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On some functional equations with a restricted domain

by

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Abstract. The functional equations considered are of the form (1) and (2) where f, g, h map an abelian group G into the other abelian group H. We assume their validity for almost all $(x, y) \in G\times G$ and investigate the question whether there exist functions F, G, H almost equal to f, g, h respectively and fulfilling our equations everywhere. The notion "almost all" ("almost everywhere") has been introduced in an axiomatic way.

§ 1. Recently there has been an increased interest in functional equations and inequalities whose validity is postulated "almost everywhere" (abbreviated to a.e. in the sequel). This a.e. is understood in various wans (see for instance [3], [7], [9], [8] and [6]). We shall be interested here in two functional equations,

(1) $$f(x+y)\big(f(x+y)-f(x)-f(y)\big)=0 \quad \text{(of Mikusiński)}$$

and

(2) $$f(x+y)=g(x)+h(y) \quad \text{(of Pexider)},$$

related to the well-known Cauchy equation (cf. [4] and [1]), assuming their a.e. validity in the sense described explicitly below. Roughly speaking, we are going to answer the following question: does there exist a function F (or: do there exist functions F_1, F_2, F_3) such that it satisfies (1) (or: they satisfy (2)) everywhere and $f=F$ a.e. (or: $f=F_1$, $g=F_2$, $h=F_3$ a.e.)? Such a problem was first raised by P. Erdös [5] in connection with Cauchy's functional equation. Positively solved by N. G. de Bruijn [3] and independently by W. B. Jurkat [8], this problem was then investigated by M. Kuczma [9] in connection with convex functions and by the present author for polynomial functions (also with positive answers).

On account of these results we may also expect a positive answer to our question. The aim of the present paper is to show that this is really the case.

§ 2. We shall restrict our attention to the notion of a set-ideal in a group and some related questions in order to achieve greater clarity of statements in further sections. Indeed, suppose that we are given a nonempty family $\mathfrak{J}$ of subsets of a group G (1) such that

(i) $A, B \in \mathfrak{J}$ implies $A \cup B \in \mathfrak{J}$;

(ii) $A \in \mathfrak{J}$, $B \subset A$ implies $B \in \mathfrak{J}$;

(iii) $G \notin \mathfrak{J}$;

(iv) $A + a_0 := \{a + a_0 \colon a \in A\}$ as well as $-A := \{-a \colon a \in A\}$ belong to $\mathfrak{J}$ whenever $A \in \mathfrak{J}$ and $a_0 \in G$.

Then $\mathfrak{J}$ is called a *proper linearly invariant* (shortly: p.l.i.) ideal in G. The word "proper" refers here to (iii) and "linearly invariant" to (iv) (compare [10] and [6]).

A property $\mathcal{T}(x)$, $x \in G$, is said to *hold* (a.e.)$_{\mathfrak{J}}$ iff there exists a set $U \in \mathfrak{J}$ such that $\mathcal{T}(x)$ is satisfied for all $x \in G \setminus U$.

Let two p.l.i. ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ in G and $G^2 := G \times G$, respectively, be given. We say that $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are *conjugate* iff for every $M \in \mathfrak{J}_2$ there exists a set $U \in \mathfrak{J}_1$ such that the set

$$V_x(M) := \{y \in G \colon (x, y) \in M\} \tag{3}$$

belongs to $\mathfrak{J}_1$ whenever $x \notin U$.

As an instance, the family of all sets with Lebesgue measure zero (or with a finite outer Lebesgue measure) and also the collection of all first category Baire sets on the real line are p.l.i. ideals which are conjugate with their analogues on the real plane. Likewise, if G is an arbitrary group of infinite order, then the family of all its finite subsets yields a p.l.i. ideal which is conjugate with the one in G^2.

Let two subsets P and Q of a group G and a p.l.i. ideal $\mathfrak{J}$ in G be given. We say that P and Q are *congruent* (mod $\mathfrak{J}$) iff both $Q \setminus P$ and $P \setminus Q$ belong to $\mathfrak{J}$ (cf. also [10]). Then we write

$$P \equiv Q \pmod{\mathfrak{J}} \tag{4}$$

and

$$P \not\equiv Q \pmod{\mathfrak{J}}$$

whenever (4) does not hold. In the sequel we shall leave out the symbol (mod $\mathfrak{J}$) provided that $\mathfrak{J}$ will be regarded as fixed. Obviously the congruence (mod $\mathfrak{J}$) yields an equivalence relation in 2^G.

We omit the simple proof of the following

(1) All the groups occurring in this paper will be written additively.

LEMMA 1. *Let $\mathfrak{J}$ denote a p.l.i. ideal in G. Then the families*

$$\Pi(\mathfrak{J}) := \{M \subset G^2 \colon \bigvee_{U \in \mathfrak{J}} M \subset (U \times G) \cup (G \times U)\}$$

and

$$\Omega(\mathfrak{J}) := \{M \subset G^2 \colon \bigvee_{U \in \mathfrak{J}} \bigwedge_{x \in G \setminus U} V_x(M) \in \mathfrak{J}\},$$

where $V_x(M)$ is defined by (3), *yield p.l.i. ideals in G^2, both conjugate with $\mathfrak{J}$.*

COROLLARY 1. *The p.l.i. ideal $\Omega(\mathfrak{J})$ is the greatest one (in the sense of inclusion) in the family of all p.l.i. ideals in G^2 which are conjugate with $\mathfrak{J}$.*

COROLLARY 2. *Every p.l.i. ideal in G^2 which is conjugate with $\mathfrak{J}$ may be supplied to $\Omega(\mathfrak{J})$.*

LEMMA 2. *Let $\mathfrak{J}$ denote a p.l.i. ideal in a group G. Then, for every $U \in \mathfrak{J}$, the set*

$$L := \{(x, y) \in G^2 \colon x + y \in U\}$$

belongs to $\Omega(\mathfrak{J})$.

Proof. Otherwise, for every S in $\mathfrak{J}$ one can find an $x_0 \in G \setminus S$ such that

$$V_{x_0}(L) = \{y \in G \colon (x_0, y) \in L\} \notin \mathfrak{J}.$$

On the other hand,

$$\{y \in G \colon (x_0, y) \in L\} = \{y \in G \colon x_0 + y \in U\} = U - x_0 \in \mathfrak{J},$$

which is a contradiction.

LEMMA 3. *Let two subsets P and Q of a group G and a p.l.i. ideal $\mathfrak{J}$ in G be given such that*

$$P \equiv G \pmod{\mathfrak{J}} \tag{5}$$

whereas $Q \notin \mathfrak{J}$. Then

$$P + Q := \{p + q \colon p \in P,\ q \in Q\} = G.$$

Proof. Put $W := G \setminus P$. Then $W \in \mathfrak{J}$ on account of (5). Take an arbitrary member x of G. Since, evidently, $Q \setminus (x - W) \neq \emptyset$ we may find a $q \in Q$ such that $p := (x - q) \notin W$. Hence $p \in P$ and

$$x = (x - q) + q = p + q \in P + Q,$$

which was to be proved.

In the sequel we shall make use of de Bruijn's theorem [3] which (with the aid of our definitions) states that:

(*) For every pair of conjugate ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ in G and G^2, respectively, and for every function $\Phi \colon G \to H$ the relation

$$\Phi(u + v) = \Phi(u) + \Phi(v) \quad (\text{a.e.})_{\mathfrak{J}_2}\ (2)$$

(2) Then Φ is called $\mathfrak{J}_2$-*almost additive*.

implies the existence of an additive function $\Psi\colon G\to H$ for which the equality

$$\Phi(x)=\Psi(x)$$

is satisfied $(\text{a.e.})_{\mathfrak{J}_1}$. This function Ψ is unique.

§ 3. Suppose that we are given two commutative groups G and H, a p.l.i. ideal $\mathfrak{J}$ in G and a function $f\colon G\to H$ such that the relation

(6) $$f(x+y)\neq 0 \quad \text{implies} \quad f(x+y)=f(x)+f(y) \ (\text{a.e.})_{\Omega(\mathfrak{J})}$$

is satisfied. (The conditional form (6) of equation (1) enables one to avoid an additional multiplication structure in H, see [4]). Put

(7) $$Z_f := \{x\in G\colon f(x)=0\}.$$

With the use of (7) equation (6) now reads as follows:

(6a) $$x+y\notin Z_f \quad \text{implies} \quad f(x+y)=f(x)+f(y) \ (\text{a.e.})_{\Omega(\mathfrak{J})}.$$

The latter equation suggests an investigation of a slightly more general functional equation in a natural way. Namely, replacing the set Z_f in (6a) by a fixed subset Q of the group G, get

(6b) $$x+y\notin Q \quad \text{implies} \quad f(x+y)=f(x)+f(y) \ (\text{a.e.})_{\Omega(\mathfrak{J})}$$

(cf. also [2]). It turns out that, in general, the behaviour of solutions of (6b) depends essentially on whether the condition

(C[Q]) for all $a\in G$ we have $Q\cup(Q+a)\not\equiv G \pmod{\mathfrak{J}}$

is satisfied or not. For this reason we shall distinguish two cases regarding equation (6):

(A) $$(C[Z_f]);$$

(B) $$\text{non}(A).$$

Our first result refers to equation (6b). Namely, we have the following

THEOREM 1. *Let $Q\subset G$ satisfy* (C[Q]) *and let $f\colon G\to H$ be a solution of* (6b). *Then there exists a unique additive function $F\colon G\to H$ such that* $f(x)=F(x)$ $(\text{a.e.})_{\mathfrak{J}}$.

Proof. There exists a set $M\in\Omega(\mathfrak{J})$ such that (6b) is satisfied for every pair $(x,y)\in G^2\setminus M$. By the definition of $\Omega(\mathfrak{J})$ we get the existence of a set $U(M)\in\mathfrak{J}$ such that $V_x(M)$ defined by (3) belongs to $\mathfrak{J}$ whenever $x\notin U(M)$. The proof is divided into two cases.

Case 1. $0\notin U(M)$. Fix an $x\in G$ and choose $w(x)$ from the set $G\setminus\big(U(M)\cup(U(M)-x)\big)$. Now, evidently,

$$A(x):=\Big(Q-\big(x+w(x)\big)\Big)\cup\big(V_{w(x)}(M)-x\big)\cup V_{x+w(x)}(M)\not\equiv G$$

because of (C[Q]). Consequently, $G\setminus A(x)\neq\emptyset$. Fix a $z\in G\setminus A(x)$. Then

(8) $$x+w(x)+z\notin Q.$$

Moreover, since $z\notin V_{x+w(x)}(M)$ and $x+z\notin V_{w(x)}(M)$, we have

(9) $$\big(x+w(x),z\big)\notin M \quad\text{and}\quad \big(w(x),x+z\big)\notin M.$$

Now, (8), (9) and (6b) lead to the equalities

$$f\big(x+w(x)+z\big)=f\big(x+w(x)\big)+f(z),$$
$$f\big(x+w(x)+z\big)=f\big(w(x)\big)+f(x+z),$$

whence we obtain

$$f(x+z)-f(z)=f\big(x+w(x)\big)-f\big(w(x)\big),$$

which says that $f(x+z)-f(z)$ depends on x only provided $z\in G\setminus A(x)$. This enables us to define a function $F_1\colon G\to H$ by the formula

(10) $$F_1(x):=f(x+z)-f(z)$$

where z is arbitrarily taken from the complement of $A(x)$.

Note that on account of (C[Q]) the relation

(11) $$(Q-x)\cup A(x)\cup V_x(M)\not\equiv G$$

is satisfied whenever $x\notin U(M)$. Thus, for fixed $x\in G\setminus U(M)$, we are able to take $z\in G$ such that

$$x+z\notin Q, \quad z\notin A(x) \quad\text{and}\quad z\notin V_x(M),$$

whence by (6b)

$$f(x+z)=f(x)+f(z).$$

This compared with (10) gives

$$f(x)=F_1(x) \quad\text{for}\quad x\notin U(M),$$

i.e.,

(12) $$f(x)=F_1(x)\ (\text{a.e.})_{\mathfrak{J}}.$$

Now we are going to prove that F_1 is $\Omega(\mathfrak{J})$-almost additive. Indeed, fix a u and a v arbitrarily from G so that $u+v\notin -U(M)\in\mathfrak{J}$. Lemma 2 ensures that

(13) $$\{(u,v)\in G^2\colon u+v\in -U(M)\}\in\Omega(\mathfrak{J}).$$

Take an s from G such that

(14) $$s\notin A(u)\cup\big(U(M)+v\big)\cup U(M)\cup\big(U(M)-u\big).$$

Then it is possible to put

$$w(v):=s-v \quad\text{and}\quad w(u+v):=-(u+v).$$

Indeed,

$$s-v \notin U(M) \cup (U(M)-v)$$

by (14), while

$$-(u+v) \notin U(M) \cup (U(M)-(u+v))$$

as $u+v \notin -U(M)$ and $0 \notin U(M)$. Now

$$\begin{aligned}&(A(u+v)-s) \cup A(v)\\ &=\big([Q \cup (V_{-(u+v)}(M)-(u+v)) \cup V_0(M)]-s\big) \cup (Q-s) \cup (V_{s-v}(M)-v) \cup V_s(M)\\ &= (Q-s) \cup E, \quad E \in \mathfrak{J},\end{aligned}$$

which together with (C[Q]) and (14) easily implies that one is able to find a $t \in G$ such that

$$\begin{aligned}t &\notin (Q-s) \cup (Q-(u+v+s)) \cup (A(u+v)-s) \cup A(v) \cup V_s(M) \cup (V_{u+s}(M)-v)\\ &= (Q-s) \cup (Q-(u+v+s)) \cup E_1, \quad E_1 \in \mathfrak{J},\end{aligned}$$

whence we infer that

(15) $\quad (s,t) \notin M, \quad s+t \notin Q, \quad (u+s, v+t) \notin M, \quad u+s+v+t \notin Q$

as well as

(16) $\quad F_1(u+v) = f(u+v+s+t)-f(s+t), \quad F_1(v) = f(v+t)-f(t)$

(14) gives also

(17) $$F_1(u) = f(u+s)-f(s).$$

Relations (15), (16) and (17) together with (6b) allow us to perform the following calculation:

$$\begin{aligned}&F_1(u+v)-F_1(u)-F_1(v)\\ &= f(u+v+s+t)-f(s+t)-f(u+s)+f(s)-f(v+t)+f(t)\\ &= f(u+s)+f(v+t)-f(s)-f(t)-f(u+s)+f(s)-f(v+t)+f(t) = 0.\end{aligned}$$

Thus we have just proved that (see (13))

(18) $$F_1(u+v) = F_1(u)+F_1(v) \ (\text{a.e.})_{\Omega(\mathfrak{J})}.$$

Since $\mathfrak{J}$ and $\Omega(\mathfrak{J})$ are conjugate (Lemma 1), we may apply de Bruijn's result (*): there exists a unique additive function $F\colon G \to H$ such that

$$F(u) = F_1(u) \ (\text{a.e.})_{\mathfrak{J}}.$$

The latter relation compared with (12) immediately leads to

$$f(x) = F(x) \ (\text{a.e.})_{\mathfrak{J}}.$$

Case 2. $0 \in U(M)$. Take an arbitrary $x_0 \in G \setminus U(M)$ and put $x = s+x_0$ in (6b). Then we get

(19) $\quad s+y+x_0 \notin Q \quad$ implies $\quad f(s+y+x_0) = f(s+x_0)+f(y)$

for all $(s,y) \notin M-(x_0, 0) \in \Omega(\mathfrak{J})$.

Write

(20) $$g(s) := f(s+x_0), \quad s \in G.$$

With the aid of (20) relation (19) assumes the form

(21) $\quad s+y+x_0 \notin Q \quad$ implies $\quad g(s+y) = g(s)+g(y-x_0),$

$$(s,y) \notin M-(x_0, 0).$$

Now put $y = t+x_0$ in (21). Then

$$s+t+x_0+x_0 \notin Q \quad \text{implies} \quad g(s+t+x_0) = g(s)+g(t),$$
$$(s,t) \notin M-(x_0,x_0) \in \Omega(\mathfrak{J}),$$

which with the use of the definitions

(22) $$Q_0 := Q-(x_0+x_0), \quad M_0 := M-(x_0, x_0)$$

may be written in the form

(23) $\quad s+t \notin Q_0 \quad$ implies $\quad g(s+t+x_0) = g(s)+g(t),$

$$(s,t) \notin M_0 \in \Omega(\mathfrak{J}).$$

Observe that

(24) $$V_0(M_0) = \{t \in G\colon (0,t) \in M_0\} \in \mathfrak{J}.$$

In fact, $V_0(M_0) = \{t \in G\colon (x_0, t+x_0) \in M\} = \{t \in G\colon (x_0, t) \in M\}-x_0$, which belongs to $\mathfrak{J}$ because of $x_0 \notin U(M)$.

Putting $s = 0$ in (23), we find that

(25) $\quad t \notin Q_0 \quad$ implies $\quad g(t+x_0) = c_0+g(t)$

for all $t \notin V_0(M_0)$. Here we have put $c_0 := g(0)$.

Consider the set

$$L := G^2 \setminus \big(M_0 \cup \{(s,t) \in G^2\colon s+t \in V_0(M_0)\}\big).$$

The congruence

(26) $$L \equiv G^2 \ (\text{mod}\ \Omega(\mathfrak{J}))$$

is then fulfilled (this can easily be obtained by making use of the fact that $M_0 \in \Omega(\mathfrak{J})$, (24) and Lemma 2). Take an arbitrary pair (u,v) from L. Then we may apply (25) with $t = u+v$ as well as (23) with $(s,t) = (u,v)$ whence

(27) $$\begin{aligned}u+v \notin Q_0 \quad &\text{implies} \quad g(u+v+x_0) = c_0+g(u+v),\\ u+v \notin Q_0 \quad &\text{implies} \quad g(u+v+x_0) = g(u)+g(v).\end{aligned}$$

As a consequence of (27) we find that

$$u+v \notin Q_0 \quad \text{implies} \quad g(u+v)+c_0 = g(u)+g(v).$$

Putting

(28) $$h(s) := g(s) - c_0\,, \quad s \in G\,,$$

we infer that

$$u+v \notin Q_0 \quad \text{implies} \quad h(u+v) = h(u)+h(v)\,.$$

Moreover, (22) and (C[Q]) imply (C[Q_0]). Finally, let us note that $L' := G^2 \setminus L \in \Omega(\mathfrak{J})$ (see (26)). Thus there exists a set $U(L') \in \mathfrak{J}$ such that for all $x \in G \setminus U(L')$ the set $\{y \in G\colon (x, y) \in L'\}$ belongs to $\mathfrak{J}$. Without loss of generality we may assume that $0 \notin U(L')$. Indeed, one can easily check that we simply have $\{y \in G\colon (0, y) \in L'\} = V_0(M_0) \in \mathfrak{J}$.

Summing up, we have shown that

$$u+v \notin Q_0 \quad \text{implies} \quad h(u+v) = h(u)+h(v) \quad \text{for} \quad (u, v) \notin L' \in \Omega(\mathfrak{J})\,,$$

(C[Q_0]) is satisfied and $0 \notin U(L')$. Consequently, we have Case 1 for the function h. Hence we derive the existence of a unique additive function $F\colon G \to H$ such that

(29) $$h(x) = F(x) \ \text{(a.e.)}_{\mathfrak{J}}\,.$$

By (28), (20) and (29) we get

(30) $$f(x) = F(x) - F(x_0) + c_0 \ \text{(a.e.)}_{\mathfrak{J}}\,.$$

Moreover, (28) and (29) imply the existence of a set $E \in \mathfrak{J}$ such that

(31) $$g(x) = F(x) + c_0 \quad \text{for} \quad x \in G \setminus E\,.$$

Now take a $t_0 \notin Q_0 \cup V_0(M_0) \cup (E - x_0) \cup E$. Then (25) applied for $t = t_0$ gives in view of (31)

$$F(t_0+x_0) + c_0 = c_0 + F(t_0) + c_0\,,$$

i.e.,

$$F(x_0) = c_0\,,$$

whence the relation

$$f(x) = F(x) \ \text{(a.e.)}_{\mathfrak{J}}$$

follows immediately on account of (30). This completes the proof of our assertion.

Remark 1. Putting $Q = Z_f$ in Theorem 1, we obtain the description of the behaviour of a solution of (6) in Case (A).

Remark 2. The phrase $\text{(a.e.)}_{\Omega(\mathfrak{J})}$ in (6b) may be replaced by $\text{(a.e.)}_{\mathfrak{J}_2}$ where $\mathfrak{J}_2$ denotes an arbitrarily fixed p.l.i. ideal in G^2 which is conjugate with $\mathfrak{J}$, since $M \in \mathfrak{J}_2$ involves $M \in \Omega(\mathfrak{J})$ (cf. Corollary 2).

THEOREM 2. *Suppose that we are given two abelian groups G and H, a p.l.i. ideal $\mathfrak{J}$ in G, a set $W \in \mathfrak{J}$ and a function $f\colon G \to H$ such that*

(32) $$f(x+y) \neq 0 \quad \textit{implies} \quad f(x+y) = f(x)+f(y)\,, \quad x, y \in G \setminus W\,.$$

Then there exists exactly one additive function $F\colon G \to H$ such that the set

$$S := \{x \in G\colon f(x) \neq F(x)\}$$

belongs to $\mathfrak{J}$ provided Case (A) *occurs. Moreover, in that case $f|_S = 0$, $0 \notin S$ and $S \subset W$.*

Proof. By the definition of $\Pi(\mathfrak{J})$ (see Lemma 1), (32) simply means that

$$f(x+y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y) \ \text{(a.e.)}_{\Pi(\mathfrak{J})}\,.$$

In order to get the first part of our assertion it suffices to make use of Theorem 1 with $Q = Z_f$ (see also Remark 2). To check the equality $f|_S = 0$ take an arbitrary u and an arbitrary v from G. Obviously, the set $W_1 := \big(u - (W \cup S)\big) \cup \big((W \cup S) - v\big)$ belongs to $\mathfrak{J}$. For $s \notin W$ we get

(33) $$u - s \notin W \cup S \quad \text{and} \quad v + s \notin W \cup S\,.$$

In particular, we can make use of (32) with $x = u - s$, $y = v + s$, whence

(34) $$f(u+v) \neq 0 \quad \text{implies} \quad f(u+v) = f(u-s)+f(v+s)\,.$$

By (33) and (34) we find that

$$f(u+v) \neq 0 \quad \text{implies} \quad f(u+v) = F(u-s)+F(v+s)\,,$$

which in view of the additivity of F shows that

(35) $$f(u+v) \neq 0 \quad \text{implies} \quad f(u+v) = F(u)+F(v)\,.$$

Since u and v were arbitrarily taken, (35) is satisfied for all pairs $(u, v) \in G^2$. Putting $v = 0$ in (35), we obtain

(36) $$f(u) \neq 0 \quad \text{implies} \quad f(u) = F(u)$$

whence $f|_S = 0$ follows by contraposition. Setting $u = 0$ in (36) we infer that $f(0) = 0$. Thus 0 cannot be a member of S.

Finally, suppose that the inclusion $S \subset W$ is not true. Hence, one can find an $x_0 \notin W$ such that $f(x_0) \neq F(x_0)$, whence, on account of $f|_S = 0$, we get

(37) $$f(x_0) = 0 \neq F(x_0)\,.$$

Take $y \in G \setminus \big((Z_f - x_0) \cup (S - x_0) \cup S \cup W\big)$ (which is possible because of (A)). Then (32) implies

$$f(x_0+y) = f(x_0)+f(y) = f(y)\,,$$

which, by (37) and the way we have chosen y, leads to

$$F(x_0+y) = F(y)\,.$$

Now, $F(x_0) = 0$, which contradicts (37).

Conversely, every function $f\colon G\to H$ given by the formula

$$f(x) := \begin{cases} F(x) & \text{for} \quad x\notin T, \\ 0 & \text{for} \quad x\in T, \end{cases} \tag{38}$$

where T stands for an arbitrarily fixed subset of W and F denotes an arbitrary additive function from G into H, satisfies the conditional equation (32). In fact, it suffices to consider only such pairs $(x, y)\in G^2$ that $x\notin T$, $y\notin T$ and $x+y\notin T$ and to apply the additivity of F.

Thus, we have just proved the following

THEOREM 3. *The general solution of the functional equation* (32) *in Case* (A) *is prescribed by the formula* (38).

§ **4.** We proceed with an investigation of the behaviour of functions satisfying Mikusiński's functional equation almost everywhere in Case (B). At first, we exclude the trivial case $Z_f\equiv G \pmod{\mathfrak{J}}$, i.e., the hypothesis

$$Z \not\equiv G \pmod{\mathfrak{J}} \tag{39}$$

will be permanently valid.

Let $a_0\in G$ be such that

$$Z_f\cup(Z_f+a_0)\equiv G \pmod{\mathfrak{J}}. \tag{40}$$

THEOREM 4. *Let $f\colon G\to H$ satisfy* (6). *Then there exists a constant $c\in H\setminus\{0\}$ such that*

$$f(x)\in\{0, c\} \quad (\text{a.e.})_{\mathfrak{J}}$$

provided Case (B) *occurs.*

Proof. The hypothesis (40) states that there exists a set $E\in\mathfrak{J}$ such that for all $x\notin E$ the alternative

$$f(x)=0 \quad \text{or} \quad f(x-a_0)=0 \tag{41}$$

holds. Assuming, as before, that (6) is satisfied except for $(x, y)\in M\in\Omega(\mathfrak{J})$, choose arbitrarily a $b\in G\setminus\big(U(M)\cup\big(U(M)-a_0\big)\cup(E-a_0)\big)$. Then, in particular, $V_b(M)$ and $V_{a_0+b}(M)$ are members of $\mathfrak{J}$, whence the congruence

$$X := \big((Z_f\setminus V_b(M))+b\big)\cup\big((Z_f\setminus V_{a_0+b}(M))+(a_0+b)\big)\equiv G \tag{42}$$

can easily be deduced. Thus $T:=G\setminus X$ belongs to $\mathfrak{J}$. Take an arbitrary element $z_0\in G\setminus(Z_f\cup T)$. For such z_0 we must have

$$z_0=z_1+b \quad \text{or} \quad z_0=z_2+(a_0+b), \tag{43}$$

where $z_1\in Z_f\setminus V_b(M)$ and $z_2\in Z_f\setminus V_{a_0+b}(M)$, i.e., $(b, z_1)\notin M$ and $(a_0+b, z_2)\notin M$. By (43), (6) and (7)

$$f(z_0)=f(b) \quad \text{or} \quad f(z_0)=f(a_0+b).$$

Hence

$$f(x)\in\{0, f(b), f(a_0+b)\} \quad (\text{a.e.})_{\mathfrak{J}}. \tag{44}$$

Now, our assertion follows immediately from (39) and (44) by putting $x=a_0+b$ in (41).

LEMMA 4. *Suppose that $f\colon G\to H$ satisfies* (6) *for $(x, y)\notin M\in\Omega(\mathfrak{J})$. For a given set $T\in\mathfrak{J}$ there exists a set $Z(T)\subset Z_f$ such that $Z(T)\equiv Z_f$, $Z(T)\cap\big(U(M)\cup T\big)=\emptyset$ and $Z(T)=-Z(T)$.*

Proof. $Z(T):=Z_f\setminus\big(U(M)\cup\big(-U(M)\big)\cup T\cup(-T)\big)$. Only the latter equality in the Lemma requires a motivation. Indeed, take an $x\in Z(T)$ and $y\notin(Z_f-x)\cup V_x(M)\cup\big(V_{-x}(M)-x\big)$. Then

$$x+y\notin Z_f, \quad (x, y)\notin M \quad \text{and} \quad (-x, x+y)\notin M,$$

whence by (6) we obtain

$$f(x+y)=f(x)+f(y)=f(y)=f(-x+x+y)=f(-x)+f(x+y),$$

i.e., $-x\in Z_f$. On the other hand, $-x\notin U(M)\cup\big(-U(M)\big)\cup T\cup(-T)$. Thus $-x\in Z(T)$, which proves that $Z(T)\subset -Z(T)$. The inverse inclusion is now obvious.

LEMMA 5. *Suppose that $f\colon G\to H$ satisfies* (6) *for $(x, y)\notin M\in\Omega(\mathfrak{J})$ and Case* (B) *occurs. Then G possesses a subgroup K of index 2 with respect to G such that $K\equiv Z_f \pmod{\mathfrak{J}}$.*

Proof. Fix a set $T\in\mathfrak{J}$ and take $Z(T)$ constructed in the preceding Lemma. Put

$$K := \bigcup_{n=1}^{\infty}\big(\underbrace{Z(T)+\ldots+Z(T)}_{n\ \text{summands}}\big). \tag{45}$$

Evidently, K yields a subgroup of G. At first we shall prove that

$$K\neq G. \tag{46}$$

Indeed, suppose that (46) is not true. Since $Z(T)\equiv Z_f$ and (B) is assumed, we easily obtain

$$Z(T)\cup\big(Z(T)+a_0\big)\equiv G. \tag{47}$$

By (45) and the equality $K=G$, a_0 may be represented in the form

$$a_0=z_1+\ldots+z_n, \quad z_i\in Z(T) \quad \text{for} \quad i=1, \ldots, n.$$

Consequently

$$Z(T)\cup\big(Z(T)+z_1+\ldots+z_n\big)\equiv G. \tag{48}$$

Let n be the smallest positive integer for which the congruence (48) holds. (48) implies immediately

$$(49)\qquad \big(Z(T)-z_n\big)\cup\big(Z(T)+z_1+\ldots+z_{n-1}\big)\equiv G$$

(in the case where $n=1$ we put $z_0:=0$).

We shall show that for an arbitrary $z\in Z(T)$ the congruence

$$(50)\qquad Z(T)+z\equiv Z(T)$$

is fulfilled. To this aim, observe that in view of the relation $Z(T)\equiv Z_f$ it suffices to prove that

$$(50a)\qquad Z(T)+z\equiv Z_f$$

for $z\in Z(T)$. Suppose that $\big(Z(T)+z\big)\setminus Z_f\notin\mathfrak{J}$. Hence

$$A:=\Big(\big(Z(T)\setminus V_z(M)\big)+z\Big)\setminus Z_f\notin\mathfrak{J}$$

in view of the fact that $z\notin U(M)$. Take an $x\in A$. Then $x=\tilde{z}+z$ where $\tilde{z}\in Z(T)\setminus V_z(M)$ and $f(x)\neq 0$. In particular, $(z,\tilde{z})\notin M$. Hence, (6) and the inclusion $Z(T)\subset Z_f$ imply

$$0\neq f(x)=f(z+\tilde{z})=f(z)+f(\tilde{z})=0\,,$$

which is a contradiction. Likewise, if we had $Z_f\setminus\big(Z(T)+z\big)\notin\mathfrak{J}$, then we would get $Z_f\setminus(Z_f+z)\notin\mathfrak{J}$ (on account of $Z(T)\equiv Z_f$) and consequently $(Z_f-z)\setminus Z_f\notin\mathfrak{J}$, whence

$$B:=\Big(\big(Z_f\setminus V_{-z}(M)\big)-z\Big)\setminus Z_f\notin\mathfrak{J}$$

since $-z\notin U(M)$. Taking an $x\in B$, we get $x=\tilde{z}-z$, where $\tilde{z}\in Z_f\setminus V_{-z}(M)$ and $f(x)\neq 0$. In particular, $(-z,\tilde{z})\notin M$, which on account of (6), the equality $Z(T)=-Z(T)$ and the inclusion $Z(T)\subset Z_f$ involves

$$0\neq f(x)=f(-z+\tilde{z})=f(-z)+f(\tilde{z})=0\,.$$

This contradiction ends the proof of (50a). Consequently (50) holds for all $z\in Z(T)$. If we apply it for $z=-z_n$, then on account of (49) we obtain

$$Z(T)\cup\big(Z(T)+z_1+\ldots+z_{n-1}\big)\equiv G\,.$$

which is incompatible with the minimality of n assumed in (47). Thus (46) is satisfied.

On the other hand, (45) and (47) imply $K\cup(K+a_0)\equiv G$, whence, by Lemma 3, $G=K+\big(K\cup(K+a_0)\big)=K\cup(K+a_0)$. (46) excludes here the possibility of $a_0\in K$. Finally, K yields a subgroup of index 2 with respect to G.

It remains to prove that $K\equiv Z_f$. Observe that $K\equiv Z(T)$. Actually, if we had $K\setminus Z(T)\notin\mathfrak{J}$, then, because of $K\cap(K+a_0)=\emptyset$ and $Z(T)\subset K$,

$$\begin{aligned}\mathfrak{J}\in G\setminus\Big(Z(T)\cup\big(Z(T)+a_0\big)\Big)&=\big(K\cup(K+a_0)\big)\setminus\Big(Z(T)\cup\big(Z(T)+a_0\big)\Big)\\&=\big(K\setminus Z(T)\big)\cup\Big(\big(K\setminus Z(T)\big)+a_0\Big)\notin\mathfrak{J}\,,\end{aligned}$$

which is a contradiction. The congruence $K\equiv Z_f$ now follows immediately from the fact that $Z(T)\equiv Z_f$, which finishes the proof of our Lemma.

THEOREM 5. *Let $f\colon G\to H$ satisfy (32) and let hypotheses (39) and (40) be fulfilled. Then there exists a constant $c\in H\setminus\{0\}$ such that $f(x)\in\{0,c\}$ for all $x\in G$. Moreover, G possesses a subgroup K of index 2 such that $K\subset Z_f$, $K\equiv Z_f$ and $f(x)=c$ for $x\in K\cup W$.*

Proof. On account of Theorem 4 there exist a constant $c\in H\setminus\{0\}$ and a set $I\in\mathfrak{J}$ such that $f(x)\in\{0,c\}$ for $x\in G\setminus I$. Take an arbitrary x from G and $s\in(x-Z_f)\setminus\big((x-W)\cup W\cup I\big)$. Then, by (32),

$$0\neq f(x)=f(x-s+s)\quad\text{ implies }\quad f(x)=f(x-s)+f(s)=f(s)\in\{0,c\}\,,$$

which ends the proof of the first part of our assertion.

In virtue of Lemma 5, G possesses a subgroup K of index 2 such that $K\equiv Z_f$. This group is of the form

$$K=\bigcup_{n=1}^{\infty}\big(\underbrace{Z(W)+\ldots+Z(W)}_{n\text{ summands}}\big)\,,$$

where $Z(W):=Z\equiv Z_f$, $Z\subset Z_f$, $Z\cap W=\emptyset$, $Z=-Z$ and $Z+z\equiv Z$ for all $z\in Z$. This results from (45) and (50) by putting $T=W$. In particular, we have

$$(Z+z_1)\cap(Z+z_2)\notin\mathfrak{J}\quad\text{ for all } z_1,z_2\in Z\,.$$

Now, suppose that $K\not\subset Z_f$, i.e. that there exist points $z_i\in Z$, $i=1,\ldots,n$ such that

$$K\ni z_1+\ldots+z_n\notin Z_f\,.$$

Clearly, $n\geqslant 2$. Take an

$$s\in\big((Z-z_n)\cap(Z+z_{n-1})\setminus\bigcup_{i=1}^{n-1}\big((z_{n-i}+\ldots+z_{n-1})-W\big)\,.$$

For such an s we get $z_n+s\in Z$, $z_{n-1}-s\in Z$, $z_{n-i}+\ldots+z_{n-1}-s\notin W$, $i=1,\ldots,n-1$, and

$$\begin{aligned}0\neq f(z_1+\ldots+z_n)&=f\big((z_1+\ldots+z_{n-1}-s)+(z_n+s)\big)\\&=f(z_1+\ldots+z_{n-1}-s)+f(z_n+s)=f(z_1+\ldots+z_{n-1}-s)\\&=f(z_1)+f(z_2+\ldots+z_{n-1}-s)=f(z_2+\ldots+z_{n-1}-s)=\ldots=f(z_{n-1}-s)=0.\end{aligned}$$

This contradiction proves that $K\subset Z_f$.

It remains to show that

$$f(x)=c\quad\text{ for }\quad x\notin K\cup W.$$

At first, observe that the quotient group G/K is equal to $\{K,K'\}$, where we have put $K':=G\setminus K$. Since every two groups of order 2 are isomorphic, we infer that

$$(\mathrm{K})\qquad K+K=K,\quad K+K'=K',\quad K'+K'=K\quad\text{ and }\quad K'=-K'.$$

Now assume that there exists an $x_1 \notin K \cup W$ such that $f(x_1) = 0$. Take an $x \in G \setminus (Z_f \cup W \cup (W+x_1))$. Since $x, x_1 \notin K$, we infer from (K) that $x-x_1 \in K$ and, in particular, $f(x-x_1) = 0$. In view of the relations $x-x_1 \notin W$ and $x_1 \notin W$, (32) gives

$$0 \neq f(x) = f(x-x_1)+f(x_1) = 0\,,$$

which is a contradiction. This completes the proof.

Conversely, every function $f\colon G \to H$ given by the formula

$$(51) \qquad f(x) = \begin{cases} 0 & \text{for} \quad x \in K \cup T, \\ c & \text{for} \quad x \notin K \cup T, \end{cases}$$

where K denotes a subgroup of index 2 with respect to G, T is a subset of W and c is a constant from $H \setminus \{0\}$, yields a solution of (32). In fact, take $x, y \in G \setminus W$ such that $x+y \notin K \cup T$. Then, in particular, x and y do not belong to T and, by (K), either $(x, y) \in K \times K'$ or $(x, y) \in K' \times K$. In both cases, (51) involves $f(x)+f(y) = c = f(x+y)$, i.e., (32) is satisfied.

Therefore, we have just proved

THEOREM 6. *In Case* (B) *the general nontrivial solution of the functional equation* (32) *is prescribed by the formula* (51).

Remark 3. (32) and the condition $Z_f \equiv G \pmod{\mathfrak{J}}$ imply $f = 0$. Actually, suppose that $\mathfrak{J} \in S := \{x \in G\colon f(x) \neq 0\} \neq \emptyset$. Then taking $x \in G$ and $s \in G \setminus \big((x-(W \cup S)) \cup W \cup S\big)$, we infer from (32) that

$$f(x) \neq 0 \quad \text{implies} \quad f(x) = f(x-s)+f(s) = 0\,,$$

which is a contradiction.

Remark 4. The special case $W = \emptyset$, reduces (32) to Mikusiński's equation

$$(\mathrm{M}) \qquad f(x+y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x)+f(y)\,, \qquad (x, y) \in G^2,$$

which has been investigated in [4]. The main result of [4] reads as follows. If G has no subgroups of index 2, then the family $\mathcal{F}$ of all solutions of (M) coincides with the family of all additive functions of the type: $G \to H$: If G possesses subgroups of index 2, then, besides additive solutions, $\mathcal{F}$ contains also the family of all functions which are of the form

$$(51\mathrm{a}) \qquad f(x) = \begin{cases} 0 & \text{for} \quad x \in K\,, \\ c & \text{for} \quad x \notin K\,, \end{cases}$$

where K stands for an arbitrary subgroup of index 2 while c denotes an arbitrary constant from $H \setminus \{0\}$.

This result follows simply from our Theorems 3 and 6. Indeed, if G has no subgroups of index 2 and $f\colon G \to H$ satisfies (M), then Lemma 5 excludes Case (B), whence, on account of Theorem 3, f must be additive (a set T in (38), as a subset of $W = \emptyset$, must be empty). Clearly, every additive function from G into H yields a solution of (M). If G possesses subgroups of index 2 and $f\colon G \to H$ satisfies (M), then again Case (A) admits additive solutions only, while in Case (B) with the aid of Theorem 6 we obtain the form (51a) for f (a set T in (51) must be empty). The fact that f given by (5b) fulfils (M) also follows from Theorem 6.

However, we must underline that the groups G and H were not assumed to be commutative in [4].

THEOREM 7. *Let two commutative groups G and H, two conjugate p.l.i. ideals $\mathfrak{J}$ and $\mathfrak{J}_2$ in G and G^2 resp. and a function $f\colon G \to H$ be given such that*

$$f(x+y) \neq 0 \quad \textit{implies} \quad f(x+y) = f(x)+f(y) \ \ (\text{a.e.})_{\mathfrak{J}_2}\,.$$

Then there exists a function $F\colon G \to H$ fulfilling Mikusiński's equation (M) *and such that*

$$f(x) = F(x) \ \ (\text{a.e.})_{\mathfrak{J}}\,.$$

This function F is unique.

Proof. Supplying $\mathfrak{J}_2$ to $\Omega(\mathfrak{J})$ (cf. Corollary 2), we infer that f satisfies (6). Remark 4 says that the family of all solutions of (M) coincides with the family of all additive functions of the type $G \to H$ whenever Case (A) occurs. On the other hand, in that case Theorem 1 ensures that a solution of (6) is $(\text{a.e.})_{\mathfrak{J}}$ equal to an additive function $F\colon G \to H$ (see also Remark 1). Thus our Theorem is true in Case (A).

Assume now that Case (B) occurs. Then Lemma 5 ensures the existence of a subgroup K of index 2 with respect to G with the property that $K \equiv Z_f \pmod{\mathfrak{J}}$. On the other hand, $f(x) \in \{0, c\}$ $(\text{a.e.})_{\mathfrak{J}}$, where $c \in H \setminus \{0\}$, on account of Theorem 4. Put

$$F(x) := \begin{cases} 0 & \text{for} \quad x \in K\,, \\ c & \text{for} \quad x \notin K\,. \end{cases}$$

F yields a solution of (M) (see Remark 4). Evidently, $f(x) = F(x)$ $(\text{a.e.})_{\mathfrak{J}}$.

The uniqueness in Case (A) is implied by Theorem 1, while in Case (B) it is seen from the construction. Thus the theorem has been completed.

§ 5. The aim of the present section is to give a description of the behaviour of solutions of equation (2) whose validity is postulated almost everywhere. More exactly, we assume that two additive abelian groups G and H are given, functions f, g and h map G into H, $\mathfrak{J}$ yields a p.l.i. ideal in G, $M \in \Omega(\mathfrak{J})$, and equation

$$(52) \qquad (x, y) \notin M \quad \text{implies} \quad f(x+y) = g(x)+h(y)$$

is satisfied. Then

THEOREM 8. *In the case where*

$$S(M) := \{(y, x): (x, y) \in M\} \in \Omega(\mathfrak{J}) \tag{53}$$

there exist: exactly one additive function φ: $G \to H$, *contants* $p, q \in H$ *and a set* $S \in \mathfrak{J}$ *such that*

$$\begin{aligned} F_1(x) &:= \varphi(x)+p+q = f(x), \\ F_2(x) &:= \varphi(x)+p = g(x), \\ F_3(x) &:= \varphi(x)+q = h(x) \end{aligned} \tag{54}$$

for all $x \in G \setminus S$ [3].

Proof. Put $M_1 := M \cup S(M) \in \Omega(\mathfrak{J})$. Then

$$(x, y) \notin M_1 \quad \text{implies} \quad \begin{cases} f(x+y) = g(x)+h(y), \\ f(x+y) = g(y)+h(x). \end{cases} \tag{55}$$

Take $x_0 \notin U(M_1)$ ($U(M_1)$ has here the same meaning as in the previous sections). For such an x_0 we infer from (55) that

$$\left.\begin{aligned} f(x_0+y) &= g(x_0)+h(y) \\ f(x_0+y) &= g(y)+h(x_0) \end{aligned}\right\} \quad \text{for} \quad y \notin V_{x_0}(M_1) \in \mathfrak{J},$$

i.e.,

$$\begin{aligned} g(x) &= f(x+x_0)+p_1 \ (\text{a.e.})_{\mathfrak{J}}, \\ h(y) &= f(y+x_0)+q_1 \ (\text{a.e.})_{\mathfrak{J}}, \end{aligned}$$

where we have put $p_1 := -h(x_0)$ and $q_1 := -g(x_0)$.

Consequently, we easily obtain

$$f(x+y) = f(x+x_0)+p_1+f(y+x_0)+q_1 \ (\text{a.e.})_{\Omega(\mathfrak{J})}.$$

Setting $x+x_0$ and $y+x_0$ instead of x and y, respectively, we get

$$f(x+y+x_0+x_0) = f(x+x_0+x_0)+f(y+x_0+x_0)+p_1+q_1 \ (\text{a.e.})_{\Omega(\mathfrak{J})}. \tag{56}$$

Define a function Ψ: $G \to H$ by the formula

$$\Psi(x) := f(x+x_0+x_0)+p_1+q_1.$$

Equation (56) now assumes the following form:

$$\Psi(x+y) = \Psi(x)+\Psi(y) \ (\text{a.e.})_{\Omega(\mathfrak{J})}. \tag{57}$$

De Bruijn's result (*) may now be applied to the latter equation (57): there exists exactly one additive function φ: $G \to H$ such that $\Psi(x) = \varphi(x)$ $(\text{a.e.})_{\mathfrak{J}}$. This and the definition of Ψ imply

$$f(x) = \varphi(x)-\varphi(x_0)-\varphi(x_0)-p_1-q_1 \ (\text{a.e.})_{\mathfrak{J}}. \tag{58}$$

[3] Evidently, the triplet (F_1, F_2, F_3) yields a solution of the Pexider equation (2).

Recalling the relations between f and g as well as between f and h, we get in view of (58)

$$\begin{aligned} g(x) &= \varphi(x)-\varphi(x_0)-q_1 \ (\text{a.e.})_{\mathfrak{J}}, \\ h(y) &= \varphi(y)-\varphi(x_0)-p_1 \ (\text{a.e.})_{\mathfrak{J}}. \end{aligned} \tag{59}$$

Now, it suffices to put $p := -\varphi(x_0)-q_1$, $q := -\varphi(x_0)-p_1$ and to define S as the union of the exceptional sets for functions f, g and h (these sets are implicitly introduced in the phrase $(\text{a.e.})_{\mathfrak{J}}$ occurring in each of the relations (58) and 59)).

Remark 5. Assumption (53) is essential. To show this let us first note that, in the case where (52) is satisfied, there exists a set $M_0 \in \Omega(\mathfrak{J})$ such that

$$f(x+y) = g(x)+h(y) \quad \text{if and only if} \quad (x, y) \notin M_0. \tag{52a}$$

We have simply

$$M_0 = \{(x, y) \in G^2: f(x+y) \neq g(x)+h(y)\}.$$

Evidently, $M_0 \subset M$ and hence $M_0 \in \Omega(\mathfrak{J})$.

Now, assume that for every $M \in \Omega(\mathfrak{J})$ for which (52) is satisfied we have $S(M) \notin \Omega(\mathfrak{J})$. In particular, (52a) holds and $S(M_0) \notin \Omega(\mathfrak{J})$. Suppose that there exist: an additive function φ: $G \to H$, constants $p, q \in H$ and a set $S \in \mathfrak{J}$ such that for all $x \in G \setminus S$ relations

$$f(x) = \varphi(x)+p+q, \quad g(x) = \varphi(x)+p, \quad h(x) = \varphi(x)+q \tag{60}$$

occur. Note that

$$\begin{aligned} \{(x, y): f(x+y) \neq g(y)+h(x)\} &= \{(y, x): f(x+y) \neq g(x)+h(y)\} \\ &= \{(y, x): (x, y) \in M_0\} \\ &= S(M_0) \notin \Omega(\mathfrak{J}). \end{aligned} \tag{61}$$

(61), Lemma 2, Lemma 1 and Corollary 1 ensure that

$$P := S(M_0) \setminus \big(\{(x, y) \in G^2: x+y \in S\} \cup (G \times S) \cup (S \times G)\big) \neq \emptyset.$$

For $(x, y) \in P$ (60) implies $f(x+y) = g(y)+h(x)$, which contradicts (61).

Remark 6. Obviously, the assertion of Theorem 8 remains true in the case where the validity of the equation $f(x+y) = g(x)+h(y)$ is postulated for all pairs $(x, y) \in G^2 \setminus M$ where M yields a member of an arbitrary p.l.i. ideal $\mathfrak{J}_2$ in G^2, conjugate with $\mathfrak{J}$. Clearly, also in that case the assumption $S(M) \in \Omega(\mathfrak{J})$ must be retained.

As in the previous sections, the case $\mathfrak{J}_2 = \Pi(\mathfrak{J})$ $\big(M = (W \times G) \cup (G \times W),\ W \in \mathfrak{J}\big)$ enables one to strengthen the result just obtained. Namely, we have the following

THEOREM 9. *Suppose that functions f, g and h are of the type $G \to H$ and satisfy the condition*

$$f(x+y) = g(x)+h(y) \quad \text{for} \quad x, y \in G \setminus W,\ W \in \mathfrak{I}. \tag{62}$$

Then there exist: exactly one additive function $\varphi\colon G \to H$ and constants $p, q \in H$ such that

$$f(x) = \varphi(x)+p+q \quad \text{for} \quad x \in G,$$
$$g(x) = \varphi(x)+p \quad \text{for} \quad x \in G\setminus W,$$
$$h(x) = \varphi(x)+q \quad \text{for} \quad x \in G\setminus W.$$

Proof. Putting $M := (W\times G)\cup(G\times W)$ in (52) and noting that we have here $M = S(M)$, on account of Theorem 8 we get the existence of a unique additive function $\varphi\colon G\to H$ and constants $p, q \in H$ such that (54) is satisfied for $x \in G\setminus S$, $S \in \mathfrak{I}$. Take a $u \in G$ and an

$$s \in G\setminus\big((u-(W\cup S))\cup W\cup S\big).$$

Then

$$f(u) = f(u-s+s) = g(u-s)+h(s) = \varphi(u-s)+p+\varphi(s)+q$$
$$= \varphi(u)+p+q.$$

Now, take $x \notin W$ and $y \notin W\cup S$. By (62) we get

$$f(x+y) = g(x)+h(y),$$

whence in view of (54) and the equality $f(u) = \varphi(u)+p+q$, $u \in G$, we obtain

$$g(x) = \varphi(x)+p.$$

Analogously, the equality $h(x) = \varphi(x)+q$ can be derived for $x \notin W$, which ends the proof.

As a direct consequence of this result we obtain the following

THEOREM 10. *The general solution of the functional equation* (62) *is given by the formulas*

$$f(x) = \varphi(x)+p+q \quad \text{for} \quad x \in G,$$

$$g(x) = \begin{cases} \varphi(x)+p & \text{for} \quad x \notin W, \\ \alpha(x) & \text{for} \quad x \in W, \end{cases}$$

$$h(x) = \begin{cases} \varphi(x)+q & \text{for} \quad x \notin W, \\ \beta(x) & \text{for} \quad x \in W, \end{cases}$$

where $\varphi\colon G\to H$ is an arbitrary additive function, α, β are arbitrary functions of the type: $W\to H$ and p, q are arbitrary constants from H.

Remark 7. In the particular case $W = \emptyset$ we get the general solution of the Pexider equation (2) (cf. [1]).

Remark 8. In the case where $g = h = f$ in (62) we immediately obtain $p = q = 0$, i.e., f must be additive (Hartman's result [7]).

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