A set-theoretical equivalent of the prime ideal theorem for Boolean algebras

by

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§ 0. This paper gives a set-theoretical equivalent of the prime ideal theorem which is interesting in itself. It can be used to give short proofs for equivalences between the prime ideal theorem and other known theorems, like "Every inverse limit of non-empty finite algebras is non-empty" (see Grätzer [1].) We illustrate this for the case of a very general theorem from Grätzer [1].

Remark. "Equivalent" means: "provably equivalent in ZF".

List of theorems:

BPI. *The prime ideal theorem for Boolean algebras.*

Tych(T_2). *Every product of compact T_2 spaces, with the usual topology, is compact.*

FA. *Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras of type r, $\mathfrak{A} = \langle A, \ldots\rangle$, $\mathfrak{B} = \langle B, \ldots\rangle$, B is finite. Suppose that for every finite $C \subseteq A$ there is a partial r-homomorphism from C into B. Then there exists an r-homomorphism from A into B.*

P. *Let D, E be sets, and let, for all $d \in D$, d be finite. Suppose that for all finite $F \subseteq D$ there is a S such that for all $f \in F$: $f \cap S \in E$. Then there is a S such that for all $d \in D$: $d \cap S \in E$.*

§ 1. BPI $\Leftrightarrow$ P $\Leftrightarrow$ FA.

THEOREM 1. FA $\Rightarrow$ BPI.

Proof. Trivial.

THEOREM 2. P $\Rightarrow$ FA.

Proof. Let $C \subseteq A$, C is finite, and let

$$F_C = \{f \mid f \text{ a partial } r\text{-homomorphism from some } D \subseteq C,\ D \neq \emptyset \text{ into } B\}.$$

F_C is finite because B is. Take for E the set of all non-empty sets that consist of a finite number of non-empty partial r-homomorphisms from subsets of A into B, closed under taking unions.

Take for D the set $\{F_C|\ C \subseteq A \ \&\ C \text{ finite}\}$.

The conditions of P are satisfied. Take the promised S. $\bigcup(S \cap \bigcup D)$ is an r-homomorphism from A into B.

Example. Let $a = fa_1 \dots a_n$. $S \cap F_{\{a\}}$ contains just one $\{\langle a, b\rangle\}$. $S \cap F_{\{a_i\}}$ contains just one $\{\langle a_i, b_i\rangle\}$, $1 \leqslant i \leqslant n$. $S \cap F_{\{a_1,\dots a_n, a\}}$ contains at least $\{\langle a, b\rangle\}$, $\{\langle a_i, b_i\rangle\}$, $1 \leqslant i \leqslant n$, and is closed under taking unions, so

$$\{\langle a, b\rangle\} \cup \{\langle a_i, b_i\rangle|\ 1 \leqslant i \leqslant n\} \in S \cap F_{\{a_1,\dots,a_n,a\}}$$

hence it is a partial r-homomorphism:

$$b = fb_1 \dots b_n .$$

Theorem 3. BPI $\Leftrightarrow$ Tych(T_2).

Proof. See J. Łoś and C. Ryll-Nardzewski [2].

Theorem 4. Tych$(T_2) \Rightarrow$ P.

Proof. Let $Q = \prod_{d \in D} \mathrm{P}(d)$. ($\mathrm{P}(d)$ is the power-set of d.) $\mathrm{P}(d)$ is finite, so compact and T_2 in the discrete topology. Define, for $F \subseteq D$, F finite, the set

$$K(F) = \{q \in Q|\ \text{for all } f \in F\colon (q)_f = f \cap \bigcup\{(q)_f|\ f \in F\} \ \&\ (q)_f \in E\} .$$

(1) $K(F) \neq \emptyset$.

For there is a S such that $S \cap f \in E$, all $f \in F$. So $S \cap \bigcup F$ suffices: Let

$$(q)_f = \begin{cases} (S \cap \bigcup F) \cap f, & \text{all } f \in F, \\ d, & d \in D - F. \end{cases}$$

Then $q \in K(F)$.

(2) *$K(F)$ is closed.*

Proof by cases: Let $q \notin K(F)$; then

(a) $(q)_{f_0} \notin E$, $f_0 \in F$: $q \in \{p \in Q|\ (p)_{f_0} \in \{(q)_{f_0}\}\}$ (this set is open and disjunct from $K(F)$).

(b) $f_0 \cap \bigcup\{(q)_f|\ f \in F\} \neq (q)_{f_0}$. Always:

$$(q)_{f_0} \subseteq \bigcup\{(q)_f|\ f \in F\} ,$$

so there is a $f_1 \in F$ such that

$$(q)_{f_1} \cap (f_0 - (q)_{f_0}) \neq \emptyset .$$

We have

$$q \in \{p \in Q|\ (p)_{f_0} \in \{(q)_{f_0}\} \ \&\ (p)_{f_1} \in \{(q)_{f_1}\}\}$$

(this set is open and disjunct from $K(F)$).

(3) *$K = \{K(F)|\ F \subseteq D \ \&\ F$ finite$\}$ is a set of closed sets with the* FIP.

Proof just as for (1). So (since $\prod_{d \in D} \mathrm{P}(d)$ is compact, by Tych(T_2)) we have $\bigcap K \neq \emptyset$. Choose a $q \in \bigcap K$. Then $\bigcup_{d \in D} (q)_d$ is the required S.

Remark. P for the case: *D is countable* is equivalent to Koenig's Lemma (or, if one prefers: the axiom of choice for countable (index) sets of finite non-empty sets). (Easy proof).

References

[1] G. Grätzer, *Universal Algebra*, Princeton 1968.

[2] J. Łoś and C. Ryll-Nardzewski, *Effectiveness of the representation theory for Boolean algebras*, Fund. Math. 41 (1954), pp. 49–56.

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