.*

Proof. Suppose first that $\alpha$ is admissible in $X$. Since $FX$ is compact and metric, some subsequence $\alpha_1$ of $\alpha$ converges in $FX$ to a point $a_1$, and since $\alpha\in\mathcal{A}_X$, $a_1\in EX$. If $\alpha$ does not converge to $a_1$, there is a subsequence $\alpha_2$ of $\alpha$ which converges in $FX$ to a point $a_2\neq a_1$, and $a_2\in EX$. By Theorem 2.1 (a), there exist disjoint open neighborhoods $U_1$, $U_2$ of $a_1,a_2$ respectively, in $FX$ whose boundaries $C_1$, $C_2$ are compact subsets of $X$. Then $U_1\cap X$ and $U_2\cap X$ are disjoint open and closed subsets of $X-(C_1\cup C_2)$, so $C_1\cup C_2$ separates $U_1\cap X$ and $U_2\cap X$ in $X$; but this is impossible since $\alpha$ is admissible in $X$ and each of $U_1\cap X$, $U_2\cap X$ contains an infinite subsequence of $\alpha$. Hence $\alpha$ converges in $FX$ to the point $a_1\in EX$.

Now suppose that $\alpha$ converges in $FX$ to a point $e\in EX$. If $\alpha$ is not admissible in $X$, there exist a compact subset $C$ of $X$ and infinite subsequences $\alpha_1$, $\alpha_2$ of $\alpha$ such that $X-C$ is the union of two separated sets $X_1$, $X_2$ with $\alpha_1\subset X_1$, $\alpha_2\subset X_2$. If $U=FX-C$, then $U-EX=U\cap X=X-C=X_1\cup X_2$; but this is impossible by Theorem 2.1 (b) since $U$ is an open neighborhood in $FX$ of the point $e\in EX$ and $X_1$ and $X_2$ are open subsets of $X$ each having $e$ as a limit point. Hence $\alpha\in\mathcal{A}_X$.

2.3. LEMMA. *If $X\in\Sigma$ and $\alpha,\beta\in\mathcal{A}_X$, then $\alpha\sim\beta$ if and only if $\alpha$ and $\beta$ converge in $FX$ to the same point of $EX$.*

2.4. LEMMA. *Suppose $X\in\Sigma$ and $G$ is an open subset of $X$ with compact boundary. If $\alpha$ and $\beta$ are equivalent admissible sequences in $X$ and $\alpha$ is eventually in $G$, then $\beta$ is eventually in $G$.*

For $X\in\Sigma$, let $E'X=\{[\alpha]|\ \alpha\in\mathcal{A}_X\}$, and let $\mathcal{G}$ denote the collection of all open subsets of $X$ with compact boundary. For each $G\in\mathcal{G}$, let $G^*=G\cup\{[\alpha]\in E'X|\ \alpha \text{ is eventually in } G\}$, and let $\mathcal{G}^*=\{G^*|\ G\in\mathcal{G}\}$. It is evident that for $G_1,G_2\in\mathcal{G}$, $(G_1\cap G_2)^*=G_1^*\cap G_2^*$, and hence $\mathcal{G}^*$ is a basis for a topology on the set $X\cup E'X$. Let $F'X=X\cup E'X$ with the topology generated by $\mathcal{G}^*$; clearly, $F'X$ is a Hausdorff space and $X$, as a subspace of $F'X$, retains its original topology.

2.5. THEOREM. *If $X \in \Sigma$, there is a homeomorphism $\varphi: FX \to F'X$ with $\varphi(x) = x$ for every $x \in X$.*

Proof. For $e \in EX$, let $\varphi(e) = [\alpha]$, where $\alpha$ is any sequence of points of $X$ converging in $FX$ to $e$. By Lemmas 2.2 and 2.3, $\varphi$ is a well defined 1-1 function on $EX$ and $\varphi(EX) = E'X$. Extend $\varphi$ to a function from $FX$ onto $F'X$ by setting $\varphi(x) = x$ for $x \in X$.

Suppose $G \in \mathcal{G}$ and let $U = \varphi^{-1}(G^*)$. If $U$ is not open in $FX$, there is a point $e \in U$ which is a limit point of $FX - U$; since $U \cap X = G$ and $G$ is open in $X$, $e \in EX$. Since $X$ is dense in $FX$, there is a sequence $\beta$ of points of $X - U$ which converges to $e$ in $FX$. By Lemma 2.3, $\beta \sim \alpha$ and hence by Lemma 2.4, $\beta$ is eventually in $G$. But this is a contradiction since $X - U = X - G$ and $\beta \subset X - U$. Hence $\varphi$ is continuous and since $FX$ is compact and $\varphi$ is 1-1, $\varphi$ is a homeomorphism.

Thus for $X \in \Sigma$, the Freudenthal compactification $FX$ may be identified with the space $F'X$ described above, so that an "end" of $X$ may be considered to be an equivalence class of admissible sequences in $X$. Note that a necessary and sufficient condition for a sequence $[\alpha_i]$ of ends of $X$ to converge to an end $[\alpha]$ of $X$ is that if $G \in \mathcal{G}$ and $\alpha$ is eventually in $G$, then for almost all $i$, $\alpha_i$ is eventually in $G$.

An important relation between proper maps and Freudenthal compactifications is stated in the next theorem. The general form given here follows from a theorem of Skljarenko [26]; a weaker version appears in [3 (Lemma 4.2)], and some related theorems in [19 (Theorem 5)] and [20 (Theorem 4)].

2.6. THEOREM. *If $X$ and $Y$ are locally compact Hausdorff spaces, then every proper map $f: X \to Y$ has a unique extension to a map $Ff: FX \to FY$. Moreover, $Ff(EX) \subset EY$, and if $f(X) = Y$, then $Ff(EX) = EY$.*

The following lemma, the proof of which is immediate from the definitions, is stated here for later reference. (It might also be noted that this lemma, together with the criterion mentioned above for convergence of a sequence of ends of $X$, yields an easy proof of Theorem 2.6 for $X$, $Y \in \Sigma$.)

2.7. LEMMA. *Suppose $X, Y \in \Sigma$ and $f: X \to Y$ is a proper map. If $\alpha = \{x_i\}_{i=1}^{\infty}$ is an admissible sequence in $X$, then $f(\alpha) = \{f(x_i)\}_{i=1}^{\infty}$ is an admissible sequence in $Y$.*

A proper map $f: X \to Y$, $X$ and $Y$ rim compact Hausdorff spaces, is *end preserving* [25] if $Ff|EX$ is injective; if, in addition, $Ff(EX) = EY$, then $f$ will be said to be *strongly* end preserving. If $X$ is a closed subset of $Y$, then $X$ will be said to be *(strongly) properly embedded* in $Y$ if the inclusion $j: X \to Y$ is (strongly) end preserving.



For any space $X$, let $\mathcal{S}_X$ denote the set of all sequences of points of $X$.

2.8. THEOREM. *If $X$ is a closed subset of a space $Y \in \Sigma$, then $X$ is properly embedded in $Y$ if and only if $\mathcal{A}_X = \mathcal{A}_Y \cap \mathcal{S}_X$.*

Proof. Suppose first that $\mathcal{A}_X \neq \mathcal{A}_Y \cap \mathcal{S}_X$. Since clearly $\mathcal{A}_X \subset \mathcal{A}_Y$, there is a sequence $\alpha \in \mathcal{A}_Y \cap \mathcal{S}_X$ such that $\alpha \notin \mathcal{A}_X$, and hence there are subsequences $\alpha_1$, $\alpha_2$ of $\alpha$ which converge in $FX$ to distinct ends $e_1$, $e_2$ of $X$. If $j: X \to Y$ is the inclusion, the sequences $Fj(\alpha_1)$ and $Fj(\alpha_2)$ converge in $FY$ to $e_1' = Fj(e_1)$ and $e_2' = Fj(e_2)$, and since $\alpha \in \mathcal{S}_X$, $Fj(\alpha_1) = \alpha_1$ and $Fj(\alpha_2) = \alpha_2$. Hence $\alpha_1 \to e_1'$ and $\alpha_2 \to e_2'$, and since $\alpha \in \mathcal{A}_Y$, it follows that $e_1' = e_2'$. Hence $Fj|EX$ is not injective, so $X$ is not properly embedded in $Y$.

Conversely, if $X$ is not properly embedded in $Y$, there exist distinct ends $e_1$, $e_2$ of $X$ and an end $e$ of $Y$ such that $Fj(e_1) = Fj(e_2) = e$. If $\alpha_1$, $\alpha_2$ are sequences of points of $X$ converging in $FX$ to $e_1$, $e_2$, respectively, then the sequences $Fj(\alpha_1)$ and $Fj(\alpha_2)$ each converge in $FY$ to the point $e$. Since $Fj(\alpha_1) = \alpha_1$ and $Fj(\alpha_2) = \alpha_2$, it follows that the sequence $\alpha$ obtained by alternating terms of $\alpha_1$ and $\alpha_2$ is admissible in $Y$. Hence $\alpha \in \mathcal{A}_Y \cap \mathcal{S}_X$, but $\alpha \notin \mathcal{A}_X$, so $\mathcal{A}_X \neq \mathcal{A}_Y \cap \mathcal{S}_X$.

2.9. THEOREM. *If $X$ is a closed subset of a rim compact Hausdorff space $Y$, then $X$ is properly embedded in $Y$ if and only if $\mathrm{Cl}_{FY} X$ is equivalent to $FX$.*

Proof. Let $j: X \to Y$ and $k: \mathrm{Cl}_{FY} X \to FY$ be the inclusions. Suppose first that $\mathrm{Cl}_{FY} X$ is equivalent to $FX$, and let $h: FX \to \mathrm{Cl}_{FY} X$ be a homeomorphism such that $h(x) = x$ for $x \in X$. Since $kh: FX \to FY$ is an extension of $j: X \to Y$, $kh = Fj$ and hence $Fj|EX$ is injective.

Conversely, suppose $X$ is properly embedded in $Y$ and let $Fj(FX) = K$. Since $FX$ is a compact Hausdorff space and $Fj$ is injective, $Fj$ maps $FX$ homeomorphically onto $K$. Since $X$ is dense in $FX$, $Fj(X) = X$ is dense in $K$ and hence $K = \mathrm{Cl}_{FY} X$. Since $Fj(x) = x$ for $x \in X$, it follows that $FX$ is equivalent to $\mathrm{Cl}_{FY} X$.

**§ 3. Embedding in subsets of $Q$.** The following facts are used to some advantage in [3]: Every space $X \in \Sigma$ can be embedded in $Q \times [0, 1)$, where $Q$ is the Hilbert cube, in such a way that the closure, $\overline{X}$, of $X$ in $Q \times [0, 1]$ is equivalent to $FX$, and if $X$ is so embedded, then every neighborhood of $X$ in $Q \times [0, 1)$ contains a closed neighborhood $U$ of $X$ such that the closure, $\overline{U}$, of $U$ in $Q \times [0, 1]$ is equivalent to $FU$, and such that $\overline{U} - U = \overline{X} - X$. These results will be amplified somewhat in this section.

A *tree* is a connected, simply connected, locally finite 1-complex. A *dendron* is a locally connected continuum which contains no simple closed curve. An *endpoint* of a dendron $D$ is a point $p$ of $D$ such that $p$ is an endpoint of every arc in $D$ that contains it. The set $E$ of all endpoints of a dendron $D$ is totally disconnected and does not separate $D$. (But $E$

need not be closed, and hence $D-E$ may not belong to $\Sigma$; however, $D-E$ *is* rim compact and if $X = D-E$, it follows from the characterization given in Theorem 2.1 that $FX$ is homeomorphic to $D$ under a homeomorphism which is the identity on $X$.)

A subset $A$ of a space $X$ is said to be *unstable* in $X$ [14] if there is a homotopy $\varphi\colon X\times[0,1]\to X$ such that $\varphi_1(x) = x$ for every $x \in X$, and for $0 \leqslant t < 1$, $\varphi_t(X) \cap A = \emptyset$.

If $T$ is a tree, then $FT \in \mathrm{AR}$ and $ET$ is an unstable subset of $FT$ ([23, Lemma 2.2]; no proof is given in [23], but Professor Sher has indicated to the author an elementary argument based upon a particular realization of $T$ in $E^2$). The next two lemmas establish somewhat more than the above result for trees. The first of these lemmas is essentially known (cf. [28, Theorem 5], [22, Theorem 3] and [21]), and the proof of the second resembles Sher's argument but makes use of a convex metric instead of a geometric construction.

3.1. LEMMA. *For $X \in \Sigma$, $FX$ is a dendron if and only if $X$ is connected, locally connected, and contains no simple curve.*

Proof. If $FX$ is a dendron, then $X$ contains no simple closed curve since $FX$ does not, $X$ is connected since $FX$ is, and $X$ is locally connected since $X = FX - EX$ and $EX$ is closed and totally disconnected.

Suppose then that $X$ is connected, locally connected and contains no simple closed curve. Since $X$ is locally connected and locally compact, $FX$ is locally connected and for every connected open subset $U$ of $FX$, $U \cap X$ is connected [9, 30]. It follows that if $U$ is a connected open subset of $FX$ and $e$ is a limit point of $U$, there is an arc $pe$ from $p$ to $e$ in $FX$ such that $pe-\{e\} \subset U$.

Suppose $J$ is a simple closed curve in $FX$ and $e \in J \cap EX$. There exist two disjoint connected open subsets of $FX$ each having $e$ as a limit point, and hence there exist arcs $pe$, $qe$ in $FX$ such that $pe-\{e\}$ and $qe-\{e\}$ are disjoint subsets of $X$. Let $pq$ be an arc from $p$ to $q$ in $X$, let $U$ be a connected open neighborhood of $e$ in $FX$ such that $U \cap pq = \emptyset$, and let $rs$ be an arc in $U \cap X$ from a point $r \in pe$ to a point $s \in qe$. Then $pe \cup qe \cup pq \cup rs$ contains a simple closed curve in $X$, contrary to hypothesis. Hence $FX$ is a dendron.

3.2. LEMMA. *If $D$ is a dendron and $E$ is the set of all endpoints of $D$, then $E$ is unstable in $D$.*

Proof. There exists a convex metric $\varrho$ for $D$ [17, p. 96; 15, p. 324]; since $D$ is unique arcwise connected, every arc in $D$ is isometric (with respect to $\varrho$) to a real number interval. Let $p$ be a point of $D-E$ and for each $x \in D$ let $px$ denote the arc from $p$ to $x$ in $D$. For $(x, t) \in X\times[0,1]$, let $\varphi(x, t)$ be the unique point $z \in px$ such that $\varrho(p, z) = t\cdot\varrho(p, x)$.

Since $D$ is locally connected and unique arcwise connected, it easily follows that if $\{x_i\}$ is a sequence of points of $D$ converging to a point $x \in D$, then the sequence $\{px_i\}$ of arcs converges to the arc $px$. Hence if $(x_i, t_i) \to (x, t)$ in $D\times[0,1]$ and $\varphi(x_i, t_i) \to z \in D$, then $z \in px$ and since $\varrho(p, z) = t\cdot\varrho(p, x)$, it follows that $z = \varphi(x, t)$. Hence $\varphi\colon D\times[0,1]\to D$ is continuous.

If $0 < t < 1$ and $z = \varphi(x, t)$, then $z \in px$ and $0 < \varrho(p, z) < \varrho(p, x)$. Hence $z \in px - \{p, x\}$, so $z \notin E$. Therefore $\varphi_t(D) \subset D-E$ for $0 \leqslant t < 1$, and hence $E$ is unstable in $D$.

A subset $A$ of a space $X$ is said to be a *$Z$-set* in $X$ if $A$ is closed and for each non-empty homotopically trivial open subset $U$ of $X$, $U-A$ is non-empty and homotopically trivial [1]. A useful summary of some of the basic properties of $Z$-sets may be found in [27] along with additional results; in particular, we need the fact [27, Theorem 2] that *a closed subset $A$ of $Q$ is a $Z$-set in $Q$ if and only if $A$ is unstable in $Q$* (cf. also [29, Lemma 2.2]).

3.3. THEOREM. *If $T$ is a tree and $C$ is a $Z$-set in $Q$ homeomorphic to $ET$, then $T\times Q \approx Q-C$.*

Proof. Since $FT$ is a dendron, it follows [29, Theorem 6.3] that $FT\times Q \approx Q$. Let $h\colon FT\times Q\to Q$ be a homeomorphism and let $A = h(ET\times Q)$. Since $ET$ is unstable in $FT$, $ET\times Q$ is unstable in $FT\times Q$; hence $A$ is unstable in $Q$ and therefore, by the result of Toruńczyk mentioned above, $A$ is a $Z$-set in $Q$. Since $ET$ is a deformation retract of $ET\times Q$, $\mathrm{Sh}\,A = \mathrm{Sh}(ET\times Q) = \mathrm{Sh}(ET) = \mathrm{Sh}\,C$; hence $A$ and $C$ are $Z$-sets in $Q$ having the same shape, and this implies [8, Theorem 2] that $Q-A \approx Q-C$. Since $h(T\times Q) = Q-A$, it follows that $T\times Q \approx Q-C$.

3.4. COROLLARY. *If $T$ is a tree, then $F(T\times Q) \approx Q$.*

Let $C_0$ be a Cantor set embedded as a $Z$-set in $Q$ and let $Q_0 = Q-C_0$. The set $Q_0$ will play an important role in the next section. We give one preliminary result here.

3.5. THEOREM. *Every space $X \in \Sigma$ has an end preserving embedding onto a $Z$-set in $Q_0$, and if $C$ is a closed subset of $C_0$ homeomorphic to $EX$, then $X$ has a strong end preserving embedding onto a $Z$-set in $Q-C$.*

Proof. Let $h\colon FX\to Q$ be an embedding such that $h(FX)$ is a $Z$-set in $Q$ and let $C' = h(EX)$. Since there is a homeomorphism of $Q$ onto itself taking $C'$ onto $C$, it may be assumed that $C' = C$, and hence $h(X) \subset Q_0 \subset Q-C$. Identifying $FQ_0$ and $Q$, we have $h\colon FX\to FQ_0$, and $h$ is an extension of the map $g\colon X\to Q_0$ defined by $g(x) = h(x)$ for $x \in X$. By Theorem 2.1, $h = Fg$ and hence $Fg|EX$ is injective, so $g\colon X\to Q_0$ is an end preserving embedding. Similarly, identifying $F(Q-C)$ and $Q$ and defining $f\colon X\to Q-C$ by $f(x) = h(x)$ for $x \in X$, it follows that $h = Ff$ and hence $Ff|EX$ is injective and $Ff(EX) = C = E(Q-C)$, so $f\colon X\to Q-C$ is a strong end preserving embedding.

**§ 4. Proper shape retracts.** Throughout this section, it is to be understood that *all spaces considered are in the class* $\Sigma$. We first give some properties of SUV$^\infty$ spaces [12, 23, 24] and of absolute proper retracts [25].

A closed subset $X$ of a space $Y$ is said to have *property* UV$^\infty$ *in* $Y$ if for every neighborhood $U$ of $X$ in $Y$, there is a neighborhood $V$ of $X$ in $Y$ such that $V \subset U$ and $V$ is contractible in $U$ to a point. If $X$ has property UV$^\infty$ in some ANR, then $X$ has property UV$^\infty$ in every ANR in which it is embedded as a closed subset; in this case, $X$ is said to be a UV$^\infty$ space, or to have property UV$^\infty$. It is well known that a compact space $X$ has property UV$^\infty$ if and only if $\mathrm{Sh}\,X$ is trivial; i.e., $X$ has the shape of a point. (This follows explicitly from [6, Theorem 9.1] and [7, Theorem 7.1], and is implicit in [13].)

Property UV$^\infty$ has been most useful in studying compact spaces. A related property, SUV$^\infty$, which agrees with UV$^\infty$ for compact spaces but seems to have some advantages in the noncompact case, was introduced by Hartley [12] and developed further by Sher [23, 24]. We will not repeat the definition of property SUV$^\infty$, but will rely on the characterization given below.

Let us say that a space $X$ has trivial *proper* shape if there is a tree $T$ such that $\mathrm{Sh}_p X = \mathrm{Sh}_p T$ (since every compact tree has the shape of a point, this agrees with the notion of trivial shape for compacta). Then, by Corollary 3.5 of [23], *a space $X$ has property* SUV$^\infty$ *if and only if* $\mathrm{Sh}_p X$ *is trivial.* It should be remarked that it is also a consequence of Corollary 3.5 of [23] that *if* $\mathrm{Sh}_p X \leqslant \mathrm{Sh}_p Y$ *and* $\mathrm{Sh}_p Y$ *is trivial, then* $\mathrm{Sh}_p X$ *is trivial.* Thus any space which is proper shape dominated by an SUV$^\infty$ space is itself an SUV$^\infty$ space.

A space $X$ is a *proper retract* of $Y$ if there is a proper map $r\colon Y \to X$ such that $r|X = i_X$, and $X$ is an *absolute proper retract* (APR) if $X$ is a proper retract of every space $Y$ in which $X$ is properly embedded (i.e., the inclusion of $X$ into $Y$ is end preserving; see Section 3). These notions were introduced in [25], where it is shown, among other results, that $X \in$ APR if and only if $FX \in$ AR and $EX$ is unstable in $FX$, and that $X \in$ APR if and only if $X$ is a noncompact ANR and $X \in$ SUV$^\infty$. The second characterization readily implies that any proper retract of an APR is an APR.

Let the Hilbert cube, $Q$, be regarded as $\prod_{n=1}^{\infty} I_n$, $I_n = [-1, 1]$ for all $n$. The pseudo-interior, $s$, of $Q$ is then $\prod_{n=1}^{\infty} \mathring{I}_n$, $\mathring{I}_n = (-1, 1)$ for all $n$. Every compact subset of $s$ is a $Z$-set in $Q$ and in $s$ [1, Theorem 3.3 and Theorem 9.1].

Let $C_0$ be a Cantor set in $s$ and let $Q_0 = Q - C_0$. Let $D$ be a dendron having $C_0$ as its set of endpoints, and let $T_0 = D - C_0$. Then $T_0$ is a tree, $ET_0 \approx C_0$ and, by Theorem 3.3 and Corollary 3.4, $T_0 \times Q \approx Q_0$ and $F(T_0 \times Q) \approx Q$; in fact, of course, the pairs $(F(T_0 \times Q), E(T_0 \times Q))$ and $(Q_0, C_0)$ are homeomorphic. These facts, together with the results from [25] given above, yield the following characterization of absolute proper retracts.



4.1. THEOREM. *A space $X$ is an* APR *if and only if $X$ is homeomorphic to a proper retract of* $Q_0$.

Proof. To show that every proper retract of $Q_0$ is an APR, it is only necessary to show that $Q_0 \in$ APR, and for this it suffices [25, Theorem 3.1] to show that $FQ_0 \in$ AR and $EQ_0$ is unstable in $FQ_0$. But this is immediate from the fact that $(FQ_0, EQ_0) \approx (Q, C_0)$ and that $C_0$, being a $Z$-set in $Q$, is unstable in $Q$.

Conversely, suppose $X \in$ APR. Then by Theorem 3.5, there is an end preserving embedding $g\colon X \to Q_0$, and since $X \in$ APR, $g(X)$ is a proper retract of $Q_0$ by definition.

Suppose $M$ is a locally compact AR, $Y$ is a closed subset of $M$ and $X$ is a closed subset of $Y$. A *proper fundamental retraction* of $Y$ to $X$ in $(M, M)$ is a proper fundamental net (see [3]) $\underline{r}\colon Y \to X$ in $(M, M)$, $\underline{r} = \{r_a \mid a \in A\}$, such that $r_a(x) = x$ for every $x \in X$, $a \in A$. A closed subset $X$ of a space $Y$ is called a *proper shape retract* of $Y$ if there exist a locally compact absolute retract $M$, an embedding $h\colon Y \to M$ and a proper fundamental retraction $\underline{r}\colon h(Y) \to h(X)$ in $(M, M)$. That this property is independent of $M$ and of $h$ is a consequence of the following theorem, which can be proved by a trivial modification of the argument for Theorem 2.10 of [6].

4.2. THEOREM. *Suppose $M$ and $M'$ are locally compact* AR*'s, $Y$ is a closed subset of $M$ and $h\colon Y \to M'$ is an embedding. If $X$ is a closed subset of $Y$ and there is a proper fundamental retraction of $Y$ to $X$ in $(M, M)$, then there is a proper fundamental retraction of $h(Y)$ to $h(X)$ in $(M', M')$.*

A space $X$ will be called an *absolute proper shape retract* (APSR) if $X$ is a proper shape retract of every space $Y$ in which $X$ is properly embedded.

It is easily shown (cf. [6, Proposition 2.6]) that every proper retract of a space $Y$ is a proper shape retract of $Y$, and hence every APR is an APSR.

4.3. THEOREM. *A space $X$ is an* APSR *if and only if $X$ is homeomorphic to a proper shape retract of* $Q_0$.

Proof. Suppose $X \in$ APSR. By Theorem 3.5, there is an end preserving embedding $g\colon X \to Q_0$, and since $X \in$ APSR, $g(X)$ is a proper shape retract of $Q_0$ by definition.

Suppose, conversely, that $X$ is homeomorphic to a proper shape retract of $Q_0$, and let $Y$ be a space in which $X$ is properly embedded. We must show that $X$ is a proper shape retract of $Y$.

Since $C_0$ is a compact subset of $s$, there is a Hilbert cube $Q' \subset s$ such that $C_0$ is contained in the pseudo-interior of $Q'$. Let $Q'_0 = Q' - C_0$.

Since $Q'_0 \approx Q_0$, there is an embedding $h\colon X \to Q'_0$ such that $h(X) = X'$ is a proper shape retract of $Q'_0$. Since $X$ is properly embedded in $Y$ and $Q'_0 \in$ APR, it follows from [25, Theorem 5.2] that $h$ can be extended to a proper map $f\colon Y \to Q'_0$. Let $j\colon Q'_0 \to Q_0$ be the inclusion. Then $jf\colon Y \to Q_0$ is a proper map and $jf|X\colon X \to Q_0$ is an embedding of $X$ onto $X'$; since $X' \cup C_0$ is a compact subset of $s$ and hence a $Z$-set in $Q$, $X'$ is a $Z$-set in $Q_0$. Applying [2, Theorem 3.1], there is an embedding $\hat{h}\colon Y \to Q_0$ such that $\hat{h}(x) = f(x)$ for $x \in X$. Let $Y' = \hat{h}(Y)$.

Since $Q'_0$ is an APR and is properly embedded in $Q_0$, there is a proper retraction $p\colon Q_0 \to Q'_0$. Since $X'$ is a proper shape retract of $Q'_0$ and $Q'_0 \in$ AR, there is a proper fundamental retraction $\underline{r} = \{r_\alpha|\ \alpha \in A\}$ of $Q'_0$ to $X'$ in $(Q'_0, Q'_0)$. Let $j\colon Q'_0 \to Q_0$ be the inclusion, and let $\underline{s} = \{jr_\alpha p|\ \alpha \in A\}$. Then $(\underline{s}, Q_0, X')$ is a proper fundamental net in $(Q_0, Q_0)$, and hence so is $(\underline{s}, Y', X')$. Since $jr_\alpha p(x) = x$ for every $x \in X'$, $(\underline{s}, Y', X')$ is a proper fundamental retraction. Hence $X'$ is a proper shape retract of $Y'$, so $X$ is a proper shape retract of $Y$.

4.4. COROLLARY. *If $Y \in$ APSR and $X$ is a proper shape retract of $Y$, then $X \in$ APSR.*

Proof. If may be assumed that $Y$ is properly embedded in $Q_0$. Since $Y \in$ APSR, there is a proper fundamental retraction $\underline{r}\colon Q_0 \to Y$ in $(Q_0, Q_0)$, and since $X$ is a proper shape retract of $Y$, there is a proper fundamental retraction $\underline{s}\colon Y \to X$ in $(Q_0, Q_0)$. Then $\underline{sr}$ is a proper fundamental retraction of $Q_0$ to $X$, so by Theorem 4.3, $X \in$ APSR.

4.5. THEOREM. *If $X$ is a proper shape retract of $Y$, then* $\mathrm{Sh}_p X \leqslant \mathrm{Sh}_p Y$.

Proof. It may be assumed that $Y$ is properly embedded in $Q_0$. Let $\underline{r}\colon Y \to X$ be a proper fundamental retraction of $Y$ to $X$ in $(Q_0, Q_0)$. Let $\underline{i}$ denote the degenerate net $\{i_{Q_0}\}$. The composition of the proper fundamental nets $(\underline{i}, X, Y)$ and $(\underline{r}, Y, X)$ is the proper fundamental net $(\underline{r}, X, X)$. Since $(\underline{r}, X, X)$ is generated by $i_X\colon X \to X$ as is $(\underline{i}, X, X)$, it follows [3, Lemma 3.5] that $(\underline{r}, X, X) \underset{p}{\simeq} (\underline{i}, X, X) = \underline{i}_X$, and hence $\mathrm{Sh}_p X \leqslant \mathrm{Sh}_p Y$.

The following result is given in [23]; the statement there requires that $Y$ be a tree, but this is needed only to insure that $FY \in$ AR and $EY$ is unstable in $FY$, so the argument applies equally well to any $Y \in$ APR.

4.6. LEMMA. *If $Y \in$ APR and $f, g\colon X \to Y$ are proper maps, then $f \underset{p}{\simeq} g$ if and only if $Ff|EX = Fg|EX$.*

4.7. THEOREM. *A noncompact space $X$ is an* APSR *if and only if* $X \in \mathrm{SUV}^\infty$.



Proof. Suppose $X \in$ APSR, and assume that $X$ is properly embedded in $Q_0$. Then $X$ is a proper shape retract of $Q_0$ and hence $\mathrm{Sh}_p X \leqslant \mathrm{Sh}_p Q_0 = \mathrm{Sh}_p(T_0 \times Q) = \mathrm{Sh}_p T_0$, so $X \in \mathrm{SUV}^\infty$.

Conversely, suppose $X \in \mathrm{SUV}^\infty$ and let $T$ be a tree such that $\mathrm{Sh}_p X = \mathrm{Sh}_p T$. By Theorem 3.5, it may be assumed that $X$ is a $Z$-set in $M = T \times Q$ and that the inclusion $j\colon X \to M$ is strongly end preserving. It follows from [24, Theorem 2] that there is a cofinal system $\{U_\alpha|\ \alpha \in A\}$ of closed neighborhoods of $X$ in $M$ such that for each $\alpha$, there is a homeomorphism $h_\alpha\colon M \to U_\alpha$ which leaves $X$ pointwise fixed. Let $A$ be directed by the relation $\alpha \leqslant \beta \Leftrightarrow U_\beta \subset U_\alpha$. For each $\alpha \in A$, let $r_\alpha = j_\alpha h_\alpha$, where $j_\alpha\colon U_\alpha \to M$ is the inclusion, and let $\underline{r} = \{r_\alpha|\ \alpha \in A\}$.

Suppose $\alpha, \beta \in A$ and $\alpha \leqslant \beta$. Let $j\colon U_\beta \to U_\alpha$ be the inclusion. Then $h_\alpha\colon M \to U_\alpha$ and $jh_\beta\colon M \to U_\alpha$ are proper maps and since $EX = EM$ and $h_\alpha$, $jh_\beta$ leave $X$ pointwise fixed, $Fh_\alpha$ and $F(jh_\beta)$ leave $EM$ pointwise fixed. Hence by Lemma 4.6, $h_\alpha \underset{p}{\simeq} jh_\beta$ and therefore $r_\alpha \underset{p}{\simeq} r_\beta$ in $U_\alpha$. It follows that $\underline{r}$ is a proper fundamental net from $M$ to $X$ in $(M, M)$, and since $r_\alpha(x) = x$ for every $x \in X$, $\underline{r}$ is a proper fundamental retraction. Since $M \in$ APSR, it follows from Corollary 4.4 that $X \in$ APSR.

The definitions given above for "proper fundamental retraction", "proper shape retract" and "absolute proper shape retract" are obviously modeled on Borsuk's definitions [6] of the corresponding notions for compacta, although the terminology used here is an adaptation of Mardešić's usage [16]. A connection between the notion of APSR, for local compacta, and that of *absolute shape retract* (ASR), or, equivalently, *fundamental absolute retract* (FAR), for compacta is given in the next theorem and its corollary.

4.8. THEOREM. *If $X$ is a proper shape retract of $Y$, then $FX$ is a shape retract of $FY$.*

Proof. Suppose $Y$ is properly embedded in $Q_0$ and let $\underline{r} = \{r_\alpha|\ \alpha \in A\}$ be a proper fundamental retraction from $Y$ to $X$ in $(Q_0, Q_0)$. It follows from the proof of [3, Theorem 4.5] that there is a fundamental sequence $\underline{f} = \{f_k|\ k = 1, 2, 3, \ldots\}$ from $FX$ to $FY$ in $FQ_0 = Q$ such that for each $k$, there is an $\alpha_k \in A$ such that $f_k|X = r_{\alpha_k}|X$. Since $X$ is dense in $FX$ and for each $x \in X$ and for $k = 1, 2, \ldots$, $f_k(x) = r_{\alpha_k}(x) = x$, it follows that also for each $e \in EX$ and each $k$, $f_k(e) = e$. Hence $\underline{f}$ is a fundamental retraction from $FY$ to $FX$.

4.9. COROLLARY. *If $X \in$ APSR, then $FX \in$ ASR.*

Proof. If $X$ is properly embedded in $Q_0$, then $X$ is a proper shape retract of $Q_0$ and hence $FX$ is a shape retract of $FQ_0 = Q$. It follows [6, Theorem 6.2] that $X \in$ ASR.

4.10. EXAMPLE. *The converse of Corollary 4.9 does not hold.*

Proof. Observe first that every $SUV^\infty$ space, and therefore every APSR, which has exactly one end must have the proper shape of the half-open interval $[0,1)$. If $X$ is a "$\sin(1/x)$-curve" minus one endpoint of the limit interval, then $X$ has exactly one end but [3, Example 4.14] $\mathrm{Sh}_p X \neq \mathrm{Sh}_p[0,1)$, so $X \notin$ APSR.

We conclude by pointing out some parallel equivalences, the last few of which have been established above, the remaining ones being well known.

4.11. *If $X$ is compact, the following are equivalent*:
(i) $X \in$ AR,
(ii) *$X$ is a retract of $Q$*,
(iii) $X \in UV^\infty$ *and* $X \in$ ANR,
(iv) *$X$ has trivial shape and* $X \in$ ANR,
*and so are*:
(a) $X \in$ ASR,
(b) *$X$ is a shape retract of $Q$*,
(c) $X \in UV^\infty$,
(d) *$X$ has trivial shape.*

4.12. *If $X$ is noncompact, the following are equivalent*:
(i′) $X \in$ APR,
(ii′) *$X$ is a proper retract of $Q_0$*,
(iii′) $X \in SUV^\infty$ *and* $X \in$ ANR,
(iv′) *$X$ has trivial proper shape and* $X \in$ ANR,
*and so are*:
(a′) $X \in$ APSR,
(b′) *$X$ is a proper shape retract of $Q_0$*,
(c′) $X \in SUV^\infty$,
(d′) *$X$ has trivial proper shape.*

## References


[1] R. D. Anderson, *On topological infinite deficiency*, Michigan Math. J. 14 (1967), pp. 365–383.
[2] — and T. A. Chapman, *Extending homeomorphisms to Hilbert cube manifolds*, Pacific J. Math. 38 (1971), pp. 281–293.
[3] B. J. Ball and R. B. Sher, *A theory of proper shape for locally compact metric spaces*, Fund. Math. 86 (1974), pp. 163-192. (Research announcement in Bull. Amer. Math. Soc. 79 (1972), pp. 1023–1026.)
[4] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 221–254.
[5] — *Concerning the notion of the shape of compacta*, Proc. Inter. Sympos. Topology and its applications, Herceg-Novi, 1968, Beograd 1969, pp. 98–104.
[6] — *Fundamental retracts and extensions of fundamental sequences*, Fund. Math. 64 (1969), pp. 55–85.
[7] K. Borsuk, *A note on the theory of shape of compacta*, Fund. Math. 67 (1970), pp. 265–278.
[8] T. A. Chapman, *On some applications of infinite dimensional manifolds to the theory of shape*, Fund. Math. 76 (1972), pp. 181–193.
[9] H. Freudenthal, *Über die Enden topologisher Räume und Gruppen*, Math. Z. 33 (1931), pp. 692–713.
[10] — *Neuaufbau der Endentheorie*, Annals of Math. 43 (1942), pp. 261–279.
[11] — *Kompaktisierungen und Bikompaktisierungen*, Indag. Math. 13 (1951), pp. 184–192.
[12] D. Hartley, *Quasi-cellularity in manifolds*, Ph. D. Thesis, University of Georgia, 1973.
[13] D. M. Hyman, *On decreasing sequences of compact absolute retracts*, Fund. Math. 64 (1969), pp. 91–97.
[14] W. Kuperberg, *Homotopically labile points of locally compact metric spaces*, Fund. Math. 73 (1971), pp. 133–136.
[15] C. Kuratowski and G. T. Whyburn, *Sur les éléments cyclique et leurs applications*, Fund. Math. 16 (1930), pp. 305–331.
[16] S. Mardešić, *Retracts in shape theory*, Glasnik Mat. 6 (26) (1971), pp. 153–163.
[17] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. 100 (1928), pp. 75–163.
[18] K. Morita, *On bicompactifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku, Sec. A, vol. 4, 94 (1952), pp. 221–229.
[19] — *On closed mappings*, Proc. Japan Acad. 32 (1956), pp. 539–593.
[20] K. Nowiński, *Closed mappings and the Freudenthal compactification*, Fund. Math. 76 (1972), pp. 71–83.
[21] B. J. Pearson, *Dendritic compactifications of certain dendritic spaces*, Pacific J. Math. 47 (1973), pp. 229–232.
[22] V. V. Proizvolov, *Peripherally bicompact tree-like spaces* (in Russian), Dokl. Akad. Nauk SSSR 189 (1969), pp. 724–727; translated as Soviet Math. 10 (1969), pp. 1491–1493.
[23] R. B. Sher, *Property $SUV^\infty$ and proper shape theory*, Trans. Amer. Math. Soc., to appear.
[24] — *Proper shape theory and neighborhoods of sets in Q-manifolds*, preprint.
[25] — *A theory of absolute proper retracts*, preprint.
[26] E. G. Skljarenko, *On perfect bicompact extensions* (in Russian), Dokl. Akad. Nauk SSSR 146 (1962), 1031–1034; translated as Soviet Math. 3 (1962), pp. 1455–1458.
[27] H. Toruńczyk, *Remarks on Anderson's paper "On topological infinite deficiency"*, Fund. Math. 66 (1970), pp. 393–401.
[28] L. E. Ward, *On dendritic sets*, Duke Math. J. 25 (1958), pp. 505–514.
[29] J. E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. 150 (1970), pp. 1–25.
[30] L. Zippin, *On semicompact spaces*, Amer. J. Math. 57 (1935), pp. 327–341.


*Accepté par la Rédaction le 22. 2. 1974*