Closure-preserving families of finite sets

by

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Abstract. The study of spaces which admit a closure-preserving cover of compact sets was initiated separately by Tamano and Telgársky, and a number of interesting results and examples now appear in the literature. In this article, we extend the known results. It has been shown that if a collectionwise-normal T_1 space admits a closure-preserving cover of compact sets, then it is paracompact. The unfortunate aspect of this result is that it depends so heavily on the collectionwise-normality of the space. We provide a partial remedy for this situation by applying heavy restrictions on the nature of the members of the cover, while requiring nothing more than the T_1 axiom on the entire space to obtain the following result. If X is T_1 and admits a closure-preserving cover of finite sets, then it is θ-refinable. If it should happen that the number of points in each covering member is bounded by a fixed integer, then X is metacompact.

Introduction. The study of spaces which admit a closure-preserving cover of compact sets was initiated separately by Tamano, in [5], and Telgársky in [6], and a number of results and examples now appear in the literature (see [2], [3], [4], [7], [8] and [9]. It is the purpose of this paper to give some results concerning spaces which admit a closure-preserving cover of finite sets. If X is T_1 and admits a closure-preserving cover of finite sets, then X is σ-metacompact, hence θ-refinable. If it should happen that the number of points in each covering member is bounded by a fixed integer, then X is metacompact.

Telgársky has shown in [7], that any space which admits a closure-preserving cover of finite sets has a countable closed cover, each member of which is scattered and is the union of discrete subsets. These results, together with the σ-metacompactness of such spaces, serve to give a good description of such spaces.

Definition [10]. A topological space X is *θ-refinable* if and only if for every covering H of X whose elements are open, there is a countable family C such that each F in C is a collection of open sets which is a refinement of H covering X and for every point p of X there exists on F in C such that p is a member of only finitely many members of F.

DEFINITION. A topological space X is *σ-metacompact* if it is the countable union of closed metacompact subspaces.

THEOREM 1. *Let K be a closed θ-refinable subspace of a space X. Let $\mathcal{V} = \{V(a)|\ a \in A\}$ be an open cover of X. Then there is a sequence, $\mathcal{V}(n)$, of open refinements of $\mathcal{V}$, each covering X, such that for each point $x \in K$, there is an integer $n(x)$ such that x appears in only finitely many members of $\mathcal{V}(n(x))$.*

Proof. The family $\{V(a) \cap K|\ a \in A\}$ is a cover of K by sets open in K. Hence there is a sequence of refinements $\mathcal{W}(n)$, each consisting of open subsets of K such that for each n, $\mathcal{W}(n)$ covers K and such that for each $x \in K$, there is an integer $n(x)$ such that x is an element of only a finite number of members of $\mathcal{W}(n)$. It may be assumed that each $\mathcal{W}(n)$ is indexed on A, that is, $\mathcal{W}(n) = \{W(n, a)|\ a \in A\}$ and that for each $a \in A$, $W(n, a) \subset V(a) \cap K$.

Now fix an integer n. For each $a \in A$, let $T(n, a)$ be an open subset of X such that $W(n, a) = T(n, a) \cap K$. Let $\mathcal{V}(n) = \{T(n, a) \cap V(a)|\ a \in A\}$. Then $\mathcal{V}(n)$ consists of open subsets of X, and refines $\mathcal{V}$. Moreover, if a point of K appears in only a finite number of members of $\mathcal{W}(n)$, then it appears in only a finite number of members of $\mathcal{V}(n)$.

If, to the family $\mathcal{V}(n)$, we add all sets of the form $V(a) \cap (X - K)$, then the resulting $\mathcal{V}(n)$ is a cover of X, refines $\mathcal{V}$ and has the property that if a point of K meets only finitely many members of $\mathcal{W}(n)$, then it meets only finitely many members of $\mathcal{V}(n)$. The collection $\mathcal{V}(n)$ satisfies the conclusion of the theorem.

COROLLARY 2. *Let X be a space and $\{F(i)|\ i \in Z^+\}$ a family of closed, θ-refinable subspaces of X such that $X = \bigcup_{i=1}^{\infty} F(i)$. Then X is θ-refinable.*

Proof. Let $\mathcal{V}$ be an open cover of X. For each i, let $\mathcal{V}(i, n)$ be a sequence of refinements of $\mathcal{V}$, of the type guaranteed by the previous theorem. Then the family of refinements $\{\mathcal{V}(i, n)|\ i, n \in Z^+\}$ satisfies the requirements of θ-refinability.

COROLLARY 3. *Each σ-metacompact space is θ-refinable.*

Proof. A metacompact space is θ-refinable.

LEMMA 4. *Let X be a T_1 space and let $x \in X$ such that $X - \{x\}$ is metacompact. Then X is metacompact.*

Proof. The proof is obvious.

LEMMA 5. *Let X be a space and let $\mathcal{F} = \{F(a)|\ a \in A\}$ be a closed, closure-preserving cover of X. For each $p \in X$, let*

$$K(p) = \{x \in X|\ \text{if } x \in F(a) \in \mathcal{F}, \text{ then } p \in F(a)\}.$$

Then $K(p)$ is open.

Proof. For each $x \in X - K(p)$, there is an index $a(x) \in A$ such that $x \in F(a(x))$ and $p \notin F(a(x))$. Then

$$X - K(p) = \bigcup \{F(a(x))|\ x \in X - K(p)\}$$

and since $\mathcal{F}$ is closure-preserving, $X - K(p)$ is closed.

THEOREM 6. *Let X be a T_1 space. Let n be a positive integer and let $\mathcal{F} = \{F(a)|\ a \in A\}$ be a closure-preserving cover of X such that for each $a \in A$, $|F(a)| \leqslant n$. Then X is metacompact.*

Proof. The proof is by induction. Let $P(n)$ be the proposition: Each T_1 space X which admits a closure-preserving cover, no member of the cover having more than n points, is metacompact.

$P(1)$ is true for if a T_1 space has the property that its family of singletons is closure-preserving, then the space is discrete, hence surely metacompact.

Assume now that $P(n)$ is true, and let $\mathcal{F} = \{F(a)|\ a \in A\}$ be a closure-preserving cover of the T_1 space X such that for each $a \in A$, $|F(a)| \leqslant n+1$. For each $p \in X$, let $K(p) = \{x \in X|$ if $x \in F(a) \in \mathcal{F}$, then $p \in F(a)\}$. By Lemma 5, each set $K(p)$ is open. Moreover, the family $\{K(p)|\ p \in X\}$ is point finite; in fact a point of X can appear in at most $n+1$ of these sets, for if $x \in \bigcap \{K(p_i)|\ i = 1, 2, \ldots, j\}$, then let $a \in A$ such that $x \in F(a)$. Then each point p_i, for $i = 1, 2, \ldots, j$, is also in $F(a)$. If $j > n+1$, then $|F(a)| > n+1$, which is contradictory.

To show that X is metacompact then, it is enough to show that each set $K(p)$ is metacompact.

To see this, consider the family $\{K(p) \cap F(a)|\ a \in A\}$. This is a closed, closure-preserving cover of $K(p)$. Moreover, p is an element of every non-void member of this collection, for if $x \in K(p) \cap F(a)$, then $x \in F(a)$, and since $x \in K(p)$ as well, we have by definition of $K(p)$ that $p \in F(a)$. Hence $p \in K(p) \cap F(a)$.

Finally, then, consider the collection $\{(K(p) \cap F(a)) - \{p\}|\ a \in A\}$. This is a closed, closure-preserving cover of the space $K(p) - \{p\}$. Moreover, since p is an element of each non-void member of $\{K(p) \cap F(a)|\ a \in A\}$, then the cardinality of each non-void member of this family is actually reduced by 1 when p is removed. Thus the collection

$$\{(K(p) \cap F(a)) - \{p\}|\ a \in A\}$$

is a closed, closure-preserving cover of the T_1 space $K(p) - \{p\}$, and no member of this family has more than n points. By the induction hypotheses, $K(p) - \{p\}$ is metacompact. By Lemma 4, $K(p)$ is metacompact. Since X is now exhibited as the point-finite union of a family of open, metacompact subspaces, we have that X is metacompact.

THEOREM 7. *Let X be a T_1 space, and let $\mathfrak{F} = \{F(\alpha) |\ \alpha \in A\}$ be a closure-preserving cover of X such that for each $\alpha \in A$, $F(\alpha)$ is finite. Then X is σ-metacompact.*

Proof. For each n, let $X(n) = \bigcup \{F(\alpha) |\ |F(\alpha)| \leqslant n\}$. Since $\mathfrak{F}$ is closure-preserving, each set $X(n)$ is closed. By Theorem 7, each $X(n)$ is metacompact, hence X is σ-metacompact.

COROLLARY 8. *Let X be collectionwise-normal and T_1. Suppose X has a closure-preserving cover consisting of finite sets. Then X is paracompact.*

Proof. By Theorem 7, X is σ-metacompact. By Corollary 3, X is θ-refinable. By Theorem (iii) of [10], X is paracompact.

References

[1] E. Michael, *Point-finite and locally-finite coverings*, Can. J. Math. 7 (1955), pp. 275–279.

[2] H. Potoczny, *Closure-preserving families of compact sets*, General Topology and Its Applications (3) (1973), pp. 243–248.

[3] — *On a problem of Tamano*, Fund. Math. 75 (1972), pp. 29–31.

[4] — *A non-paracompact space which admits a closure-preserving cover of compact sets*, Proc. Amer. Math. Soc. 32 (1972), pp. 309–311.

[5] H. Tamano, *A characterization of paracompactness*, Fund. Math. 72 (1972), pp. 189–311.

[6] R. Telgársky, *Concerning two covering properties*, Proc. Japan Acad. (in press).

[7] — *Closure-preserving covers*, Fund. Math. 85 (1974), pp. 165–175.

[8] — *On topological properties defined by games*, Proc. Topol. Colloq. held in Keszthely, June 19–23, 1972.

[9] — *Spaces defined by topological games*, Fund. Math. 88 (1975), pp. 193–223.

[10] J. N. Worrell and H. H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), pp. 820–830.

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Proper shape retracts

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Abstract. Absolute shape retracts, introduced by K. Borsuk, have been studied by a number of authors. The present paper is concerned with an analogous notion in the "proper shape" theory of the author and R. B. Sher, with particular attention being given to the relationship between these spaces and the absolute proper retracts recently considered by Sher.

The notion of a fundamental absolute retract (FAR) was introduced by K. Borsuk [6] for compact metric spaces, and extended to compact Hausdorff spaces by S. Mardešić, who used the term absolute shape retract (ASR) for the extended notion. These spaces are the natural analogs of compact absolute retracts in the shape theory of compacta.

In [3], the author and R. B. Sher began a study of a theory of shape for locally compact metric spaces based on proper mappings, and in [25], Sher introduced the notion of absolute *proper* retract. In the present paper, absolute *proper shape* retracts are defined in a manner entirely analogous — in the frame-work of proper shape theory — to Borsuk's definition of fundamental absolute retracts, and it is shown that these spaces are related, in proper shape theory, to absolute proper retracts in a way quite parallel to the relationship, in compact shape theory, between absolute shape retracts and absolute retracts.

§ 1. Definitions and notations. A familiarity with the basic terminology and notations of Borsuk's shape theory for compacta [4, 5] and of proper shape theory for local compacta [3] is assumed. A number of other technical or specialized definitions are given in the text, but the remaining terminology is largely standard.

The term *absolute retract*, and the notation AR, will be used to mean absolute retract for metrizable spaces, and similarly for *absolute neighborhood retract* and ANR.