in $\mathcal{P}^5$ from $\varepsilon_1$ to, say, $\varepsilon_2$ such that each component of $\mathcal{P}^5 - \beta$ contains some points of $K$. Then

$$w(f(a)) = w[\varepsilon_1(\varepsilon_2', \varepsilon_3')] \ast \beta^{-1} \ast \beta \ast w[\varepsilon_1, \varepsilon_2']$$

$$= w[\varepsilon_1(\varepsilon_2', \varepsilon_3')] \ast \beta^{-1} \ast \beta \ast w[\varepsilon_1', \varepsilon_2']$$

is the product of words in $R$. This completes the proof of the theorem.

As an example of this theorem, consider the crossecap; this is the projection, in general position, of an embedding of the projective plane in $\mathcal{P}^5$. To find its knot group we note that there is only one region, $\Sigma$, and only one arc of double points. Thus we have one generator, $\sigma$, and two relations; the first is $\sigma_1 = \sigma^{-1}$, the second is $\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} = 1$; thus the knot group is isomorphic to $\mathbb{Z}_2$. Similarly one may consider the embedding of the Klein bottle in $\mathcal{P}^5$ whose projection in $\mathcal{P}^5$ is a surface with a single circle of self-intersection and find the knot group of this embedding to be isomorphic to $\mathbb{Z}_2$.

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Two notes on abstract model theory

II. Languages for which the set of valid sentences is semi-invariantly implicitly definable

by

Solomon Feferman * (Stanford, Cal.)

Abstract. It is shown that if $L$ is an abstract model-theoretic language, the syntax of $L$ is represented in a structure $\mathcal{M} = (A, \ldots)$, the Löwenheim-Skolem property holds down to card $(A)$ and $\mu_2$ is uniformly i.d. in $L$ then the set of $L$-valid consequences of a set $S$ of sentences is a.i.d. in $L$ whenever $S$ itself is so definable. This generalizes a theorem of Kunen for admissible fragments of $L_{\omega_1, \omega}$. The final part of the paper relates this to a program of study of good properties of model-theoretic languages.

Introduction. The aim of this note is quite different from that of the proceeding [F2], though it makes use of the same general preliminaries. In content, it is a sequel to [F1], § 3 where some connections were studied, for arbitrary languages $L$, between implicit definability of the satisfaction relation $\varepsilon$, and logical properties of $L$. The basic relevant notions of [F1] are recalled below, in particular that of the syntax of $L$ being represented in a structure $\mathcal{M} = (A, \ldots)$ and (relative to any such representation) that of $\varepsilon$ being uniformly invariantly implicitly definable (u.id) in $L$. We add here a related notion of a subset $S$ of $A$ being semi-invariantly implicitly definable (s.i.d) in $L$ (1). This includes Kunen's definition in [Ku] of s.i.d for admissible structures $\mathcal{M} = (A; \varepsilon, B_1, \ldots, B_k)$ as a special case, and in the same line extends model-theoretic generalizations of recursion theory. Kunen showed that being s.i.d is equivalent to other proposed

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(1) To be more precise, several variants of the notions s.i.d are introduced and compared in § 2. In accordance with one of these, u.id is rewritten $\# - u.id$. 
1. Further general preliminaries. It is assumed throughout, as in [F2], that $L$ is any regular, $L_{\text{adm}}$-closed language. $\text{Lm}(\tau)$ denotes the class of formulas of $L(\tau)$, explained as in [F2], 1.2. This determines the meaning of $\leq^L$, the elementary substructure relation for $L$.

1.1. Löwenheim-Skolem properties. By the cardinality $\text{card}(M)$ of a structure $M = (\langle M_i \mid i \in \omega \rangle \cup \ldots)$ is meant card$(\bigcup M_i)$. Let $k$ be any infinite cardinal.

**Definition 1.** $L$ is said to have the global $L$-$S$ property down to $k$ if whenever $M \subseteq M$ and card$(M) = k \leq \text{card}(M)$ then there exists $M' \subseteq M$ such that

(i) $M' \subseteq M$, $\text{card}(M') = k$, and

(ii) $M' \subseteq^L M$.

**Definition 2.** $L$ is said to have the local $L$-$S$ property down to $k$ if whenever $M \subseteq M$ and card$(M) = k \leq \text{card}(M)$, $\varphi \in \text{Lm}(\tau)$, and $M \models \varphi$ then there exists $M' \subseteq M$ with

(i) $M' \subseteq M$, $\text{card}(M') = k$, and

(ii) $M' \models \varphi$.

**Examples.** These are with reference to the list E1-E5 of [F2], 1.3. We are particularly concerned with whether $L$-$S$ holds down to card$(A)$ for $L$ represented in $A$. (The general notion of representation, in 2.2 below, is not needed here.) Verifications of the statements are standard.

E1. $L_A$ ($A$ admissible). Global $L$-$S$ down to card$(A)$. For $A \subseteq \omega^*$, local $L$-$S$ down to $\omega$.

E2. $L(Q)$. When card$(A) < \omega$, local $L$-$S$ does not hold down to card$(A)$. Global $L$-$S$ down to card$(A)$ for admissible $A$ with card$(A) \geq \omega$.

E3. $L_{\text{adm}}(\omega)$ ($\omega$-logic). Global $L$-$S$ down to $\omega$.

E4. $L_{\omega}$ (2nd order logic). Local $L$-$S$ does not hold down to $\omega$.

E5. $L_{\omega} \times \omega$ inaccessible. Global $L$-$S$ holds down to $\omega$.

1.2. Reducing the number of sorts. It suffices to deal with single-sorted structures in the case of $L_A$ by means of the method of unification of domains which replaces consideration of $(\bigcup M_i \cup \omega \cup \ldots)$ by that of $(\bigcup M_i \cup M_0, M_1, \ldots, M_n, \ldots)$. This method can also be applied to other languages, such as the $L(\omega, Q)$; but it cannot be applied to 2nd-order logic in the treatment followed here, according to which each domain $M_i$ of "individuals" must have an associated domain $M_i = (M, M_i)$. A slightly weaker requirement is that one can trade any domain $M_i$ for a unary predicate when $M_i$ happens to be a subset of some other $M_j$.

**Definition.** $L$ is said to have the sort-reduction property if for each $\tau \in \text{Typ}_L$ with

(a) $\text{Sort}(\tau) = (M_0, M_1, \ldots, M_n)$, $n \geq 1$,

and with each $\varphi \in \text{Lm}(\tau)$ and each $M \in \text{Str}(\tau)$, for which

(b) $M = (M_0, M_1, \ldots, M_n; B_0, \ldots)$ and $M \subseteq M_0$ are associated respectively, a new $\varphi \in \text{Typ}_L$, a formula $\varphi'$ of $L(\tau)$, and an $L$-structure $\mathfrak{M}$ of type $\tau$ satisfying the following conditions:

(i) $\text{Sort}(\varphi') = \text{Sort}(\tau) - (M_0)$, $\text{Symb}(\varphi') = \text{Symb}(\tau) \cup \{M_i\}$ (considering $M_i$ as a unary relation symbol).

(ii) $M' = (M_0; M_1, M_2, B_0, \ldots)$.

(iii) The free variables of $\varphi'$ are the same as those of $\varphi$, except that each $x^{(0)}$ is replaced by a variable $x^{(i)}$.

(iv) If $x$ is any assignment to the free variables of $\varphi$ then

$M \models \varphi(x) \Leftrightarrow M' \models \varphi'(x')$.

It is easily seen that the languages $E1$-$E5$ (among others) have the sort-reduction property. Whenever this property holds for $L$ and we have local $L$-$S$ down to card$(A)$ then from existence of expansions $\mathfrak{M}$ of $\mathfrak{M}$ having certain properties we can obtain existence of such $\mathfrak{M}$ without new sorts. This idea is applied in Theorem 3 of Section 2.3 below.

1.3. Joins of indexed structures.

**Definition.** $L$ is said to have the join property if for each $\tau \in \text{Typ}_L$ is associated $\tau \times \text{Typ}_L$ having at least one new sort $i$, such that we have operations $\varphi \mapsto \varphi[i]$, $\mathfrak{M}_i : \tau \mapsto \mathfrak{M}_i$, and $\mathfrak{M} \mapsto \mathfrak{M}_i$ satisfying the following conditions:

(i) $(\text{Join}) \left( \sum_{\tau} M_i \right) \in \text{Str}(\tau)$ whenever $M_i \in \text{Str}(\tau)$ for each $i \in I$, and

$\sum_{\tau} M_i = (I, \ldots)$. 
(ii) (Projection) If $\mathfrak{M} \vDash \text{Str}(x^2)$ and $R = (I, \ldots)$ then $\mathfrak{M}_0 \vDash \text{Str}(x)$ for each $i \in I$ and $\left( \sum_{i \in I} \mathfrak{M}_i \right)_0 = \mathfrak{M}_0$ for each $k \in I$.

(iii) If $\varphi \in \text{Fm}(x^2)$ then $\varphi^x \in \text{Fm}(x^2)$ and $\varphi^x$ has the free variables of $\varphi$ together with a new free variable $x$ of sort $I$.

(iv) For any $\mathfrak{M} = (I, \ldots) \vDash \text{Str}(x^2)$, $i \in I$, $\varphi \in \text{Fm}(x)$ and assignment $\sigma$ to the free variables of $\varphi$ in $\mathfrak{M}_{0,i}$,

$\mathfrak{M} \vDash \varphi^x(t, \sigma) \iff \mathfrak{M}_0 \vDash \varphi(s)$.

All familiar languages (including $E_1$-$E_5$) have the join property. We illustrate the verification of this for $L_4$ and for $x$ single-sorted with one binary relation symbol $R$. Given $\mathfrak{M}_r = (M; R)$ for $i \in I$, take

$\sum_{i \in I} \mathfrak{M}_i = (I, \bigcup_{i \in I} M_i \cup \{i\} \times M, \bigcup_{i \in I} \{i\} \times R)$.

Thus $\text{Sort}(x^2) = \{I, M\}$ and $\text{Symb}(x^2) = \{M^r, R^r\}$ with $M^r$ binary, $R^r$ ternary. Inversely, given

$\mathfrak{M} = (I, M; M^r, R^r)$

of type $x^2$, define $\mathfrak{M}_0 = (M_0; R_0)$ by

$x \in M_0 \iff \langle i, x \rangle \in M^r$ and $\langle x, y \rangle \in R_0 \iff \langle i, x, y \rangle \in R^r$.

$\varphi^x(u, \ldots)$ is obtained from $\varphi(\ldots)$ by replacing each atomic $R(x, y)$ by $R^r(u, x, y)$ and replacing each quantifier $\forall \langle \ldots \rangle$ by $\forall \langle x \rangle$.

The join is thus a kind of disjoint union operation displaying the index set as a new sort. This particular method of realizing the join property must be modified slightly in special cases like 2nd order logic.

§ 2. Representation of syntax; implicit definability of syntactic and semantic notions.

2.1. Valid consequence.

Definition. Given $S \subseteq \text{Sto} \{x\}$ with $\text{Mod}(S) \neq 0$ and $\varphi(x_1, \ldots, x_n)$ in $\text{Fm}(x)$ write

$S \vDash \varphi$ if $\text{Mod}(S) \subseteq \text{Mod}(\bigwedge x_1 \ldots \bigwedge x_n \varphi)$.

$S \vDash \text{Vd}(\tau) \iff \langle \varphi \vDash \text{Sto}(\tau) \text{ and } S \vDash \varphi \rangle$.

when $\text{Mod}(S) = 0$ we write $S \vDash \text{Vd}(\tau) = A$. $S$ is omitted in these notations when $S$ is empty.

2.2. Representation of syntax in a structure.

Definition 1. $L$ is said to be represented in $\mathfrak{A}$ relative to $\langle \pi_a(x) \rangle_{x \in A}$ if $\mathfrak{A} = (A, \ldots)$ is an $L$-structure and each $\pi_a(x)$ is an $L$-formula such that:

(i) $\bigcup_{i \in I} \text{Sto}(\tau) \{x \in \text{Type}_L\} \subseteq A$,
(ii) $\mathfrak{A} \vDash \pi_a(a)$ for each $a \in A$,
(iii) $\forall \rho_n(a)/\pi_a(x) \rightarrow \varphi(x) = y$ for each $a \in A$.

By (ii), (iii), not only does $\pi_a$ define $a$ in $\mathfrak{A}$, but it defines a unique element $a'$ in each structure satisfying $\exists x \pi_a(x)$.

It is not required that $\mathfrak{A}$ be single-sorted; $A$ is just a distinguished domain of $\mathfrak{A}$. But we shall assume that $A$ is of maximum cardinality among the domains of $\mathfrak{A}$, i.e.

$\text{card}(A) = \text{card}(\mathfrak{A})$.

The principal example is that of set-theoretical representation, i.e., where $\mathfrak{A} = (A, \varepsilon)$ and $A$ is a transitive set. Namely, if $L$ contains $L_4$, conditions (ii) and (iii) are satisfied by use of the following (inductively defined) $L_4$-formulas:

$\mu_{\alpha}(x) = \bigwedge \{ y \mid y \varepsilon x \leftrightarrow \bigvee_{b \in \alpha} \mu_{\pi_0}(y) \}$,

$\pi_a(x) = \bigwedge \bigvee \{ y \mid \mu_{\pi_0}(y) \}$.

Roughly speaking, $\mu_{\alpha}$ describes a with respect to its members, while $\pi_a \alpha$ describes it with respect to its entire set of predecessors, taking $b \leq_{\alpha} \varepsilon \alpha \iff b \in \text{TC}(\alpha)$.

Every familiar language $L$ (including those in $E_1$-$E_5$), for which $\text{Sto}(x)$ is a set, has a representation in a transitive $A$ such that $L$ contains $L_4$.

For in every case formulas are built up from atomic formulas by repeated application of syntactic operations and constitute certain closed well-founded trees; these can be identified in a canonical manner with certain sets in the cumulative hierarchy.

Remark. It may be expected that all "natural" languages which will ever be used will have such a set-theoretical representation. One might therefore think of including this as part of the abstract definition of model-theoretical language. That has not been done here (or elsewhere) since many (if not most) results of abstract model theory do not depend on such a build-up. (Even the results below require only special aspects of such a build up, just those given in the preceding definition.) It is also possible that languages with some kind of non-well-founded formulas could be used for counter examples.

The language $L_{\text{on}}$ is not represented in a set. One could extend the notion to that for representability in classes, but it seems preferable to deal with $L_{\text{on}}$ where $\forall_{\text{on}}$ satisfies explicit closure properties (such as satisfying ZF) in place of $L_{\text{on}}$.

($^1$) Corollary to [P1], § 3.3(3); write $\forall' z$ for $\forall z$, as in formula (2) of the text here.
Before languages with infinite formulas were taken up systematically, one considered representations in the natural numbers, or in the set of all finite sequences from a finite alphabet. For representation in \( \mathcal{U} = \{o <\}\) we may take each \( m_u(x) \) \((u \in o)\) to express that \( x\) has exactly \( n-1 \) predecessors, and \( \sigma_u(x) = m_u(x) \land \land_{x \in \mathcal{Y}} \neg m_u(y) \).

We assume throughout the following that \( \mathcal{U} = \langle A, \ldots \rangle \) is any \( L \)-structure and \( \tau_a(x) \) any sequence relative to which \( L \) is representable in \( \mathcal{U} \). \( \tau_a \) denotes the type of \( x \).

**Definition 2.** For \( \mathcal{U} = \langle A', \ldots \rangle \) of type \( \tau_a \), we write \( \mathcal{U}_0 \subseteq \mathcal{U}_1 \), and call \( \mathcal{U}_1 \) a \( \pi \)-extension of \( \mathcal{U}_0 \) if

1. \( \mathcal{U}_0 \subseteq \mathcal{U}_1 \)
2. \( \mathcal{U}_1 \ni \sigma_u(x) \) for each \( x \in A_0 \).

More generally we write \( \mathcal{U}_0 \rightarrow \mathcal{U}_1 \) if

1. \( \mathcal{U}_0 \rightarrow \mathcal{U}_1 \) is an embedding and
2. \( \mathcal{U}_1 \ni \sigma_u(x) \) for each \( x \in A_0 \).

In the context of set-theoretical representation for \( \mathcal{U} = \langle A, \varepsilon \rangle \), \( A_0 \subseteq A \), \( \mathcal{U} = \langle A', \varepsilon' \rangle \), then \( \mathcal{U}_0 \subseteq \mathcal{U}_1 \) just in case \( \mathcal{U}_1 \) is an \( \pi \)-extension of \( \mathcal{U}_0 \). Thus, more generally, there exists a \( \pi \)-extension \( \mathcal{V} \) of \( \mathcal{U} \) isomorphic to an \( \pi \)-extension of \( \mathcal{U}_0 \) in that case \( \pi \) is unique. Given \( x \in A_0 \), \( a' \in A' \) we have

\( \mathcal{U}_1 \ni \sigma_u(x') \)

if and only if \( a' = \pi(x) \). Then \( \pi_1 : \text{To}(\langle A \rangle) \rightarrow \mathcal{U} \) for the restriction \( \pi_1 \) of \( \pi \) to \( A \).

2.3. **Invariant implicit definability of subsets of \( A \).** We shall have to make considerable use in the remainder of the paper of projections of \( L \)-elementary classes. Following [F2], I, 1(1), we write:

\[
\mathcal{U} \ni \exists x_{\psi} \psi \Leftrightarrow \exists x^* \psi \Leftrightarrow \exists x \exp(\mathcal{U}) \quad \text{(3.1)}
\]

whenever \( \psi \in \text{Type}_1 \), \( \psi \in \mathcal{U} \), \( \psi \in \text{Sym}(\mathcal{U}) \). When there is no ambiguity, we simply write \( \mathcal{U} \ni \exists x \) if this holds.

Some motivation for the following definitions and comparison of the terminology with previous usage will be found in the next section 2.4. We abbreviate \( (\text{semi-}) \pi \)-invariantly implicitly definable by \( (\pi) \).

The subscript \( \pi \) abbreviates extra (for the case of possibly new sorts) and \( \pi \) abbreviates sharply.

\( L_1 \) is assumed to be any language contained in \( L \) ([F1], § 3.1); for simplicity this can be taken so that

\[
\text{St}_{L_1} \subseteq \text{St}_L.
\]
\#-i id. For the latter two we can also make use of the following, which is easily proved.

**Lemma.** \( \# \text{-i id}. \) \((S \text{ is id) in } L \Leftrightarrow S \text{ and } A \Leftrightarrow S \text{ are id in } L). \) The same holds with the qualifications \( \# \) on both sides, as well as for the unrestricted versions.

All of the foregoing may be relativized to parameters \( a_1, \ldots, a_n \) in \( A \) simply by replacing \( \mathcal{M} \) throughout by \([\mathcal{M}, a_1, \ldots, a_n]\). This is unnecessary in the case \( L = L \) since there we have every \( \pi_{a} \) definable in \( a \).

All notions relativized to some choice of parameters are indicated in boldface.

### 2.4. Comparisons with previous notions.

These were introduced successively by Fraïssé [Fr], Kreisel [Kr] and Kunen [Ku] for model-theoretic generalizations of recursion theory to a variety of structures \( \mathcal{M} \). The generalized notion of \( \# \text{-i id} \) subset \( S \) of \( \mathcal{M} = (A; R_1, \ldots, R_k) \) given in [Fr] agrees with that of being id in \( L_{\omega, \omega} \) if we replace \( \mathcal{M} \subseteq \mathcal{M} \) by \( \mathcal{M} \subseteq \mathcal{M} \).

The definitions of [Kr] apply to structures \( \mathcal{M} = (A; F, R_1, \ldots, R_k) \) with \( F \) binary and make use of a relation of \( \# \text{-i id} \) generalizing that of end-extension. The notions id (i:id) of [Ku] are equivalent to those here of id, (i:id) in \( L_{\omega, \omega} \) if we replace \( \mathcal{M} \subseteq \mathcal{M} \) by \( \mathcal{M} \subseteq \mathcal{M} \). This includes as a special case the definitions given for ordinals \( (\alpha < \beta, R_1, \ldots, R_k) \) in [Kr].

The other cases of principal interest are \( \mathcal{M} = (A; \epsilon, R_1, \ldots, R_k) \) which are admissible (w.r. to \( R_1, \ldots, R_k \)). In these particular cases, being id in \( L_{\omega, \omega} \) is equivalent to the set-theoretical representation \( \pi_{a} \) and agrees with Kunen’s id because \( \mathcal{M} \subseteq \mathcal{M} \) is the same as end-extension. Kunen obtained various results on the \( \# \text{-i id} \) sets. In particular he showed for countable admissible \( \mathcal{M} \), that id agrees with \( \mathcal{M} \) (i.e. with the class of sets proposed as \( \# \text{-i id} \) by Kripke and Platek \( ^{(*)} \)).

The motivations for these definitions are perhaps better understood if one considers the ideas for the proof in the particular case that \( L = L_{\omega, \omega} \), \( \mathcal{M} = (\omega < \epsilon, R_1, \ldots, R_k) \) and \( S \subseteq \omega \) of:

\[ \begin{align*}
(1) & \quad S \text{ is id } \Leftrightarrow S \text{ is ree. enum. in } R_1, \ldots, R_k. \\
(2) & \quad (a_1 < B_1, \ldots, B_{\omega}, S, \omega, \ldots, B_{\omega}) \in \mathcal{M}. 
\end{align*} \]

\(^{(*)}\) Kunen also obtained characterizations for uncountable \( \mathcal{M} \) as well, in some cases as \( \mathcal{M} \) and in other cases as \( \mathcal{M} \). An elegant refinement of these for arbitrary admissible \( \mathcal{M} \) was found by Barwise [B2].

Whenever it is verified on the basis of \( \varphi \) that \( a \in S \) (where of course \( a \in A \)) we make use only of some finite segment of \( \omega \). Hence for each \( a \in S \) there exists \( b \in \omega \) with \( b \geq a \) such that if

\[ \begin{align*}
(3) & \quad (A', <', R_1, \ldots, S', S_1', \ldots) \in \mathcal{M} \quad \text{and} \quad (A'; <') \text{ is an end-extension of} \\
& \quad (b, <) \quad \text{then} \quad a \in S'. 
\end{align*} \]

It follows that \( S \) is id.

In fact we can obtain here the stronger conclusion that \( S \) is \( \# \text{-i id} \) simply by adjoining to \( \varphi \) the statements \( \lambda \) and \( \sigma \) that there is a least element and that every element has a unique successor (resp.). Once the position of \( a \) is fixed by \( \pi_{a} \), the verification of \( a \in S' \) holds equally in any structure \( (3) \) satisfying \( \varphi \land \lambda \land \sigma \).

### 2.5. Invariant implicit definability of semantic notions.

The following definition was given in [El] \(^{(3)} \). Here \( \bar{a} \) abbreviates \( \bar{a} \).

**Definition.** Suppose \( T: \text{Str}_T(\tau) \rightarrow \bar{S}(\Lambda) \) where \( \tau \in \text{Typ}_T, \tau_{\bar{a}} \cap \tau = 0 \).

\( T \) is \#-i unid in \( L \) if for some \( \tau^* \) and \( \theta \) we have:

\[ \begin{align*}
(1) & \quad \tau^* \supseteq T \cup \langle{T} \rangle \quad \text{and} \quad \theta \in \text{St}_{\text{Typ}}(\tau^*), \\
(2) & \quad \text{for each } \mathcal{M} \in \text{St}_{\text{Typ}}(\tau), \langle \mathcal{M}, \mathcal{M}, T(\mathcal{M}) \rangle \vdash \delta, \text{ and} \\
(3) & \quad \text{if } a \in A \text{ and } (\mathcal{M}, \mathcal{M}, T(\mathcal{M})) \vdash \delta \text{ and } \mathcal{M}, \pi_{a}(a') \text{ then } a' \in T \Rightarrow a \leq \langle{T} \rangle. 
\end{align*} \]

It is clear how one would define notions of asid and uaid or without \# or \( \bar{a} \). However, only the preceding will be used below.

The main application of interest is to the function \( T_{L_\omega} \) or more precisely \( T_{L_\omega} \) for any \( \tau \in \text{Typ}_T \) (renamed as so as to be disjoint from \( \tau_{\bar{a}} \))

\[ \begin{align*}
(1) & \quad \text{Tr}_{L_\omega}(\mathcal{M}) = \langle \varphi \rangle : \forall \varphi \in \text{St}_{\text{Typ}}(\tau) \text{ and } \mathcal{M} \vdash \delta. 
\end{align*} \]

**Definition.** \( L_\omega \) is \#-i id in \( L \) for each \( \tau \in \text{Typ}_T \).

It was shown in [E1], § 3 ("Adequacy Theorem") that \( L_\omega \text{ is}\) adequate to truth in itself, for any admissible \( A \); the same method of proof also works for all familiar languages (including \( \mathbb{E}_1 - \mathbb{E}_5 \)). Roughly speaking, to prove this we first considered the usual implicit definition relation

\[ \begin{align*}
\text{Sat}_{L}(\mathcal{M}) = \langle \langle \varphi \rangle, \bar{\varphi} \rangle : \forall \varphi \in \text{Fn}_{\text{Typ}}(\tau) \text{ and } \bar{\varphi} \in \text{St}_{\text{Typ}}(\tau) \text{ and } \mathcal{M} \vdash \delta(\varphi) \}
\end{align*} \]

\(^{(3)}\) The symbol \# and subscript \( a \) were not used in designating this notion in [F1]; they are added here to be in accord with Definitions 1 and 2 of § 2.5.
where $\mathfrak{M}^{<\omega}$ is the set of all finite (or eventually constant) sequences from $\mathfrak{M}$. $\mathfrak{M}^{<\omega}$ is introduced as an auxiliary sort. This is a sharp (â) implicit definition because the answer to each question is: $\langle \varphi, \psi \rangle \in \text{Sat}(\mathfrak{M})^+$ for proper subformulas $\varphi$ of $\varphi$ (so $\varphi < \text{yo} \varphi$). The uniformity lies in using the same $\theta$ to express the inductive clauses independent of $\mathfrak{M}$. In contrast to the syntactic role of $\mathfrak{M}$, $\mathfrak{M}$ must be kept fixed in $[\mathfrak{M}^0, \mathfrak{M}^0, T', ...]$ in order that quantification be absolute.

It is seen from the proof that already $L_a$ is adequate to truth in $L_a$. In fact, the same holds for the languages $E^3 E^3$, i.e. the implicit definition of truth of these $L$ can always be given by an ordinary finite formula, using of course the restriction of satisfaction to $L_a$-admissible structures. This gives interest to the following observation.

**Theorem 3.** Suppose $L_a$ is $L_{a,a^*}$-closed and that $L_a$ is adequate to truth in $L$. Then whenever $S$ is sid$_a$ in $L_a$ it is also sid$_a$ in $L_a$.

**Proof.** Let $\varphi \in \text{St}_{L_a}(\tau)$ satisfy the conditions to make $S$ sid$_a$ in $L_a$, and $\theta$ the conditions to make $T(L)(\tau)$ in $L$. We add the parameter $\theta$ to $\mathfrak{M}$ considered as an element of $A$. Then the sentence $\theta \land \exists L$ expresses that $\varphi$ is true and serves to make $S$ sid$_a$ in $L_a$.

**Corollary 4.** If, in addition to the hypotheses of the preceding theorem:

(i) $L_a$ satisfies the sort-reduction property, (ii) $L_a$ satisfies local $L$-S down to card$(A)$, then $S$ sid$_a$ in $L_a$ implies $S$ sid$_a$ in $L_a$.

**Corollary 5.** If further, $L_a$ has the set-theoretical representation in $\mathfrak{M} = (A, \tau)$ and $L_a \subseteq L_a$ then $S$ sid$_a$ in $L_a \Rightarrow S$ sid$_a$ in $L_a$.

In particular, as we have seen, all these hypotheses apply to $L = L_a$ (admissible) and $L_a = L_{a,a^*}$.

§ 3. The principal theorems.

**Theorem 1.** Suppose that $L_a$ satisfies the join property and that $L_a$ is adequate to truth in $L$. Then for each $a \in \text{Type}_L$, $\text{Vd}_L(\tau)$ is sid$_a$ in $L_a$.

**Theorem 2.** Under the same conditions as Theorem 1, if $S_a \subseteq \text{St}_{L}(\tau)$ and $S_a$ is sid$_a$ in $L$ then $S_a - \text{Vd}_L(\tau)$ is sid$_a$ in $L_a$.

**Theorem 3.** Under the same conditions as Theorem 1, if in addition

(i) the local $L$-S property holds down to card$(A)$ and

(ii) $L_a$ satisfies the sort-reduction property then we can conclude that $\text{Vd}_L(\tau)$ is sid$_a$ in $L_a$ and that $S_a - \text{Vd}_L(\tau)$ is sid$_a$ in $L$ whenever $S_a$ is sid$_a$ in $L_a$.

**Proofs.** We begin with a proof of Theorem 2. We then see how to get the stronger conclusion of Theorem 1 in case $S_a = 0$.

Let $S = S_a - \text{Vd}_L(\tau)$ be the class of valid consequences of $S_a$. Since by definition $S = A_0$ when $\text{Mod}(S_0) = 0$, we may assume $\text{Mod}(S_0) \neq 0$. Thus for each $a \in A$,

(1) $a \in S \iff a \in \text{St}_{L}(\tau) \land \forall \mathfrak{M} \in \mathfrak{M}_a \exists \mathfrak{M} \in \mathfrak{M}_a \text{Mod}(\mathfrak{M}) = 0 \text{Mod}(S_0) = \mathfrak{M} \ni a$.

Then $a \in S \iff \exists \mathfrak{M} \in \mathfrak{M}_a \ni S_0$ and $S_0 \ni a$. For each $a \in A$ choose $\mathfrak{M}_a$ in such a way that

(2) $\text{Mod}(\mathfrak{M}_a) = \mathfrak{M}_a \ni S_0 , \text{Mod}(\mathfrak{M}_a) = \mathfrak{M}_a \ni a$.

Take $\mathfrak{M}_a$ to be arbitrary of type $\tau$ when $a \in S$. Thus

(3) $a \in S \iff (\forall b \in \mathfrak{M}_a \mathfrak{M}_a \ni b) \Rightarrow \mathfrak{M}_a \ni a$.

Let $\theta$ make $T(L)(\tau)$ sid$_a$ in type $\tau$. In particular,

(4) (i) for each $a \in A$, there is a $\tau^*_{\theta}$-expansion $S^*_{\theta}$ of $[\mathfrak{M}, \mathfrak{M}_a, \text{Tr}(\mathfrak{M}_a)]$ such that $S^*_{\theta} \ni \theta$ and

(ii) if $[\mathfrak{M}, \mathfrak{M}_a, T'] \ni \theta$ and $\mathfrak{M}_a \ni \theta(b')$ then $b' \in T' \Rightarrow b \in \text{Tr}(\mathfrak{M}_a)$.

We write $T$ as the unary relation symbol for $\text{Tr}$ in $\theta$. Using the join property, take

(5) $\mathfrak{M}^\tau = \sum \mathfrak{M}^\tau_{\theta}$, $\mathfrak{M}^\tau = \text{type of } \mathfrak{M}^\tau$.

$\mathfrak{M}^\tau$ may be considered to be an expansion of $\mathfrak{M}, \mathfrak{M}^\tau = [\mathfrak{M}, ...]$. For each formula $\varphi(a_1, ..., a_n)$ of type $\tau^*$ we have a formula $\varphi^*(a_1, ..., a_n)$ of type $\tau^*$ such that for each $a \in A$, $b \in \mathfrak{M}$

(6) $\mathfrak{M}^\tau \ni \varphi^*(a, b) \iff \mathfrak{M} \ni a \ni \mathfrak{M} \ni \varphi(b)$.

More generally, given any $\mathfrak{R}$ of type $\tau^*$ with $\mathfrak{R} = [\mathfrak{M}, ...], \mathfrak{A} = (A', ...)$ and given any $a' \in A'$ we can form $\mathfrak{R}_{a'}$ so that for each assignment $s$ in $\mathfrak{R}_{a'}$

(7) $\mathfrak{R} \ni \varphi(a', s) \iff \mathfrak{R}_{a'} \ni \varphi(s)$.

Note for the following that $[\mathfrak{T}(\tau)]^\tau$ has the form $\mathfrak{T}^\tau(a, s)$. Let $\varphi$ give the sid$_a$ for $S_a$ in type $\tau^*$. We may assume $\tau^* = \tau^\tau$. Thus there exists

(8) $\mathfrak{M}^\tau = [\mathfrak{M}, S_a, ...]$ with $\mathfrak{M}^\tau \ni \psi$, such that whenever $[\mathfrak{M}, S'] \ni \psi$ and $\mathfrak{A} \subseteq \mathfrak{M}$ then $S_a \subseteq S'$.

$\psi$ has a symbol $S_a$ for $S_a$. Both $\mathfrak{M}^\tau, \mathfrak{M}^\tau$ have $\mathfrak{A}$ as their $\tau^\tau$-reducts, and their union forms a structure that we denote $\mathfrak{M}^\tau$:

(9) $\mathfrak{M}^\tau = \mathfrak{M}^\tau \cup \mathfrak{M}^\tau = [\mathfrak{M}, S_a, ..., ...]$. 

The following sentence $\Gamma$ will be shown to provide an $\#-$sidi$_S$ of $S_4$; it contains the $\#$-sidi$_S$ of $S_4$, together with the statement expressing that $\theta$ holds in each $\mathfrak{M}_S$ together with a statement expressing (3). This uses a new symbol $\mathcal{S}$.

(10) $\Gamma = \psi \land \bigwedge u \theta^2(u) \land \bigwedge u [S(u) \leftrightarrow [\bigwedge v [S(v) \rightarrow T^2(u, v)] \rightarrow T^2(u, w)]]$.

Here $u, v$ range over the sort $A$. By all of the preceding, $[\mathfrak{M}, S, S_4, \ldots] \models \Gamma$, i.e.

(11) $[\mathfrak{M}, S, S_4] \models \Gamma$.

Suppose now that

(12) $[\mathfrak{M}, S', S_4] \models \Gamma$ with $\mathfrak{M} \subseteq \mathfrak{M}'$.

Thus $\mathfrak{M}, S', S_4$ has an expansion in type $\tau \cup \tau'$ which satisfies $\Gamma$; this expansion is a union of two structures $[\mathfrak{M}, S_4, \ldots]$ and $[\mathfrak{M}, \ldots]$, of types $\tau$, $\tau'$ resp. We have $A \subseteq A'$, $\mathfrak{M} \models \pi_a(a)$ for each $a \in A$ and

(13) (i) $[\mathfrak{M}, S_4] \models \exists y$

(ii) $[\mathfrak{M}, \ldots] \models \bigwedge u \theta^2(u)$.

By (i), $S_4 \subseteq S_4'$. By (ii) we can form structures $\mathfrak{M}_{a(A)}$ for each $a' \in A'$, of the form

(14) $\mathfrak{M}_{a(a)} = [\mathfrak{M}, \mathfrak{M}_{a(A)}(a)]$, $T_{a(A)}(a)$, ...] and $\mathfrak{M}_{a(A)} \models \psi$.

From (4)(ii) we have for any $b \in A$ and $a' \in A'$,

(15) $b \in T_{a(A)} \Rightarrow b \in T_{b(a)}$.

Note that $T_{a(a)}(b') \cap b' \in T_{a(A)}$. To complete the proof, we show for each $a \in A$ that

(16) $a \in S \Leftrightarrow a \in S'$.

Suppose $a \in S$. By (10), (12) it suffices to show that

(17) the assignment $a$ to $u$ in $[\mathfrak{M}, S', S_4', \ldots]$ satisfies $\forall v [S(v) \rightarrow T^2(u, v)]$.

Here the hypothesis is equivalent to

$\forall v [v \in S_4' \Rightarrow \exists b' \in T_{a(A)}]$.

Assuming this we have in particular $\forall v [v \in S_4 = b \in T_{a(A)}]$, so by (15) $\mathfrak{M}_{a(A)} \models \psi$. Then also $\mathfrak{M}_{a(A)} \models a$ since $a \in S = S_4 \cup \forall v (\psi)$. Again by (15), $a \in T_{a(A)}$, i.e. $a$ satisfies $T_{a(A)}(u, v)$, q.e.d.

For the proof of Theorem 1 where $S_4 = 0$ we can simply take

(18) $\Gamma = \bigwedge u \theta^2(u) \land \bigwedge u [S(u) \leftrightarrow T^2(u, w)]]$. 

This is satisfied in $[\mathfrak{M}, S] = [\mathfrak{M}, S, \ldots]$. To show that we have $\#-$sidi$_S$ here, suppose

(19) $[\mathfrak{M}, S'] \models \Gamma$ and $a' \in A'$ and $\mathfrak{M} \models \pi_a(a')$.

We still have (14) and (15) as above. Thus if $a \in S$, i.e. $a$ is valid, then $\mathfrak{M}_{a(A)} \models b \Rightarrow a' \in A'$, hence $T_{a(A)}(a, a')$ is satisfied and $a' \in S'$ by (18), (19).

Thus (19) = $[a \in S' \Rightarrow a' \in S']$ which is as required for $\#-$sidi.

Finally, Theorem 3 is a direct corollary of Theorems 1.2 and § 2.5, Theorem 3. By the results of § 2.5 this generalizes [Ku] Theorem 3.2.

4. Good properties of model-theoretic languages.

4.1. Background. This follows on the discussions of Barwise [B1], Introduction, and as length in Kreisel [Kr2], concerning

(i) good properties of languages $L_a$ and

(ii) suggestions for finding new languages with such properties. I use "good" to cover what they had variously described as: useful, simple, basic, pleasing, balanced, coherent, etc. As examples of such properties one particularly had in mind the holding of suitable generalizations of

(a) the compactness theorem,
(b) recursive enumerability of the valid sentences, and
(c) the interpolation theorem.

At the time of those discussions there was reason for optimism due to the achievements on (a)-(c) for countable admissible $L_a$ of Barwise [B1], which at the same time showed the superiority of generalizations obtained by definability criteria in place of the crude cardinality criteria initially considered. This optimism was bolstered by the progress being made on (a), (b) for uncountable $L_a$ in [Ku] and [B2]. It continued with Kreisel's results [Ku] giving (a), (b) for fragments of $L_{\omega_1^{\omega_2}}(Q_0)$. However, the failure of interpolation in all these languages was annoying. Since then there have been no evident successes, let alone with the fairly specific scheme proposed by Kreisel [Kr2] (cf. 4.4 (1) below). This has led to pessimism as some as the prospects for further progress.

Abstract model theory should provide the proper setting in which to give precise formulations of the desired properties of languages.

It is conceivable one could then use such formulations to obtain definitive negative results, thereby justifying the current pessimism. Personally, I do not think such will be found, rather that abstract negative results will be useful to expand wide classes of languages from consideration. An example of this kind is already to be found in Barwise [B3].

The following is an attempt to formulate good properties of languages in abstract terms. As emphasized by Kreisel, the progress with the $L_a$ consisted in generalizing the notion of finiteness alongside generalizations
of being recursive and semi-recursive (recursively enumerable) for sets of syntactic objects. This notion is to be chosen not only with the compactness theorem in mind, but also with the idea that the syntactic objects themselves, considered as sets in the cumulative hierarchy $V$, should be generalized finite. Since we are after syntactically natural languages here, it is appropriate to restrict attention to $L$ having a set-theoretical representation in some $W = (\mathcal{A}; \varepsilon, \ldots)$ with $A \in V$, $A$ transitive.

Remarks. (i) As already stated, the syntactic objects in a natural language are canonically represented by coded well-founded trees, and hence may be identified with elements of $V$. It should not be expected, though, that every element of $A$ corresponds to a syntactic object.

(ii) The aim here relates to the problem of finding good generalizations of recursion theory (g.r.t.), but only for structures $W$ of sets. It is expected that a good g.r.t. should also be able to provide suitable notions for any structure $W$.

4.2. Some good properties (preliminary). These are formulated in terms of $L$, $W$, and three abstract classes $F$, $R$ and $\varepsilon R$ of subsets of $A$ called respectively the $W$-finite, $W$-recursive and $W$-semi-recursive sets. The first task is to set down desirable properties which interrelate all five; then the question will be how to choose $L_1$, $\ldots$, $\varepsilon R$ so as to satisfy them. This section will concern recursion theoretic properties, which do not involve $L$ explicitly. (They could involve $L$ implicitly in those cases where $F$, etc. are defined in terms of $L$, as, for example, suggested in 4.4(2) below.)

It is assumed that

1. (i) $A \in V$, $A \neq 0$, $A$ is transitive, and
2. (ii) $A$ is closed under $\{, \}$, $\cup$ and $\cap$.

Then sub-relations of $A$ are certain subsets of $A$; also functions are identified with their graphs. The following hypotheses are fairly standard.

2. (i) $F \subseteq R \subseteq \varepsilon R \subseteq F(A)$.
2. (ii) $R = \varepsilon R \cap \varepsilon R$.
2. (iii) $F$ is closed under $\cup$, $-A$ and $\varepsilon R$ is closed under $\cup, \cap$.
2. (iv) $\varepsilon R$ is closed under substitution by $A$-recursive functions.
2. (v) The functions $\{, \}$, $\cup$ and $\cap$ are $W$-recursive.
2. (vi) The $\varepsilon$-relation on $A$ is $W$-recursive.

These are not intended to exhaust desirable recursion-theoretic properties.

Remark. In the search for stronger languages than presently known, $A$ will likely satisfy very strong closure conditions, including being admissible. But languages represented in some $A$ satisfying weak closure conditions could be useful in other ways, e.g. in proof theory. For this reason closure of $R$ under bounded quantification is not listed as an hypothesis.

A principal condition which ought to be satisfied is that inductive definitions with $W$-recursive clauses always lead to $W$-semi-recursive sets. We formulate this rather strictly. By a rule on $A$ we mean a relation $R \subseteq A^2$ such that each $a \in \text{Dom}(R)$ is a function $a: \text{Dom}(a) \rightarrow A$ with $\text{Dom}(a) \neq \emptyset$. Whenever $R((a_n)_{n \in \mathbb{N}}, b)$ we write

$$\begin{array}{c}
\ldots \ a_n \ldots (\varepsilon I) \quad (R) \\
\frac{}{b}
\end{array}$$

A collection $(R_{n \in \mathbb{N}})$ of rules is said to be $W$-recursive if we have $R \in W$ where $R(a, b, c) \equiv R_{n}(a, b) \& c \in c$. Given any $X \subseteq A$ there is a least set $\text{Der}_W(X)$ of objects derivable from $X$ by the rules $(R_{n \in \mathbb{N}})$. Then we take

$$x \in F \cdot \text{Der}_W(X) \equiv \exists d (d \in F \& d \text{ is a derivation of } x \text{ from } X \text{ by means of the rules } (R_{n \in \mathbb{N}})),$$

where the notion of derivation is explained as usual. We should then require:

3. (i) If $X$ is $W$-finite and $(R_{n \in \mathbb{N}})$ is an $W$-recursive collection of rules then $F \cdot \text{Der}_W(X)$ is $W$-semi-recursive.

Remark. More generally, one should formulate a notion of uniform $W$-recursive (or $W$-semi-recursive) monotone operator $\Gamma(X)$ and require that $\Gamma(X) \subseteq X$ is always $W$-semi-recursive. This will not be done here.

4.3. Some good properties (cont.). We now turn to properties which explicitly concern $L$. These are given in groups which seem to me to correspond to some degree of reasonableness for demands on $L$, with those of group I being in the nature of minimum requirements. The later groups contain additional or stronger properties. (This ranking is to be considered as tentative.)

I. (i) $\text{St}_F \subseteq A$.
II. (ii) $\text{St}_F \in W$ and $\text{St}_F \in W$ for each $\tau \in \text{Typ}_L$.
III. (iii) $\text{St}_F \subseteq W$, i.e. every $L$-sentence considered as a set is $W$-finite.
IV. (iv) $L$ is regular, has the join property and the sort-reduction property.
V. (v) $\text{St}_F \subseteq \text{St}_F$ for each $\tau \in \text{Typ}_L$ and satisfaction for $L$ agrees with that for $L$ on admitted $L$ structures.
VI. (vi) The function $a: \text{Ext}_a$ is $W$-recursive where $\text{Ext}_a(x)$ is the definition of $a$ in $L_a$ given in § 2.2(3).
VII. (vii) $\text{Val}_L(\tau) \in W$ for each $\tau \in \text{Typ}_L$ i.e. the valid sentences form an $W$-semi-recursive set.
VIII. (viii) If $(T(C)(a))$ is $L$-categorical (as defined below) then $a \in F$.  

3 = Fundamenta Mathematicae, T. LXXXIX
II. (i) \( L \) is \( L_{\text{ac}} \)-closed.
(ii) \( S \in \mathcal{R} \) and \( S \subseteq \text{Stac}(\tau) \Rightarrow S = \text{Vul}(\tau) \in \mathcal{R} \); i.e. the set of consequences of an \( \mathcal{A} \)-semi-recursive set is again \( \mathcal{A} \)-semi-recursive.
(iii) If \( S \in \mathcal{F} \) then \( S \) is aid in \( L \) (as defined below).
(iv) If \( S \in \mathcal{R} \) then \( S \) is id in \( L \).
(v) If \( S \in \mathcal{R} \) then \( S \) is aid in \( L \).
(vi) \( L \) is adequate to truth in itself, i.e. \( Tr_L \) is \( \equiv \)-uid in \( L \).
(vii) \( L \) is truth-maximal; i.e. if \( Tr_L \), is \( \equiv \)-uid in \( L \) then \( L' \subseteq L \).

III. (i) \( \text{Vul}(\tau) = F \rightarrow \text{Dec}(\tau) \) for some \( \mathcal{A} \)-finite \( \tau \) and \( \mathcal{A} \)-recursive collection \( E \) of rules (completeness).
(ii) The local \( L \cdot S \) property holds down to \( \text{card}(A) \).
(iii) \( L_{\text{ac}} \) is adequate to truth in \( L \).

IV. (i) \( (F, sR) \) forms a compactness property, as defined in [F2], § 1.5; in particular if \( S \) is an \( \mathcal{A} \)-semi-recursive set of \( L \)-sentences and every \( \mathcal{A} \)-finite subset of \( S \) has a model then \( S \) has a model.

V. (i) \( F = A \).
(ii) For any \( S \in \mathcal{R} \), \( S = \text{Vul}(\tau) = F \rightarrow \text{Dec}(\tau) \) for some \( \mathcal{A} \)-recursive collection \( R \) of rules (strong completeness).

With reference to I(viii), \( \mathfrak{M} \) is said to be \( L \)-categorical if for some \( \varphi \in \text{Stac} \),
(i) \( \mathfrak{M} \models \varphi \)
(ii) \( \mathfrak{M} \models \varphi \) whenever \( \mathfrak{M}' \models \varphi \).

The definition of aid (absolutely implicitly definable) used in II(iii) is given exactly like that for aid above (Definition 1, § 2.3) except that we replace \( S \subseteq S' \) by \( S = S' \) in the conclusion of (iii). More generally, in the same way we define: \( S \) is aid in \( L_0 \). Being aid in the sense of [Ku] is then the same as being aid in \( L_{an} \).

Under the hypothesis of Corollary 5 of § 2.5, we also get:
\[ S \text{ is aid in } L \Leftrightarrow S \text{ is aid in } L_{an}. \]

These hypotheses are certainly met if all the properties I-III are satisfied.

4.4. The problem of choosing \( L, \mathcal{A}, F, R, sR \). This is the difficult part, if the aim is to satisfy a substantial portion of \( L \)-V in a language stronger than \( L_{an} \) or \( L_{an}(Q_0) \). The following is a scattered collection of proposals, remarks, examples, and questions.

1. Kreisel's scheme [Kr2]. The following identification was proposed:

\[ \mathcal{A} \text{-finite} = \text{aid in } L_{an}, \]

(ii) \( \mathcal{A} \)-recursive = \( \text{id in } L_{an} \),
(iii) \( \mathcal{A} \)-semi-recursive = \( \text{id in } L_{an} \).

In addition, Kreisel assumed that \( A \) satisfies elementary closure conditions guaranteeing at least representation of \( L_d \) in \( A \), and formal derivations, if used at all, are to be \( \mathcal{A} \)-finite. The following problem was stated (loc. cit.) p. 145: "What further conditions must \( A \) satisfy in order that the basic properties of PC generalize to \( L_d \) for the translation given [above]? For what extensions of \( L_d \) do these properties persist?"

This problem has been studied with reference principally to the properties of completeness in the form I(vii) and II(ii) and compactness, IV(i). The main results of § 3 of this paper are of interest with respect to the first of these. These and the stability result of § 2.5, Corollary 5 show that II(ii) is a consequence of the proposed definition of \( sR \) and, basically, the properties that local \( L \cdot S \) holds down to \( \text{card}(A) \) and that \( L_{an} \) is adequate to truth in \( L \). But the results do not seem to yield, for example, completeness of an \( \mathcal{A} \)-recursive collection of rules (III(i)).

The problem of [Kr2] with reference to the compactness property IV(i) has been studied only for the \( L_d \). Here the examples of Gregory [G] show that the above scheme definitely fails to give compactness on some (necessarily) uncountable admissible \( A \). (In addition, it is consistent with ZFC to assume \( F = A \).
Thus stronger closure conditions on \( \mathcal{A} \) than admissibility would certainly be necessary to insure IV(i). Kunen [Ku] and Barwise [B2] only found strong conditions which guarantee some partial compactness results.

Remark. Variants may be worth considering in place of the above scheme. For example, one may restrict to those \( S \) where the auxiliary relations are themselves already \( S \) and another notion intermediate between \( S \) and \( \equiv \)-aid was suggested near the end of § 2.4, to correspond to the idea that \( S \in B \) then this fact can be verified using only an \( \mathcal{A} \)-finite part of \( B \). One simple way to interpret this is that there exists \( b \in A \) such that whenever \( \mathfrak{M} \models \varphi \) and \( \mathfrak{M} \models \varphi \) then \( (a \in B = a' \in B') \). Another would use the proposed definition of \( F \) itself.

2. Generalized-finiteness from the \( L \)-categorical sets. The preceding scheme gives a general explanation of \( F, R, sR \) which depends only on \( \mathcal{A} \); they may then be used in the search for good \( L \). But one can well imagine alternative schemes where these notions depend essentially on \( L \). As an example, the following is immediately suggested by I(viii). For \( L \), define \( L \cdot \text{Cat} = \{ a \in (\text{TC}(a)), s \text{ is } L \cdot \text{categorical} \} \), and then take

\[ (i) F = L \cdot \text{Cat} \text{ and } A = \text{TC}(F). \]

This leaves open the choice of \( R \) and \( sR \). A natural companion choice by 4.2.5 would be

\[ * \]
(ii) \( sR = \text{all sets of the form } F - \text{Der}_R(X) \) where \( X \in F \) and \( R \) is an \( \mathfrak{K} \)-finite collection of rules, and

\[ R = sR \cap s\mathfrak{K}. \]

But one could also consider taking sid or the \( L \)-representable sets as defined in (3) below, for \( sR \).

The problem here is how to choose new \( L \) so that if we take \( F, A \) as in (i) then we have some good properties; at the very least we should want \( \text{St}_L \subseteq A \) and \( L_A \subseteq L \). The following is an imprecise conjecture: for any given reasonable \( L \), we can build up a least \( L \)-closed language with

\[ L_{\text{TCU-CMP}} \subseteq L \quad \text{and} \quad \text{St}_L \subseteq \text{TO}(L-\text{Cnt}). \]

Then we could start with any \( L \) whose properties are not satisfactory and try to obtain a good \( L \) from it in this way.

**Examples.** It may be of interest to compare the definitions of \( F \) proposed in (1), (2) on familiar languages. In the case of \( L_\epsilon \), it is countable and there is an \( \epsilon \)-admissible set that can be shown under (1) that \( A = F \) and that the same holds under (2) when \( A \) is also locally countable. (I do not know whether that hypothesis is essential.) For \( L_{\text{eq}} = L_{\text{EQ}}, \) Kunen [Ku] pointed out that

\[ \text{HC} \cup (\text{HC}) \subseteq F \]

on the definition \( F = \text{aid in } L_{\text{eq}} \). But definition (2) gives

\[ \text{HC} = F, \]

simply by applying local \( L \)-S down to \( \kappa \). It should be noted with this definition of \( F \), and those just above of \( sR \) and \( R \), that \( L_{\text{eq}} \) has all the properties I-III as well as IV(ii) and IV(i).

(3) **Generalised G"odel-Mostowski recursion theory** [M]. Let \( T_\epsilon \) be the set of sentences \( \bigwedge \pi_{\kappa}(x) \) for \( \kappa \times A \) together with \( \text{Diag}(\mathfrak{K}) \). The work in [M] suggests defining \( R, sR \) in terms of \( L \) by:

(i) \( S \in sR \iff \) for some \( \varphi, \psi, \)

\[ \forall a \in A \, \alpha \in S \iff \varphi \cup T_\epsilon \uplus \bigwedge \pi_{\kappa}(x) \rightarrow \text{Diag}(\mathfrak{K}) \]

and

(ii) \( S \in R \iff \) for some \( \varphi, \psi, \)

\[ \forall a \in A \, \alpha \in S \iff \varphi \cup T_\epsilon \uplus \bigwedge \pi_{\kappa}(x) \rightarrow \psi(x). \]

It is easily seen that every set which is \( sR \)-admissible is \( sR \) in this sense. But one would hope to do better and show that a single \( \varphi \) could be used independently of \( S \) (analogous to the system \( Q \) in arithmetic).

This approach leaves completely open how to determine \( F \). Again the problem is how to choose new \( L \) so that it has good properties at least with \( \mathfrak{K}, R, sR \).

4. **Strengthening the \( Q_{\mathfrak{K}} \) languages.** These would have been examples of good \( \mathfrak{K} \)-admissible languages if one had interpolation. The counter-example to this (due to Keisler) suggests introducing a certain quantifier with stronger expressiveness power, which directly gives the missing interpolant.

We add to any \( L_A \) a new operator \( Q^\varphi \) which is a quantifier binding pairs \( x, y \) of variables \( (E \) for Equivalence relation). Consider formulas built up using the operations of \( L_A \) and \( Q^\varphi \), \( \varphi \) for any \( \varphi \). Satisfaction is defined as follows:

\[ M \models Q^\varphi x, \forall y \varphi(x, y) \iff (\equiv) \]

has at least \( \kappa \) distinct equivalence classes, where \( (\equiv) \) is the equivalence relation:

\[ x_1 \equiv x_2 \iff x_1 \in \bigwedge \varphi [\varphi(x_1, y) \leftrightarrow \varphi(x_2, y)]. \]

(We may read \( x_1 \equiv x_2 \) as: \( x_1, x_2 \) are \( \varphi \)-indistinguishable).

Then for \( \varphi \) in \( L_{\mathfrak{K}} \) \( \exists R, M \models Q^\varphi x, \forall y \varphi(x, y) \) is in \( PC \cap \dot{E}^* \) but is not in general in \( E \) for \( L_{\mathfrak{K}}(Q_\mathfrak{K}) \); this is the counter-example to (even Soslin-Kleene) interpolation. Note that \( Q_{\mathfrak{K}}^\varphi(x) \) is definable using \( Q^\varphi \).

**Conjecture.** \( L_{\mathfrak{K}}(Q^\varphi) \) has a complete axiomatization for recursive rules of inference, so its valid sentences are recursively enumerable.

**Question.** What are the good properties of \( L_{\mathfrak{K}}(Q^\varphi) \) and of the \( L_{\mathfrak{K}}(Q^\varphi) \) more generally?

Subsequent to the writing of this paper, I learned that the conjecture above for \( L_{\mathfrak{K}}(Q^\varphi) \) is correct. This was established independently by J. A. Makowsky and J. Stavi; they have also obtained axiomatization and compactness results for a number of related stronger languages. Their work is to appear in a paper jointly with S. Shelah.

**Correction to [T1].** J. Stavi has brought to my attention that the argument for the Corollary (*) in [T1] § 3.4 is incorrect at one point. For, it follows from work of Cohen that there exist uncountable well-founded transitive models of ZF with only countably many ordinals; cf. Katriel, Ann. Math. Log. 1 (1970), p. 42. Certainly then there exist admissible \( A \subseteq H(\omega_1) \) for which \( \omega_1 \in A \). Thus the only the "if" part of the Corollary is justified by the proof. However, Stavi has observed that if \( CH \) is assumed then the "only if" part is also correct. He shows that in this case for each \( A \subseteq H(\omega_1) \) there exists \( \alpha \in A \) with card(\( \alpha \)) = \( \kappa \). Such \( \alpha \) may be used to produce a counter-example to Soslin-Kleene interpolation \( (\text{I}_{\mathfrak{K}})^{\text{Soslin}} \). It is of interest to consider the status of both card(\( \alpha \)) for \( \alpha \in A \) and \( (\text{I}_{\mathfrak{K}})^{\text{Soslin}} \) in the case that \( CH \) is false.

(*) This reads: For \( A \) admissible, \( L_A \) is truth-complete if and only if \( A \subseteq H(\omega_1) \).
On some functional equations with a restricted domain

by

Roman Ger (Katowice)

Abstract. The functional equations considered are of the form (1) and (2) where f, g, h map an abelian group G into the other abelian group H. We assume their validity for almost all (x, y) ∈ G × G and investigate the question whether there exist functions F, G, H almost equal to f, g, h respectively and fulfilling our equations everywhere. The notion “almost all” (“almost everywhere”) has been introduced in an axiomatic way.

§ 1. Recently there has been an increased interest in functional equations and inequalities whose validity is postulated “almost everywhere” (abbreviated to a.e. in the sequel). This a.e. is understood in various ways (see for instance [3], [7], [9], [8] and [6]). We shall be interested here in two functional equations,

\[ f(x+y)(f(x)+y) - f(x) - f(y) = 0 \]  \hspace{1cm} \text{(of Mikusiński)}

and

\[ f(x+y) = g(x) + h(y) \]  \hspace{1cm} \text{(of Pexider),}

related to the well-known Cauchy equation (cf. [4] and [1]), assuming their a.e. validity in the sense described explicitly below. Roughly speaking, we are going to answer the following question: does there exist a function F (or: do there exist functions \( F_1, F_2, F_3 \)) such that it satisfies (1) (or: they satisfy (2)) everywhere and \( f = F \) a.e. (or: \( f = F_1, g = F_2, h = F_3 \), a.e.)? Such a problem was first raised by P. Erdős [5] in connection with Cauchy's functional equation. Positively solved by N. G. de Bruijn [3] and independently by W. B. Jurkat [5], this problem was then investigated by M. Kuczma [9] in connection with convex functions and by the present author for polynomial functions (also with positive answers).